# Superconducting $p$-branes and extremal black holes

A. Chamblin
*DAMTP, Silver Street, Cambridge, CB3 9EW, England*

R. Emparan
*Department of Mathematical Sciences, University of Durham, Durham DH1 3LE, England*

G. W. Gibbons
*DAMTP, Silver Street, Cambridge, CB3 9EW, England*

(Received 4 June 1998; published 31 August 1998)

In Einstein-Maxwell theory, magnetic flux lines are ''expelled'' from a black hole as extremality is approached, in the sense that the component of the field strength normal to the horizon goes to zero. Thus, extremal black holes are found to exhibit the sort of ''Meissner effect'' which is characteristic of superconducting media. We review some of the evidence for this effect and present new evidence for it using recently found black hole solutions in string theory and Kaluza-Klein theory. We also present some new solutions, which arise naturally in string theory, which are *non*-superconducting extremal black holes. We present a nice geometrical interpretation of these effects derived by looking carefully at the higher dimensional configurations from which the lower dimensional black hole solutions are obtained. We show that other extremal solitonic objects in string theory (such as $p$-branes) can also display superconducting properties. In particular, we argue that the relativistic London equation will hold on the world volume of ''light'' superconducting $p$-branes (which are embedded in flat space), and that minimally coupled zero modes will propagate in the adS factor of the near-horizon geometries of ''heavy,'' or gravitating, superconducting $p$-branes. [S0556-2821(98)09518-6]

PACS number(s): 04.70.Bw, 04.40.Nr, 04.50.+h, 11.27.+d

## I. INTRODUCTION

As is well known, the phenomenon known as ''superconductivity'' was first discovered (and named) in 1911 by H. Kammerlingh-Onnes. Kammerlingh-Onnes, in the course of studying the electric resistance of certain metals which were cooled to liquid helium temperatures, found that the resistance of mercury dropped drastically as the temperature was reduced from 4 K to 3 K. Later authors found that the temperature range over which the drop in resistivity occurs is extremely small. Thus, scientists were led to discover the first well-understood property of superconducting media: Below a certain critical temperature ($T_c$), the electric resistance of the medium is zero (to within experimentally relevant bounds). This behavior is of course the origin of the term, ''superconductor.''

On the other hand, given a superconducting medium at some temperature $T<T_c$, it is always possible to get rid of the superconductivity by applying a minimum magnetic field $B>B_c$, where $B_c(T)$ is some critical value of the magnetic field which depends on the temperature $T$. The destruction of superconductivity by a sufficiently strong magnetic field, together with the fact that the superconductor has zero resistance, leads one inevitably to the conclusion that the magnetic induction must vanish inside a superconductor, i.e., $B=0$. This property of superconductors, which is actually experimentally observed (i.e., a magnet will ''float'' above a superconducting medium), is known as the ''Meissner effect.'' The Meissner effect is succinctly expressed by the statement that a superconductor displays perfect diamagnetism. It is this property of superconducting media which is the principal focus of this paper. In fact, in this paper we shall use the terms ''perfect diamagnet'' and ''superconductor'' interchangeably, even though technically perfect conductivity is only a necessary (not sufficient) condition for perfect diamagnetism.

One may view superconductivity at various levels. One may begin by constructing a purely phemonological macroscopic theory in which Maxwell's equations are taken as fundamental and one supplements them with constitutive relations, of which the most useful is the London equation. One may then pass to a classical thermodynamic formulation of the phenomenon. Finally one may attempt to identify the quantum mechanical microscopic degrees of freedom responsible. In this paper we shall mainly be concerned with the phemonological theory. We will establish the existence in classical supergravity theories of an analogue of the usual Meissner effect. We will also have some suggestions as to how the purely phenomenological theory may be extended to a thermodynamic and quantum mechanical theory.

In fact the behavior of magnetic field lines in the presence of strong gravitational fields has been under investigation for some time by many authors (see, e.g., [1–5]). In particular, in 1974 Wald [1] studied the behavior of Maxwell test fields in the presence of a rotating black hole described by the Kerr solution. Using the fact that a Killing vector in a vacuum spacetime acts as a vector potential for the Maxwell test field, it is not hard to see that as the hole is ''spun up'' and approaches extremality, the component of the magnetic field $B$ normal to the horizon tends to zero; thus, the flux lines are expelled in the extremal limit and the hole behaves like a perfect diamagnet.

 

This effect was noticed and then confirmed in Einstein-Maxwell theory, to linear order in the magnetic field, by Bičák and Dvořák [5]. In particular, they studied Reissner-Nordström holes in the presence of magnetic fields induced by current loops. In [5] very nice pictures are presented for the field lines around a hole as it approaches extremality, so that the emergence of the Meissner effect can actually be seen. More recently, the authors of [6] considered an Abelian Higgs vortex in the Reissner-Nordström background. It was shown that in the extreme limit (but not near extremality) *all* of the fields associated with the vortex (both the magnetic and scalar degrees of freedom) are expelled from the horizon of the black hole. The magnetic and scalar fields always ''wrap around'' the horizon in the extremal limit.

In this paper we shall first review the evidence that (light) $p$-branes are superconducting (Sec. II), and then attempt to extend the analysis to include the effect of self-gravitation (Sec. III). The appearance of a form of the Meissner effect on the extremal horizon of a brane (Sec. IV) leads us to perform a comprehensive analysis of magnetic fields in the vicinity of extremal horizons (Sec. V). We establish the existence of this effect in widely generic settings, which include Kaluza-Klein and string theories. Moreover, we also present some exact solutions for extremal black holes in external fields which exhibit this Meissner effect. These should serve to dispel the notion that the effect is an artifact of the linearized approximation to the theory which could disappear after including the back reaction. We also address (Appendix B) some subtle examples where apparently the field expulsion breaks down. A closer examination shows, however, that in those examples one should not have expected the expulsion to happen in the first place, because of an interaction induced by the presence of a Chern-Simons term.

## II. SUPERCONDUCTING EXTENDED OBJECTS: LIGHT BRANES

We begin with a description of the superconducting properties of *light* branes. That is, in this section we ignore the coupling of the $p$-branes to gravity, so that we may think of the branes as extended, sheet-like objects (of zero thickness) moving in a flat spacetime background, with dynamics described by a Dirac-Born-Infeld action. In the next section, we will consider the superconducting properties of spacetimes describing gravitating branes. The superconducting properties of light branes have been discussed previously by Nielsen and Olesen [7,8] and by Balachandran *et al.* [9] (superconducting vortices with non-zero thickness, such as those examined in [10], will not be discussed here). Before reformulating their ideas in a geometrical language which generalizes to the case of heavy branes we recall for the readers' convenience some basic facts about the Meissner effect.

Phenomenological accounts of superconductivity distinguish carefully between *perfect conductivity*, i.e. $\sigma\to\infty\Leftrightarrow\mathbf{E}=\mathbf{j}/\sigma=0$, and *perfect diamagnetism*, i.e. $\mu\to\infty\Rightarrow\mathbf{B}=0$. The former merely implies that $\partial\mathbf{B}/\partial t=0$ which in turn implies that an arbitrary amount of flux may be frozen into the sample depending upon initial conditions. The latter however goes some way to implying the Meissner effect, i.e. that flux is expelled from the material so that the superconducting state is independent of initial conditions.

