Interiors of Vaidya's radiating metric: Gravitational collapse

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Using the Darmois junction conditions, we give the necessary and sufficient conditions for the matching of a general spherically symmetric metric to a Vaidya radiating solution. We present also these conditions in terms of the physical quantities of the corresponding energy-momentum tensors. The physical interpretation of the results and their possible applications are studied, and we also perform a detailed analysis of previous work on the subject by other authors.

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I. INTRODUCTION

Gravitational collapse is one of the fundamental problems for which general relativity can be of some importance. The problem has very interesting applications in astrophysics, where the formation of compact stars is usually preceded by an epoch of radiative collapse. From the theoretical point of view, this subject must be studied taking into account that the surface of the star divides the whole space-time into two different regions: the region inside the surface of the star, called the interior region, filled with matter and radiation; and the region outside the surface, called the exterior region, which will usually be filled with all types of radiation coming out of the star itself. These two regions must be matched smoothly across the star surface.

Therefore, in order to study gravitational collapse, it is necessary to describe adequately the geometry of the interior and exterior regions and to give the conditions which allow the matching of them. To describe the geometries, the assumption of spherical symmetry has been widely used because it gives a very good approximation to most physical situations, the departure from sphericity being seldom relevant for practical purposes. Thus, the exterior region is usually described by the Vaidya radiating metric [1], which is the only spherically symmetric metric with a pure radiation energy-momentum tensor. In most work on the interior region a more or less general energy-momentum tensor is assumed, and then the resolution of Einstein's equations provides the geometry which must be matched to the exterior. However, as we shall prove in this work, this procedure is not necessary at all, because *every* spherically symmetric metric bounded by a surface interpretable as the star surface (in a sense that will be made precise later) *can* be matched with a Vaidya exterior. This important fact is in accordance with the uniqueness of the Vaidya solution and permits us to avoid the always difficult resolution of Einstein's equations. We shall also give a satisfactory physical interpretation of these results by proving that the only requirement for performing the matching is that the surface of the star be defined by the condition of vanishing of the total radial pressure (or equivalent quantities as explained in the discussion) on the surface.

Historically, the pioneering work on gravitational collapse appeared in the famous paper by Oppenheimer and Snyder [2] in which they studied the collapse of dust with a static Schwarzschild exterior. Much later, the case with a static exterior was studied by Misner and Sharp [3], for a perfect fluid in the interior, and by Bel and Hamoui [4] in the general case without a flux of energy across the surface of the star. After the appearance of Vaidya's solution, Vaidya himself tried to find interior radiative solutions [5], and Misner generalized his previous work in Ref. [6], where the condition of vanishing pressure at the star surface appears for the first time. The fact that Vaidya's metric is naturally written in radiative coordinates led other authors to study the problem with the interiors expressed in similar coordinates. The first work in this line was presented by Bondi [7]; he found part of the matching conditions by eliminating unbounded discontinuities in the energy-momentum tensor, which was assumed not to have anisotropic pressures. With the same type of energy-momentum tensor, Herrera and collaborators [8] developed further this work, and they constructed a wide class of models for collapsing stars in a series of subsequent papers (see, for example, [9–11]) which are summarized in an interesting report by Herrera and Núñez [12]. On the other hand, the use of the intrinsic junction conditions of Darmois allowed these studies to be performed with different types of coordinates in the interior and the exterior. Glass started this path in a short Letter [13] which was followed and much improved in the work of Santos and collaborators

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[14–16]. Santos studied the case in which the collapsing fluid is shear-free and has heat conduction but no anisotropic pressures. These works were collected in an extensive report [17] in which one can find most of the references pertinent to the matter.

In this paper, we shall treat the problem at a very theoretical level. Our aim is to provide the most general framework and to give the physical conditions for the matching to be possible. In Sec. II we present the matching conditions in the general case, and we also outline the way it must be followed to actually match any interior metric to a Vaidya exterior. These results are physically interpreted in Sec. III, where we decompose the energy-momentum tensor in an appropriate way which allows us to express our main result in the form of a fundamental theorem. Finally, we devote Sec. IV to discussing the meaning of our results and to comparing other work with the contents of this paper. We shall then adapt our fundamental theorem to the different possible coordinates or particular cases which have been studied by other authors.

II. MATCHING OF THE VAIDYA AND SPHERICALLY SYMMETRIC METRICS

Let us consider a four-dimensional spherically symmetric space-time $\mathcal{V}$ divided into two regions $\mathcal{V}_I$ and $\mathcal{V}_E$ by an ordinary three-dimensional timelike surface Σ preserving the symmetry of $\mathcal{V}$. By an ordinary surface we mean one that does not concentrate energy-momentum, so that all quantities appearing in the energy-momentum tensor may only have bounded discontinuities across Σ. For the sake of convenience we shall talk of the interior region $\mathcal{V}_I$ and the exterior region $\mathcal{V}_E$, but this distinction can be interchanged depending on the concrete physical situation one wishes to study.