One may regard the Meissner effect as a consequence of the so-called Becker-Heller-Sauter equation

$$\mathbf{E}=\lambda^2\partial\frac{\mathbf{j}}{\partial}t \tag{2.1}$$

for some constant $\lambda$. This yields (on use of charge conservation) the freezing of magnetic flux:

$$\frac{\partial}{\partial t}(\mathbf{B}+\lambda^2\mathrm{curl}\ \mathbf{j})=0. \tag{2.2}$$

The strictly stronger non-relativistic London equation

$$\lambda^2\mathrm{curl}\ \ \mathbf{j}+\mathbf{B}=0 \tag{2.3}$$

implies the Meissner effect more directly and yields, on use of Faraday's law curl $\mathbf{E}=-\partial\mathbf{B}/\partial t$,

$$\mathrm{curl}(\mathbf{E}-\lambda^2\mathbf{j})=0\Rightarrow\mathbf{E}-\lambda^2\mathbf{j}=-\mathrm{grad}\ \psi \tag{2.4}$$

for some scalar field $\psi$.

In a relativistic generalization of the London equation is

$$-\frac{1}{\lambda^2}F_{\mu\nu}=\partial_\mu J_\nu-\partial_\nu J_\mu \tag{2.5}$$

or

$$J_\mu=-\frac{1}{\lambda^2}A_\mu+\partial_\mu\Lambda \tag{2.6}$$

for some function $\Lambda$. Because $\nabla_\mu F_{\mu\nu}=-J_\nu$, we have

$$-\nabla^2 J-\frac{1}{\lambda^2}=0 \tag{2.7}$$

and so the mass of the vector field is given as $1/\lambda^2$. If $\Lambda=0$ and in the absence of charges Eq. (2.5) is equivalent to Eqs. (2.1) and (2.3). In what follows we shall adopt Eq. (2.5) as our criterion for superconductivity.

Balachandran *et al.* [9] have argued that Eq. (2.5) typically holds on the world volume $\Sigma$ of extended objects and Nielsen has shown, in the context of Kaluza-Klein theory, that the relativistic London equation will hold on the world volume of extended objects carrying Kaluza-Klein currents [7]. The basic idea behind Nielsen's observation is that if $K^a$ is a Killing vector field generating a circle subgroup of the Kaluza-Klein group $G$ of isometries of a higher dimensional Kaluza-Klein manifold $\mathcal{E}$, $\pi:\mathcal{E}\to\mathcal{M}$ is the projection onto the spacetime manifold $\mathcal{M}$ and

$$F^{ab}=\nabla^a K^b-\nabla^b K^a, \tag{2.8}$$

then $\pi_\star F^{ab}$ is the Kaluza-Klein field strength on spacetime $\mathcal{M}$. Now if $x:\Sigma\to\mathcal{E}$ is an immersion or embedding of a $(p+1)$-dimensional submanifold or brane $\Sigma$ and $x_\pi=\pi\circ x$ the projection down to spacetime $\mathcal{M}$, then the pullback $J$

$=x^{\star}K$ of the Killing vector field $K$ to the world volume $\Sigma$ yields (via Noether's theorem and the field equations for the embedding $x$) a conserved current $J$ on the world volume. But clearly pulling back Eq. (2.8) to the world volume shows that $\pi_{\star}F$ and $J$ satisfy the London equation on $\Sigma$; i.e., $\Sigma$ is superconducting with respect to the the Kaluza-Klein current. We shall refer to this type of superconductivity as Nielsen superconductivity.

So far we have not used any field equations, either for the brane or for the background in which it moves. For light branes in some fixed background the equations of motion of a brane with vanishing Born-Infeld field on the world volume and vanishing Ramond-Ramond fields in the bulk require that it be a minimal submanifold, a particular case of which is a totally geodesic submanifold. In the next section we shall see that some self-gravitating branes satisfying the Einstein equations may be identifed with totally geodesic submanifolds. We can then see to what extent they exhibit Nielsen superconductivity.

## III. SUPERCONDUCTING SELF-GRAVITATING EXTENDED OBJECTS

In the last section we investigated the superconducting aspects of extended objects which have decoupled from gravity. This limit, where the branes are "light" so that one may focus strictly on the world volume terms in the action, has been extensively studied by recent authors [11]. In this section we consider the complementary description of extended objects in supergravity theories, which comes from focusing on the "bulk" action terms, which describe the fields which propagate in the bulk away from the brane. These bulk terms are of course just the effective supergravity Lagrangian terms which are obtained from the low energy limit of string theory and/or M-theory. One may therefore approximate the gravitational fields of $p$-branes, at least semi-classically, by looking for solutions of the supergravity equations of motion with the relevant symmetries.

Generically, these solutions will have event and Cauchy horizons, and there will no longer exist any "brane world volume." A natural question, then, is where the degrees of freedom associated with the brane are located. Before tackling that question we shall consider some examples where the location of the brane is relatively unambigous.

One of the simplest such self-gravitating brane solutions is the 6-brane of 11-dimensional supergravity. Geometrically this is a product

$$\mathcal{E}\equiv M_{TN_k}\times \mathbf{E}^{6,1}, \tag{3.1}$$

where $M_{TN_k}$ is the multi-Taub-Newman-Unti-Tamburino (multi-Taub-NUT) metric with $k$ centers,

$$ds^2=V^{-1}(d\tau+\omega_i dx^i)^2+V dx^i dx^i, \tag{3.2}$$

with $V=1+\Sigma[1/(|\mathbf{x}-\mathbf{x}_i|)]$. The group $G=\mathrm{U}(1)$. The 6-branes are located at $\mathbf{x}=\mathbf{x}_i$. These are fixed point sets of the the Killing field $\partial/\partial\tau$ and hence, by a standard result, totally geodesic submanifolds. Not only does the Killing field vanish on the branes but so does the two-form (2.8).

Consider now two orthogonally intersecting sets of 6-branes. Geometrically we have the product

$$\mathcal{E}\equiv M^{x}_{TN_k}\times M^{x'}_{TN_{k'}}\times \mathbf{E}^{2,1}. \tag{3.3}$$

There are now two Kaluza-Klein U(1) Killing fields, i.e. $G=\mathrm{U}(1)\times\mathrm{U}(1)'$. One Killing field vanishes at $\mathbf{x}=\mathbf{x}_i$ and the other at $\mathbf{x}'=\mathbf{x}'_i$. However, apart from at the intersection, one U(1) Killing vector potential and the associated two-form (2.8) are non-vanishing on the 6-brane of the other type.

Clearly, away from the intersection, there is no expulsion of a gauge field from the brane of the other type. The intersection, which is itself a brane, is superconducting relative to both types of flux.

The example we have just given may be readily extended to the case of configurations of branes intersecting at angles discussed in [12].

So far we have not used the Einstein equations. To do so, we suppose that the Killing vector field $K$ is everywhere tangent to some submanifold $\mathcal{B}$ of $\mathcal{E}$. We may regard $K$ as a Killing field of $\mathcal{B}$. Of course $\mathcal{B}$ could be all of the spacetime manifold.

We now apply the Ricci identity to the Killing vector field $K$ to give

$$\nabla_i F^{ij}=-R^{ij}_{\mathcal{B}}K_j, \tag{3.4}$$

where[1] $R^{ij}_{\mathcal{B}}$ is the Ricci tensor of $\mathcal{B}$. Thus on $\mathcal{B}$ we have the London-like relation

$$J^i=2R^{ij}_{\mathcal{B}}K_j. \tag{3.5}$$

Moreover,

$$\nabla^2 K^i=-R^{ij}_{\mathcal{B}}K_j. \tag{3.6}$$

As an example, suppose that the spacelike submanifold $\mathcal{B}$ is spacelike, compact and has negative Ricci curvature; then a simple integration by parts argument shows that $K$ must vanish everywhere on $\mathcal{B}$. If $\mathcal{B}$ is Ricci-flat, then $K$ need not vanish, but if it does not, then it must be covariantly constant. This means that locally at least $\mathcal{B}$ is the metric product of a circle with a submanifold of one dimension less than that of $\mathcal{B}$.

The result we have just sketched is responsible for the well-known fact that closed Einstein manifolds with negative cosmological constant do not admit any Killing fields. However, we would like to view it in a a different way.

If $K$ vanishes on $\mathcal{B}$, then necessarily the restriction to $\mathcal{B}$ of $F=dK$ must also vanish. Thus the submanifold $\mathcal{B}$ might be said to exhibit a kind of Meissner effect. Because the math-

[1] Our conventions are that the signature is $(-++\cdots+)$, and that the sign of the curvature is given by $(\nabla_i\nabla_j-\nabla_j\nabla_i)K^m=R^m_{nij}K^n$.

ematical theorem we are appealing to is a particular case of a Bochner vanishing theorem, it seems appropriate to refer to this effect as the Bochner-Meissner effect.

We now turn to spacetimes with event horizons.

Clearly the brane is located somewhere in the vicinity of the horizon. For a generic non-dilatonic $p$-brane, the near-horizon geometry is a standard compactification of the form $(\mathrm{adS})_{p+2} \times S^{d_T-1}$, where $d_T$ is the dimension of the transverse space [13,14] (far from the brane the geometry is usually asymptotically flat, unless some global identification has been performed).

Now the metric on $(\mathrm{adS})_{p+2}$ may be written in so-called *horospherical* coordinates $(t,\mathbf{x}_p,z)$:

$$ds^2=\frac{1}{z^2}[-dt^2+d\mathbf{x}_p d\mathbf{x}_p+dz^2]. \tag{3.7}$$

These coordinates then provide a foliation of $(\mathrm{adS})_{p+2}$ by flat timelike hypersurfaces $z=\mathrm{const}$, which are called the "horospheres." If one embeds $(\mathrm{adS})_{p+2}$ as a quadric in $\mathbf{E}^{p+1,2}$, then the horospheres are the intersection of the quadric with a family of null hyperplanes.

[The notation here reflects the fact that in the case of hyperbolic space $H^{p+2}$, which is the Euclidean section of anti–de Sitter (adS) space, the analytic continuation of the constant $z$ slices of Eq. (3.7) is literally flat spheres, termed horospheres in the mathematics literature years ago. If one regards $H^{p+2}$ as the mass-shell in $(p+3)$-dimensional Minkowski spacetime $\mathbf{E}^{p+2,1}$, then horospheres are also the intersections of the quadric with a family of null hyperplanes.]

Now each horosphere may be thought of as a static test $p$-brane which solves the Dirac-Born-Infeld equations of motion of a $p$-brane coupled to the $p+1$ potential $A^{p+1}$ whose $p+2$ field strength $F^{p+2}=dA^{p+1}$ is proportional to the volume form of $(\mathrm{adS})_{p+2}$ [15]. In this way we obtain a particularly vivid picture of how the heavy supergravity brane is composed of many stacked light branes.

The limiting brane as $z\to 0$ corresponds to the causal boundary of $(\mathrm{adS})_{p+2}$. This conformal boundary has the topology of $S^1 \times S^p$, where the $S^1$ is the timelike factor and the $S^p$ is spacelike. In fact the boundary coincides (possibly up to a discrete identification) with the conformal compactification of $(p+1)$-dimensional Minkowski space $\mathbf{E}^{p,1}$ and the isometry group $\mathrm{SO}(p+1,2)$ of $(\mathrm{adS})_{p+2}$ acts by conformal transformations on the boundary. Thus, one is led to study the singleton and doubleton representations[2] of the group $\mathrm{SO}(p+1,2)$, in the hope of understanding the conformally invariant quantum field theory (QFT) on the boundary. In fact, this boundary QFT has precisely the same degrees of freedom as the world volume fields of the corresponding $p$-brane. A natural proposal is then that the lowest scalar component of the boundary field theory represents the transverse fluctuations of the $p$-brane. Indeed, most recently it has been conjectured [16] that information about the dynamics of superconformal field theories (in the large $N$ limit) may be obtained by studying the region near the horizon of certain $D(p)$-branes. Thus, the conjecture implies a correspondence between gauge theories in the large $N$ limit and compactifications of supergravity theories. The correspondence is often called "holographic" [17] because the superconformal field theory (SCFT) resides on the causal boundary of adS space.

It is now natural to propose that a gravitating $p$-brane is "superconducting" if the field theory on the boundary of the adS factor of the near-horizon geometry exhibits behavior characteristic of a superconducting phase. Typically, given any specimen in a superconducting phase we expect to find zero modes, i.e. minimally coupled eigenmodes of some wave operator which correspond to the unimpeded movement of charge in the medium. Thus, we are led to look for zero modes which "skim along" the horospheres in the adS factor.

From what we have said above, it is natural to look for such zero modes in the singleton (or doubleton) supermultiplets. After all, the singleton (or doubleton) field theories generically contain a number of massless scalar and spinor fields, which are trapped on the boundary of adS space (the "core" of the brane). (For an explicit discussion of the matter content of the superconformal multiplet of the M5-brane see e.g. [18].) The precise form of these multiplets is not important. What is important is that these massless modes skimming along the horosphere at infinity will naturally couple to any Kaluza-Klein currents on the brane. Put another way, if we wrap the brane on a circle (taking care to avoid any fixed-point singularities [19]), then the massless fermions on the dimensionally reduced brane will naturally couple to the Kaluza-Klein charge—these modes will induce a superconducting current on the reduced brane.

We are thus led to a pleasing microscopic description of the superconducting properties of self-gravitating branes. Since the supercurrent seems to reside right at the horizon of the brane, we would expect the horizon to display the Meissner effect. In the next few sections we will present a number of examples which confirm this effect for the horizons of extreme black holes. It would be interesting to perform similar tests for higher dimensional extremal self-gravitating branes.

Of course, all of this structure will break down for *non*-extreme black branes. As you approach the outer horizon, there is no splitting of the spacetime geometry into an adS factor and a compact factor. Furthermore, it is not possible to think of a non-extreme black brane as a stack of light branes, all hovering just outside of the horizon. We would not expect the outer horizon of a non-extreme brane to support a superconducting current, and therefore we would not expect such an object to display superconducting properties. These expectations are borne out when we consider non-extreme black holes. It is always possible to penetrate non-extreme black hole horizons with magnetic flux; superconductivity, it seems, is generically broken whenever we break extremality.

[2]Singleton representations of the adS group require a *single* set of oscillators transforming under the fundamental representation of the maximal compact subgroup of the covering group of the adS group; doubletons require two such sets of oscillators.