We choose for the metric ds_I^2 acting on $\mathcal{V}_I$ the most general spherically symmetric metric, which can be expressed in isotropic coordinates $\{x_I^\mu\}=\{t,r,\theta,\phi\}$, $\mu=0,1,2,3$, as

$$ds_I^2=-A^2dt^2+B^2(dr^2+r^2d\Omega^2)\ , \tag{1}$$

where A and B are positive functions of t and r, and $d\Omega^2=d\theta^2+\sin^2\theta\, d\phi^2$. For the exterior region $\mathcal{V}_E$ we take the Vaidya radiating solution [1], which is given in radiative coordinates [7] $\{x_E^\mu\}=\{T,R,\theta,\phi\}$, $\mu=0,1,2,3$, by

$$ds_E^2=-\chi\, dT^2-2\,dR\, dT+R^2d\Omega^2\ , \tag{2}$$

where $\chi=1-2m/R$, m being a function only of T.

In order to match the metrics (1) and (2) across Σ we use the Darmois junction conditions [18], which can be written

$$ds_I^2\overset{\Sigma}{=}ds_E^2\ , \tag{3}$$

$$\mathbf{K}_I\overset{\Sigma}{=}\mathbf{K}_E\ , \tag{4}$$

where $\overset{\Sigma}{=}$ means that the equality must be evaluated on Σ. $ds_I^2|_\Sigma$ and $ds_E^2|_\Sigma$ are the first fundamental forms or intrinsic metrics of Σ seen from $\mathcal{V}_I$ and $\mathcal{V}_E$, respectively, and $\mathbf{K}_I$ and $\mathbf{K}_E$ are the second fundamental forms or extrinsic curvatures of Σ seen from $\mathcal{V}_I$ and $\mathcal{V}_E$.

If $x_I^\mu(\xi^i)$ and $x_E^\mu(\xi^i)$ are the parametric expressions of Σ on $\mathcal{V}_I$ and $\mathcal{V}_E$, respectively, where ξ^i, $i=0,2,3$ are coordinates for Σ, the intrinsic metric and extrinsic curvature have the expressions

$$ds_{IE}^2|_\Sigma\equiv g_{IEij}\,d\xi^i d\xi^j\overset{\Sigma}{=}g_{IE\mu\nu}\frac{\partial x_{IE}^\mu}{\partial\xi^i}\frac{\partial x_{IE}^\nu}{\partial\xi^j}d\xi^i d\xi^j\ , \tag{5}$$

$$\mathbf{K}_{IE}\equiv K_{IEij}\,d\xi^i d\xi^j \overset{\Sigma}{=}-n_{IE\mu}\left[\frac{\partial^2 x_{IE}^\mu}{\partial\xi^i\partial\xi^j}+\frac{\partial x_{IE}^\nu}{\partial\xi^i}\frac{\partial x_{IE}^\sigma}{\partial\xi^j}\Gamma_{IE\nu\sigma}^\mu\right]d\xi^i d\xi^j\ , \tag{6}$$

where $n_{IE\mu}$ are the outward unit one-forms normal to Σ seen from $\mathcal{V}_{IE}$, respectively.

In the case under consideration, we choose $\{\xi^i\}=\{\lambda,\theta,\phi\}$ as coordinates of the surface Σ, where λ is a time coordinate defined only on Σ and such that $t(\lambda)$, $r(\lambda)$ and $T(\lambda)$, $R(\lambda)$ are the nontrivial parametric equations of Σ on the interior and exterior regions, respectively. After a long calculation with multiple substitutions taking into account equal relative orientation of the time coordinates t, T, and λ, the junction conditions (3) and (4) [see the Appendix for the explicit expressions for the quantities appearing in (3) and (4)] become

$$R\overset{\Sigma}{=}Br\ , \tag{7}$$

$$\dot T\overset{\Sigma}{=}\frac{AB(A\dot t-B\dot r)}{Br\dfrac{\partial B}{\partial t}+A\dfrac{\partial(Br)}{\partial r}}\ , \tag{8}$$

$$m\overset{\Sigma}{=}\frac{r^3B}{2A^2}\left[\frac{\partial B}{\partial t}\right]^2-r^2\frac{\partial B}{\partial r}-\frac{r^3}{2B}\left[\frac{\partial B}{\partial r}\right]^2\ , \tag{9}$$