## IV. MEISSNER EFFECT FOR SUPERCONDUCTING STRINGS

In the preceding sections we have seen that the world volume of $p$-branes behaves like a superconducting medium with respect to gauge fields of Kaluza-Klein origin. In particular, a form of the London equation appears which implies the possibility of stationary currents in the absence of an external electric field. Another consequence of these macroscopic equations is that magnetic fields vanish inside the world volume, i.e. the Meissner expulsion of magnetic fields. As a matter of fact, the magnetic fields have a vanishing normal component to the world volume. Of course, in order for the magnetic field to be interpreted as a vector field, we must restrict ourselves to four spacetime dimensions.

When the effects of self-gravitation are included, it becomes less clear where the brane is localized. Thus, it is not so evident where the Meissner surface, where magnetic field expulsion takes place, should be located. The arguments in the previous section suggest that, at least for non-dilatonic branes, this should be in the near-horizon adS throat. Dilatonic branes are singular at the horizon, and the adS-SCFT correspondence becomes less clear, but the singular horizon (or the close vicinity of it) would be the natural place for the brane. In this and the following sections we will argue that the Meissner surface is always precisely at the horizon.

The reader may feel that there is an apparent conflation of objects of different dimensionalities here. Consider a string, which we will wrap on a circle in the Kaluza-Klein fashion. The world volume viewpoint of the previous sections would lead to the conclusion that the string carries a superconducting current along itself. In the reduced spacetime the string world volume will look like a point, and it does not make much sense to speak about the field being expelled from a point. However, when we include gravity in the picture, the string will develop a horizon, which (in $D=4$) will be seen as a 2-sphere (the fact that this might be singular will be dealt with later). Our claim is that magnetic Kaluza-Klein fields are expelled from the horizon.

Hence, our starting point is a string in $D=5$ which is wrapped to yield a black hole. The metric, in the Einstein frame, in $D=5$ is

$$ds^2=H^{-1/3}(-f dt^2+dz^2)+H^{2/3}\left(\frac{dr^2}{f}+r^2 d\theta^2+r^2\sin^2\theta d\varphi^2\right), \qquad (4.1)$$

where

$$H=1+\frac{q}{r}, \quad f=1-\frac{r_0}{r}. \qquad (4.2)$$

For $r_0\neq 0$ there is an event horizon at $r=r_0$. When $r_0=0$ the string is extremal.

If we compactify this geometry along the string direction $z$, we obtain a dilatonic black hole solution in $D=4$. In the previous sections we have seen that the string is superconducting with respect to the Kaluza-Klein gauge field $\mathcal{F}$ generated along this isometry.[3] Our aim is to show that the horizon behaves as a Meissner surface for this field in the extremal limit.

There is an obvious point of concern when dealing with the extremal limit of the solution (4.1): the proper size of the horizon is zero as measured in the Einstein frame. However, in four dimensions the gauge field equation is conformally invariant. This means that the field does not distinguish whether we are working in the Einstein, string, or any other conformal frame related to the one above by an overall rescaling of the metric by a factor of the dilaton. In particular, there exists a frame, namely $H^{4/3}ds^2$, in which the metric does not become singular at the horizon. In this frame it makes perfect sense to consider whether the field penetrates or not the horizon.

There is a well-known procedure to generate, upon reduction, an exact solution with an axisymmetric magnetic Kaluza-Klein field (see, e.g., [20] or [21]). Instead of identifying points along the orbits of $\partial/\partial z$, we twist the compactification direction to be along orbits of

$$\mathbf{q}=\frac{\partial}{\partial z}+B\frac{\partial}{\partial\varphi}. \qquad (4.3)$$

This is most easily done by changing to the adapted coordinate $\varphi\rightarrow\varphi-Bz$, such that $\mathbf{q}\varphi=0$. Here $B$ will be the asymptotic value of the magnetic field along the axis of the tube. The Kaluza-Klein gauge potential $\mathcal{A}_\mu$ reads, in terms of the original metric,

$$\mathcal{A}=\frac{q_\varphi}{|\mathbf{q}|^2}d\varphi=B\frac{g_{\varphi\varphi}}{g_{zz}+B^2 g_{\varphi\varphi}}d\varphi. \qquad (4.4)$$

This is clearly a conformally invariant expression. For the case under consideration,

$$\mathcal{A}=B\frac{Hr^2\sin^2\theta}{1+B^2Hr^2\sin^2\theta}d\varphi. \qquad (4.5)$$

We want to find the magnetic flux across a portion $\Sigma$ of the black hole horizon. This is given by the line integral $\int_{\partial\Sigma}\mathcal{A}$ on the horizon. If the horizon is at $r=r_0\neq 0$, then we find a non-vanishing flux across any portion of it. But in the extremal limit the horizon is at $r=0$, where $\mathcal{A}$ vanishes. So no magnetic flux penetrates the extremal horizon. The field is expelled from it: this is the Meissner effect. In Fig. 1 we have plotted the lines of force of the magnetic field for non-extreme and extreme configurations.

We would like to emphasize the fact that this analysis has been carried out at a level where the supergravity equations have been treated in an exact form. In particular, the field (4.5) is an exact field configuration in $D=4$ (together with the corresponding metric and string winding field).

[3] In order to avoid confusion with other gauge fields that may appear, throughout this and the following sections we will consistently use script letters for the field $\mathcal{F}$ that experiences the Meissner expulsion and its potential $\mathcal{A}$.

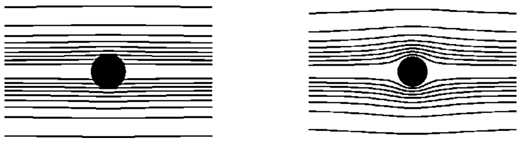

FIG. 1. Field lines of the Kaluza-Klein magnetic field $\mathcal{F}$ for the exact solution (4.5), for the black holes that result from compactification of non-extremal and extremal strings. The radius in Eq. (4.5) has been changed to ''Schwarzschild radius'' $r \to r-q$.

## V. MEISSNER EFFECT IN EXTREMAL BLACK HOLES

It is remarkable that this Meissner effect is not unique to extremal geometries derived from $p$-branes. In fact, as we argue below, it appears to be a rather generic feature of extremal black holes. Typically, the lines of force of a magnetic field penetrate the horizon of a non-extremal black hole. However, we will see that the lines of force fail to penetrate extremal horizons. Instead, they tightly wrap the black hole. *The horizon of an extremal black hole behaves like the surface of a perfectly diamagnetic object.*

To be more precise, in a superconducting material the magnetic field penetrates to some small distance from the surface: this is the penetration depth. For extremal black holes the penetration depth appears to be zero. Also, the perfectly diamagnetic state of the black hole breaks down at any finite temperature, i.e. for any deviation from extremality.

To our knowledge, this phenomenon was first pointed out in the literature by Bičák and Dvořák in [5], in the context of Einstein-Maxwell theory. We believe this to be a generic phenomenon for black holes in theories with more complicated field content, although a precise specification of the dynamical situations where this effect is present seems to be out of reach. The results below constitute very strong evidence that it is true whenever the gauge field couples minimally to the geometry, or possibly includes dilatonic couplings.

### A. Field expulsion from extremal rotating black hole

A first example (also noticed in [5]) of this Meissner effect follows from Wald's analysis [1] of a test magnetic field in the background of the neutral Kerr black hole. In [1] a solution for a field aligned with the axis of the black hole is constructed, by using the isometries of the Kerr background. Let us denote the axial and temporal Killing vectors of the Kerr solution by $\psi \equiv \partial/\partial\varphi$ and $\eta \equiv \partial/\partial t$. Then a test gauge field can be constructed as

$$\mathcal{A}_\mu = B\left(\psi_\mu + \frac{2J}{M}\eta_\mu\right) - \frac{Q}{2M}\eta_\mu . \tag{5.1}$$

$B$ is the magnetic field along the axis, and $Q$ is the charge that the black hole acquires, which we want to be zero. The field can be conveniently written in terms of the vector $\chi = \Omega_H \psi + \eta$, which is tangent to the null geodesic generators of the horizon. Here $\Omega_H$ is the angular velocity of the horizon. We find (with $Q=0$)

$$\mathcal{A}_\mu = \frac{B}{\Omega_H}\left[\chi_\mu - \left(1 - \frac{2\Omega_H J}{M}\right)\eta_\mu\right]. \tag{5.2}$$

In the extremal limit $2\Omega_H J = M$, and therefore $\mathcal{A}_\mu \propto \chi_\mu$, which vanishes precisely at the horizon. As in the preceding section, the flux along any portion $\Sigma$ of the horizon, $\int_{\partial\Sigma}\mathcal{A}$, vanishes. Again, the extremal horizon behaves like a perfect diamagnet.

This solution involved the magnetic field as a test field only. But it is possible to find an exact generalization of it within Kaluza-Klein theory. Start with the product of the (neutral) $D=4$ Kerr solution with a five dimensional direction $x^5$. We can now apply the ''twisted reduction'' procedure described in Sec. IV to put the $D=4$ neutral Kerr black hole in the background of an axisymmetric Kaluza-Klein magnetic field in an exact way. In order to avoid the presence of electric charge in the black hole, the compactification direction must also involve a twist in the time coordinate. Specifically, we identify points along orbits of the vector

$$\mathbf{q} = \frac{\partial}{\partial x^5} + B\left(\psi + \frac{2J}{M}\eta\right). \tag{5.3}$$

The *exact* Kaluza-Klein gauge field that follows is

$$\mathcal{A}_\mu = B\,\frac{\psi_\mu + \frac{2J}{M}\eta_\mu}{|\mathbf{q}|^2}, \tag{5.4}$$

which reduces to Wald's field in the linear approximation, and in the same way can be seen to exhibit the Meissner effect in the extremal limit. The reader may have noticed that Wald's solution does not contain any dilaton field, whereas the Kaluza-Klein solution does. But to linearized order in the test gauge field there is no contribution from a test dilaton [see, e.g., Eq. (5.13) below]. Therefore Wald's solution is the linear approximation to the axial field configuration for *all* Einstein-Maxwell-dilaton theories.[4]

Finally, in the solutions we have been considering the magnetic field is aligned with the rotation axis of the black hole. According to [5], the Meissner expulsion can also be seen for fields where no alignment is assumed.

### B. Field expulsion from spherically symmetric extremal throats

Now we would like to consider other classes of extremal black holes, and the most obvious candidates are charged (Reissner-Nordström) black holes. However, several subtleties arise that need to be dealt with care. Consider, as the simplest example that comes to one's mind, an electrically charged Reissner-Nordström black hole in the background of a magnetic field. This configuration was analyzed, in an ex-

[4]Actually, the Kaluza-Klein perspective provides a simple way to rederive, by linearization in the gauge field, the general technique used in [1] to construct solutions for test Maxwell fields in backgrounds with isometries.

act way, in [3]. Naively, according to our conjecture the magnetic field should be expelled from the horizon in the extremal limit in this configuration. However, this does not happen. The puzzle is solved [5] when one notices that the solution in [3] is actually *rotating.* A rotating electric charge generates a magnetic dipole moment. The black hole is therefore the source of a magnetic dipolar field. This is actually the field across the extremal horizon of the solution in [3].[5] The authors of [5] then went on to construct a linearized solution where the rotation of the charged black hole in the external field could be set to zero, and found it exhibited the Meissner expulsion of the field in the extremal limit.

In this example, the complication arises due to gravitationally induced non-linear interactions between the electric field of the black hole and the external magnetic field. However, notice that our main reason to have a charge on the black hole is to provide a means to reach the extremal limit. In other words, we are not particularly interested in the dynamical aspects associated with the charge of the black hole. Rather, we want to isolate the behavior of the magnetic field in the *gravitational* field created by the black hole. As a way to disentangle the effect of the charge of the black hole from that of the magnetic field, we can think of the charge of the black hole as being coupled to a gauge field that is different from the external magnetic gauge field. In other words, we work with a U(1)× U(1) gauge theory, with two Maxwell fields. The black hole will be charged with respect to one of the U(1) fields, while the other gauge field will be the magnetic field that experiences the Meissner effect. This introduction of a second gauge field may seem unrealistic, but we should view it as simply a device that provides us with a way to achieve extremality for the black hole. In particular, it will be clear in our analysis below that the dynamics of the gauge field associated with the charge of the black hole plays no essential role. Besides, theories with more than one gauge field arise quite naturally in string theory and related contexts.

We will start our analysis by treating the magnetic field as a test field in the background of the black hole geometry. Therefore, we want to solve the equation

$$\partial_\mu(\sqrt{-g}\mathcal{F}^{\mu\nu})=0, \tag{5.5}$$

in some fixed background geometry $g_{\mu\nu}$.

For starters, take the Reissner-Nordström metric

$$ds^2=-Vdt^2+V^{-1}dr^2+r^2(d\theta^2+\sin^2\theta d\varphi^2),$$

$$V=1-\frac{2M}{r}+\frac{Q^2}{r^2}. \tag{5.6}$$

The outer (event) horizon is at $r=r_h=M+\sqrt{M^2-Q^2}$, and extremality is achieved by setting $Q=M$.

[5]It is even clearer that, for similar reasons, we should not expect the extremal Kerr-Newman black hole, which has a magnetic dipole by itself, to expel the magnetic field [4].

For the test field $\mathcal{F}=d\mathcal{A}$ we will assume the ansatz

$$\mathcal{A}=f(r)\sin^2\theta\, d\varphi, \tag{5.7}$$

in terms of which the magnetic flux crossing any surface $\Sigma$ is given by $\int_{\partial\Sigma}\mathcal{A}$.

With the ansatz (5.7), the field equation (5.5) becomes

$$\frac{d}{dr}\left(V\frac{df}{dr}\right)=\frac{2f}{r^2}. \tag{5.8}$$

This is easily solved as

$$f(r)=r^2-Q^2, \tag{5.9}$$

up to a multiplicative constant, related to the value of the magnetic field at infinity, which we have arbitrarily fixed. According to Eq. (5.7), the magnetic flux crossing the horizon is proportional to $f(r_h)$. This is non-zero for black holes with $M>Q$, but *it vanishes precisely in the extremal limit* $r_h=Q$.

Now, we want to consider non-rotating extremal black holes in more generality. In order to simplify the analysis, we will focus only on the region near the horizon of the black hole, since it is there where the Meissner effect is exhibited. As the most generic characterization of this region for spherically symmetric extremal black holes, we will take the following:

For some choice of conformal frame, the region near the extremal horizon becomes asymptotically an infinite throat of constant radius. This is, if we choose the horizon to be at $r=0$, then

$$ds^2\simeq-\left(\frac{r}{\ell}\right)^{4\alpha}dt^2+\ell^2\left[\frac{dr^2}{r^2}+d\theta^2+\sin^2\theta d\varphi^2\right]. \tag{5.10}$$

The freedom in choosing coordinates has been used to simplify the possible forms of the metric and bring the horizon to $r=0$. The parameter $\ell$ fixes the scale of the geometry (and is typically related to the charge and mass of the black hole). The exponent $\alpha$ is an arbitrary real number. Within this class we find, for example, the extremal dilatonic black holes of [22] or the stringy black holes in [23].

As in Sec. IV, the reference to the conformal frame is motivated by the fact that, in the presence of scalar (dilaton) fields, when we write the metric in the canonical Einstein frame, the throat at $r=0$ typically pinches down to zero size in a singular way. But then we can use the dilaton to perform a conformal rescaling of the metric to yield the regular throat (5.10). Since the Maxwell field equation (5.5) is, in four dimensions, invariant under such conformal rescalings, we are allowed to choose to work in the conformal gauge fixed by Eq. (5.10). In fact, we may want to consider an equation slightly more general than Eq. (5.5),

$$\partial_\mu(\sqrt{-g}e^{-a\phi}\mathcal{F}^{\mu\nu})=0, \tag{5.11}$$

where we allow for a coupling of the test field to a dilatonic field $\phi$ with non-constant background value near the horizon, $r\simeq 0$,

$$e^{-a\phi}\simeq\left(\frac{\ell}{r}\right)^{2\beta}. \tag{5.12}$$

As a further minor generalization, we could consider the test gauge field to be coupled to a *test* scalar $\sigma$, with the standard action (we suppress inessential factors),

$$I\sim\int(\partial\sigma)^2+e^{-\sigma}\mathcal{F}^2. \tag{5.13}$$

However, the field equation for $\sigma$ implies that if $\mathcal{F}$ is linear in the (small) applied magnetic field, then $\sigma$ only enters at quadratic order and is therefore negligible in the approximation we are working. Hence we need not consider explicitly such scalars.

In order to solve Eq. (5.11), we consider again the ansatz (5.7) for the magnetic field, and we find the equation

$$\frac{d}{dr}(r^{2(\alpha-\beta)+1}f')=2r^{2(\alpha-\beta)-1}f. \tag{5.14}$$

This is a homogeneous equation, which we can solve by choosing (up to a multiplicative constant)

$$f(r)=r^{\gamma}, \tag{5.15}$$

with

$$\gamma=\sqrt{(\beta-\alpha)^2+2}+\beta-\alpha>0. \tag{5.16}$$

What is important here is that $\gamma$ is never zero. Since the flux crossing the horizon is proportional to $f(r=0)$, in order to have a finite, non-vanishing flux we should have $\gamma=0$. Instead, we find that the flux always vanishes at the horizon $r=0$.[6] The Meissner effect, therefore, is a common characteristic of extremal throats. For completeness, we show in Appendix A that the Meissner effect never takes place on non-extremal horizons.

Finally, notice that in order to solve the equations and exhibit the Meissner effect we have only needed the metric of the black hole solution. That is, the fact that we may need the black hole to be charged for it to be extremal plays no essential role. Besides this, we have assumed that the interactions of the gauge field $\mathcal{F}$ are essentially given by Eq. (5.11). More complicated situations could be envisaged, but from the evidence we have presented here we believe that the phenomenon is generic. If other couplings of the field $\mathcal{F}$ were considered, care should be exercised to ensure that the additional interactions do not indirectly generate source terms for the field $\mathcal{F}$, which would produce an outgoing flux of the field across the horizon. These cases, of course, cannot be used to disprove our conjecture, which clearly requires the absence of magnetic sources inside the black hole. A subtle example of how flux can penetrate a horizon of the type (5.10), if the theory contains Chern-Simons couplings involving the field $\mathcal{F}$, is discussed in Appendix B.

[6]The solutions with $\gamma<0$ have been discarded as pathological.

### C. Some further exact solutions

In the preceding subsection we have found evidence that magnetic fields are expelled from the horizon of spherically symmetric extremal black holes. However, the magnetic field has been treated as a test field, and its effect on the geometry of the black hole has been neglected. One could worry that, if the back reaction effect of the magnetic field on the geometry were accounted for, the behavior of the horizon might change and the magnetic field would perhaps penetrate into the black hole, thereby evading the Meissner effect. This, however, is rather unlikely: the fact that the magnetic field vanishes near the horizon leads us to expect a negligible back reaction in that region. This expectation is confirmed in all cases where exact solutions have been constructed.

We have already presented two exact solutions, in Secs. IV and V A, using the Kaluza-Klein ansatz, where we have introduced an axisymmetric magnetic field which exhibits Meissner expulsion. Similar exact fields can be introduced, for different values of the dilaton coupling, by applying ''Harrison-like'' [24] solution-generating transformations [20,25,26] (dilatonic Melvin flux tubes were discussed in [22]). In particular, the behavior of black holes in magnetic fields, for essentially any value $a>0$ of the dilaton coupling, can be readily analyzed using the solutions in [26]. We will not give any details, but in all such cases the Meissner effect can be seen to be present as well. Here we will display another sort of magnetic fields that can, in a sense, be considered as curved space generalizations of the uniform magnetic field in flat space. These are the covariantly constant fields, exemplified by the Bertotti-Robinson solution of Einstein-Maxwell theory. There do exist generalizations of such solutions for the $U(1)^2$ theory of [27] or the $U(1)^n$ theories in [26].

One should be careful, however, in constructing the solutions. The field in the Bertotti-Robinson solution is spherically symmetric, and ''emanates'' from an origin, which nevertheless is non-singular since the geometry develops an infinite throat. In the analogous dilatonic solutions, the field similarly emanates from an origin, which now is singular in the Einstein frame. In any case, our point here is that, if we want the extremal black hole to expel the field, then it is clear that the ''source'' should not be *inside* the black hole. In other words, the Bertotti-Robinson-like field and the black hole must *not* be concentric.

With this proviso, the theory we will consider will be [27]

$$I=\int d^4x\sqrt{-g}\left[R-\frac{1}{2}(\partial\phi)^2-\frac{e^{-\phi}}{2}\mathcal{F}^2-\frac{e^{\phi}}{2}G^2\right], \tag{5.17}$$

and the solution we are interested in is, in the Einstein conformal gauge,

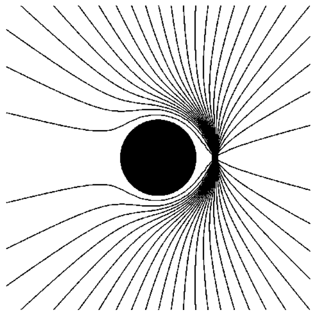

FIG. 2. Field lines of the magnetic field $\mathcal{F}$ for the exact configuration (5.18). The radius in Eq. (5.18) has been changed to ''Schwarzschild radius'' $r \to r-q$. The ''origin'' of the covariantly constant field has been put at $l=q/2$.

$$ds^2=-\frac{1}{\Delta_{\mathcal{F}}\Delta_G}dt^2+\Delta_{\mathcal{F}}\Delta_G(dr^2+r^2d\theta^2+r^2\sin^2\theta d\varphi^2),$$

$$\mathcal{F}=d\mathcal{A},\quad \mathcal{A}=b\frac{r\cos\theta-l}{r_2}d\varphi,\quad G=q\sin\theta d\theta\wedge d\varphi, \tag{5.18}$$

$$e^{\phi}=\frac{\Delta_G}{\Delta_{\mathcal{F}}},\quad \Delta_G=1+\frac{q}{r},\quad \Delta_{\mathcal{F}}=\frac{b}{r_2},$$

$$r_2\equiv\sqrt{r^2+l^2-2lr\cos\theta}.$$

In this form of the solution, both fields are of magnetic type. The black hole is extremal from the outset, with horizon at $r=0$ and charge $q$. The ''origin'' of the magnetic $\mathcal{F}$ field is at a coordinate distance $l$ along the axis $\theta=0$, i.e., at $r_2=0$. Setting $q=0$ yields a geometry that is conformally equivalent to the product of the linear dilaton vacuum of $D=2$ string theory with a sphere $S^2$ and a covariantly constant field $\mathcal{F}$. The degenerate horizon at $r=0$ is singular. The proper size of the extremal black hole is zero if measured in the Einstein metric. However, as discussed in the previous sections, for the purpose of studying the gauge fields we could just as well work in a conformally related metric where the extremal horizon is non-singular. The ''preferred'' frame is $e^{\phi}ds^2$, in which the extremal black hole area is equal to $4\pi q^2$.

Once again, the exact value of the flux across constant $r$ surfaces, given by

$$\mathcal{F}_{\theta\varphi}=b\sin\theta\frac{r^2(r-l\cos\theta)}{r_2^3}, \tag{5.19}$$

vanishes at the horizon of the black hole, $r=0$, as we claimed. The lines of force for the field $\mathcal{F}$ are plotted in Fig. 2.

With little extra effort we can consider a slightly different situation, where we have two extremal black holes, each with charge coupled to different gauge fields. As before, if we do not want to find a trivial penetration of flux, we have to consider a two-center solution.

We can analyze in this way whether the field created by the black hole with charge $q_2$ in $\mathcal{F}$ penetrates the horizon of the black hole with charge $q$. The solution is just like Eqs. (5.18) above, but now with

$$\Delta_{\mathcal{F}}=1+\frac{q_2}{r_2}. \tag{5.20}$$

The horizon of this second black hole is at $r_2=0$. The field created by it is exactly the same as in the previous example, Eq. (5.19), only changing $b\to q_2$. Thus we find another exact solution exhibiting the Meissner effect at the extremal horizon at $r=0$. Evidently, by symmetry, the flux created by the black hole with charge $q$ does not penetrate the horizon, at $r_2=0$, of the other extremal black hole.

In these examples the black hole under study has been the ''$a=1$ dilatonic black hole.'' In terms of the test field analysis performed in the previous subsection, the relevant parameters are $\alpha=0$, $\beta=1/2$, which yield $\gamma=2$ for Eq. (5.15). This is in precise agreement with the expansion for small magnetic field $b$ (and $r$) of the exact result (5.19). Different values of the dilaton coupling (essentially, any value $a>0$) can be readily analyzed using the solutions in [26], with no qualitative differences.

## VI. CONCLUSIONS

Superconductivity is a rich and multifaceted subject, with applications in a variety of physical models, from condensed matter physics to QCD. It is therefore natural to investigate how superconducting phenomena may emerge from the rich structure described by M-theory; after all, M-theory is our only real candidate for a unified description of all physical phenomena.

In this paper, we have described the superconducting phases of the solitonic objects of M-theory, the $p$-branes. In order to perform such a description, we have concentrated on three of the most elementary and well-known aspects of superconducting media: the Meissner effect, London theory and the existence of minimally coupled zero modes.

With respect to the Meissner effect, we have presented a number of exact solutions which demonstrate that Kaluza-Klein magnetic flux is expelled from the horizon of a generic extreme black hole. We have extended this analysis to the case of a black string in $D=5$, and again found that Kaluza-Klein flux is expelled. It would be interesting to perform similar tests for the Meissner effect for higher-dimensional extreme branes. It would also be interesting if we could understand *precisely* when and how the Meissner effect is broken.

Strictly speaking, the Meissner effect follows from the fact that inside a superconductor the field has to be pure gauge. This, however, is not true for the field in the interior of the extremal black hole, as can be readily seen from the examples above. We are not claiming, therefore, that the black hole interior is in a superconducting state. Our statements refer to the horizon or, at most, to the near-horizon region.

Of course, the Meissner effect is just one property exhibited by superconducting media; ultimately, we want to construct a phenomenological model which attempts to describe what is going on. The theory of London goes beyond the simple *observation* of the Meissner effect, and proposes a set of field equations which imply various things about the microscopic theory which underlies the entire phenomenon. Thus, in order to have a macroscopic phenomenological description of a superconducting $p$-brane, we have followed Nielsen, Balachandran *et al.* and others by proposing that a $p$-brane is in a superconducting phase if and only if the relativistic London equation holds on the world volume of the brane. For a test brane, this definition is not ambiguous since it is clear where the brane is located; i.e., the brane is just some extended object moving in a background spacetime, from which it has decoupled. The motivation for our definition is then clear, since the London equation will hold on the world volume of any extended object which is carrying Kaluza-Klein currents. For self-gravitating branes, we have proposed that a brane is in a superconducting phase if and only if this ''Nielsen'' type condition holds on the boundary of the adS factor of the near-horizon geometry of the brane.

Given all of this structure, it is then natural to propose that the microscopic degrees of freedom which lead to $p$-brane superconductivity are precisely the zero modes, associated with the singleton superconformal multiplets, which propagate on the boundary of the adS factor of the near-horizon geometry. These zero modes naturally couple to any Kaluza-Klein currents, and so they literally represent the unimpeded flow of charge far down the throat of a self-gravitating brane.

Of course, in this analysis we have neglected a number of other theories and approaches to superconductivity. It would be interesting to investigate whether or not it is possible to define $p$-brane superconductivity using the ideas of these other theories. Research on these and related problems is currently underway.

## ACKNOWLEDGMENTS

A.C. is supported by Pembroke College, Cambridge, England. R.E. has been partially supported by EPSRC through grant no. GR/L38158 (UK) and by UPV through grant no. 063.310-EB225/95 (Spain).

## APPENDIX A: ABSENCE OF THE MEISSNER EFFECT IN NON-EXTREMAL HORIZONS

In order to complete our general analysis of *test* magnetic fields in the vicinity of spherically symmetric black holes, here we solve the equations in the presence of non-extremal horizons. In this case, close enough to the horizon the geometry is of the Rindler form

$$ds^2=-\rho^2 d\tau^2+d\rho^2+R^2(d\theta^2+\sin\,\theta^2 d\varphi^2). \qquad \text{(A1)}$$

$R$ is a constant measuring the radius of the horizon, which is at $\rho=0$. We now solve Eq. (5.5) for a test Maxwell field in this background using the same ansatz (5.7).[7] The solution

$$\mathcal{A}=I_0\left(\frac{\sqrt{2}\rho}{R}\right)\sin^2\theta\ d\varphi \qquad \text{(A2)}$$

is expressed in terms of the Bessel function of order zero, such that $I_0(0)=1$; i.e., there is a non-vanishing flux crossing any portion of the horizon. There is no Meissner expulsion from non-extremal horizons.

## APPENDIX B: A ''COUNTEREXAMPLE'' TO THE MEISSNER EFFECT AND ITS RESOLUTION

Consider the five-dimensional action

$$I_5=\int d^5x\sqrt{-\hat{g}}\left\{\hat{R}-\frac{1}{2}(\hat{\partial}\phi)^2-\frac{1}{12}e^{-\sqrt{2/3}\phi}\hat{H}^2 -\frac{1}{4}e^{+\sqrt{2/3}\phi}\hat{F}^2\right\}. \qquad \text{(B1)}$$

Five-dimensional quantities will be careted. $\hat{H}$ and $\hat{F}$ are 3-form and 2-form field strengths, obtainable from the 2- and 1-form potentials $\hat{B},\hat{A}$, $\hat{H}=d\hat{B}$, $\hat{F}=d\hat{A}$. Very similar (but not exactly the same) actions can be derived from compactified string–M-theory. The fields $\hat{H}$ and $\hat{F}$ admit the interpretation of fields with string and particle sources. Actually, the solution we discuss below can be seen as a bound state (at threshold) of a string and a particle.

The equations of motion of this theory admit the solution

$$d\hat{s}^2=-\frac{dt^2}{\Delta^2}+\Delta^2(dr^2+r^2d\Omega_2^2)+dx_5^2,$$

$$\Delta=1+\frac{q}{r}, \qquad \text{(B2)}$$

$$\hat{B}=\Delta^{-1}dt\wedge dx_5, \quad \hat{A}=\Delta^{-1}dt. \qquad \text{(B3)}$$

The scalar $\phi$ is zero (or constant) for this solution.[8] The metric is precisely equal to the product of the $D=4$ extremal, electric Reissner-Nordström (RN) black hole with the real line $-\infty<x_5<\infty$. Hence, Kaluza-Klein reduction along $x_5$ yields the extremal RN black hole, with no electromagnetic Kaluza-Klein field.

We can now generate a background Melvin flux tube by performing a Kaluza-Klein reduction as described in Sec. IV: change the polar variable to $\varphi\rightarrow\varphi-Bx_5$, and reduce to $D=4$ by consistently identifying points along $x_5$. The Kaluza-Klein gauge potential is

$$\mathcal{A}=B\frac{\Delta^2r^2\sin^2\theta}{1+B^2\Delta^2r^2\sin^2\theta}d\varphi. \qquad \text{(B4)}$$

[7]We could also have included scalar fields, as in Eq. (5.11), but these typically take finite, non-zero values on non-extremal horizons and do not alter the results.

[8]It would be easy to construct a more general solution with different harmonic functions $\Delta_F,\Delta_H$ for the particle and string that would yield non-constant $\phi$, but we prefer to keep things simpler at this level.

This does *not* vanish on the extremal horizon $r=0$. The Meissner effect is not present for this solution. Nevertheless, the geometry near the horizon is of the form required in Eq. (5.10).

The resolution of this puzzle comes from examining the actual couplings of the Kaluza-Klein gauge field $\mathcal{A}$ in the effective $D=4$ theory. For details of the reduction procedure, see, e.g., [28]. The important point here is that the non-vanishing component of the field $\hat{B}$ along $x_5$, $\hat{B}_{\mu 5} \equiv B_\mu$, yields a Chern-Simons-like coupling in the $D=4$ action of the form

$$(dB \wedge \mathcal{A})^2 \tag{B5}$$

(times factors involving the scalar $\phi$ and Kaluza-Klein scalar, which are inessential for this discussion). The consequence is that the effective equation for $\mathcal{F}$ in $D=4$ differs now from Eq. (5.11) by the presence of an extra source term. In this indirect way, the $\hat{H}$-charge of the black hole is responsible for the appearance of an induced magnetic dipole for the black hole in the presence of an external field $\mathcal{F}$. This is the source of the flux coming out of the horizon. This is, in a way, similar to the absence of the Meissner effect in the solutions considered in [3], in that subtle non-linear interactions induce dipolar sources for the black hole.

This extra term is also present in the compactification of the string that we analyzed in Sec. IV. However, in that case its value in the extremal limit is zero, and so it does not spoil the Meissner effect.


[1] R.M. Wald, Phys. Rev. D **10**, 1680 (1974).

[2] A.R. King, J.P. Lasota, and W. Kundt, Phys. Rev. D **12**, 3037 (1975).

[3] F.J. Ernst, J. Math. Phys. **17**, 54 (1976).

[4] F.J. Ernst and W.J. Wild, J. Math. Phys. **17**, 182 (1976).

[5] J. Bičák and L. Dvořák, Phys. Rev. D **22**, 2933 (1980).

[6] A. Chamblin, J.M.A. Ashborun-Chamblin, R. Emparan, and A. Sornborger, Phys. Rev. Lett. **80**, 4378 (1998); ''Can extreme black holes have (long) Abelian Higgs hair?,'' gr-qc/9706004 [Phys. Rev. D (to be published)].

[7] N.K. Nielsen, Nucl. Phys. **B167**, 149 (1980).

[8] N.K. Nielsen and P. Olesen, Nucl. Phys. **B291**, 829 (1987).

[9] A.P. Balachandran, B.S. Skagerstam, and A. Stern, Phys. Rev. D **20**, 439 (1979).

[10] E. Witten, Nucl. Phys. **B249**, 557 (1985).

[11] G.W. Gibbons, Nucl. Phys. **B514**, 603 (1998); C.G. Callan, Jr. and J.M. Maldacena, *ibid.* **B513**, 198 (1998).

[12] J.P. Gauntlett, G.W. Gibbons, G. Papadopoulos, and P.K. Townsend, Nucl. Phys. **B500**, 133 (1997).

[13] G.W. Gibbons and P.K. Townsend, Phys. Rev. Lett. **71**, 3754 (1993).

[14] G.W. Gibbons, G.T. Horowitz, and P.K. Townsend, Class. Quantum Grav. **12**, 297 (1995).

[15] P.Claus , R. Kallosh, J. Kumar, P.K. Townsend, and A. Van Proeyen, ''Conformal Theory of M2, D3, M5 and 'D1+D5' branes,'' hep-th/9801206.

[16] J.M. Maldacena, ''The Large *N* Limit of Superconformal Field Theories and Supergravity,'' hep-th/9711200.

[17] E. Witten, ''Anti-de Sitter space and Holography,'' hep-th/9802150.

[18] P. Claus, R. Kallosh, and A. Van Proeyen, Nucl. Phys. **B518**, 117 (1998).

[19] G.W. Gibbons, ''Wrapping branes in space and time,'' hep-th/9803206.

[20] F. Dowker, J.P. Gauntlett, D. Kastor, and J. Traschen, Phys. Rev. D **49**, 2909 (1994).

[21] H.F. Dowker, J.P. Gauntlett, G.W. Gibbons, and G.T. Horowitz, Phys. Rev. D **53**, 7115 (1996).

[22] G.W. Gibbons and K. Maeda, Nucl. Phys. **B298**, 741 (1988).

[23] M. Cvetic and D. Youm, Phys. Rev. D **53**, 584 (1996); ''BPS saturated and non-extreme states in Abelian Kaluza-Klein theory and effective $N=4$ supersymmetric string vacua,'' hep-th/9508058.

[24] B.K. Harrison, J. Math. Phys. **9**, 1744 (1968).

[25] S.F. Ross, Phys. Rev. D **49**, 6599 (1994).

[26] R. Emparan, Nucl. Phys. **B490**, 365 (1997).

[27] R. Kallosh, A. Linde, T. Ortín, A. Peet, and A. Van Proeyen, Phys. Rev. D **46**, 5278 (1992).

[28] K. Stelle, ''BPS branes in supergravity,'' hep-th/9803116.