$$\begin{aligned}0\overset{\Sigma}{=}B\dot r\left\{-A^2\left[2Br\frac{\partial^2B}{\partial r^2}-r\left[\frac{\partial B}{\partial r}\right]^2+4B\frac{\partial B}{\partial r}\right]+3rB^2\left[\frac{\partial B}{\partial t}\right]^2\right\}+2B(A\dot t+B\dot r)\left[A\frac{\partial B}{\partial r}\frac{\partial B}{\partial t}-AB\frac{\partial^2B}{\partial t\,\partial r}+B\frac{\partial A}{\partial r}\frac{\partial B}{\partial t}\right]\\ +A^2\dot t\left\{\left[Ar\left[\frac{\partial B}{\partial r}\right]^2+2AB\frac{\partial B}{\partial r}+2Br\frac{\partial A}{\partial r}\frac{\partial B}{\partial r}+2B^2\frac{\partial A}{\partial r}\right]+B^2r\left[2B\frac{\partial A}{\partial t}\frac{\partial B}{\partial t}-2AB\frac{\partial^2B}{\partial t^2}-A\left[\frac{\partial B}{\partial t}\right]^2\right]\right\}\ ,\end{aligned} \tag{10}$$

where an overdot denotes $d/d\lambda$. These are the necessary and sufficient conditions for the matching of a spherically symmetric metric and a Vaidya metric across a (unspecified) timelike spherically symmetric surface. From now on, we will refer to them as the *matching conditions*.

Let us see what the meaning of the matching conditions is and what we can learn from them. First of all, it is interesting to note that the right-hand side of (9) is the well-known mass function introduced by Cahill and McVittie [19], which represents the total gravitational mass inside Σ. Thus, (9) simply reflects that the total gravitational mass as seen from the exterior must coincide with that seen from the interior on Σ, as should be expected. In general, the matching conditions (7)–(9) should be seen as mathematical relations between the relevant quantities at both sides of Σ. However, the matching condition (10) involves quantities of the interior but not the exterior. Then, this equation is a real condition on the interior and Σ, and it has a slightly different meaning, which is very important as we shall see in what follows. In the next section we shall give an interesting physical interpretation of Eq. (10) in terms of physical quantities of the interior metric.

The path that we must follow in order to extract all information from the matching conditions will be different depending on the known data. For example, an interesting problem appears when we know $A(t,r)$ and $B(t,r)$, i.e., when the interior geometry is given, and we wish to determine the function $m(T)$ and the surface Σ. To treat this problem, and bearing in mind the above comments about Eq. (10), we proceed with the following steps.

(i) Equation (10) becomes a first-order ordinary differential equation for $r(t)$ of the form $dr/dt=F(r,t)$. If the functions $A(t,r)$ and $B(t,r)$ are such that $F(t,r)$ satisfies the Lipschitz condition, then we will find the solution for $r(t)$ up to a constant c_1. This function $r(t)$ defines the surface Σ on the interior, and therefore we can check whether or not Σ is timelike. If it is not, the matching is not possible. However, in most cases Σ will be timelike and we can go to the next step.

(ii) Once $r(t)$ is known, from (9) we can determine the function $m \stackrel{\Sigma}{=} m(t)$ up to a constant c_1.

(iii) Now, from (8), we can obtain $T(t)$ up to constants c_1, c_2, and from (7) we get the last function $R(t)$. These two functions define the surface Σ on the exterior.

(iv) Finally, and due to the fact that m is a function only of T, we can get $m(T)$ from the found solutions $m(t)$ and $T(t)$, and the problem is finished.

Therefore, we have proven that for every pair of functions $A(t,r)$ and $B(t,r)$, we can determine Σ and $m(T)$ if and only if (10) has a solution for $r(t)$ that defines a timelike surface. In other words, given any spherically symmetric interior, we can always match it to a Vaidya exterior provided that the surface defined by the solution $r(t)$ of the matching condition (10) is timelike. In the next section, we shall give an alternative and more physical version of this important result.

Let us consider now the converse problem: that is, in what way do (7)–(10) restrict the possible interior geometries if $m(T)$ and Σ seen from $\mathcal{V}_E$ are given? And if we only know $m(T)$? In these cases, by analyzing the mathematical structure of the matching conditions it can be seen that the problem is clearly undetermined and the resulting equations are, in fact, functional relations for the unknown variables. This is not a surprising result because, as is well known (remember, for instance, Birkhoff's theorem), a fixed geometry can be the exterior of many different interior metrics.

III. PHYSICAL INTERPRETATION

Our purpose now is to find the physical interpretation of the results of the previous section. To that end, let us consider the problem of radiative gravitational collapse, and more generally, of spherical and radiating stars in evolution. In this case, the matching conditions (7)–(10) become surface equations of the spherically symmetric radiating star, and the matching surface must be identified with the star surface. Therefore, the nonradiative part of the star matter must be comoving with the matching surface and its unit velocity vector $\mathbf{u}$ will then be given by

$$\mathbf{u} \stackrel{\Sigma}{=} \frac{1}{\sqrt{A^2-B^2r'^2}}\left[\frac{\partial}{\partial t}+r'\frac{\partial}{\partial r}\right], \tag{11}$$

which is defined only on Σ and tangent to it, and where we have put $r'\equiv dr/dt=\dot{r}/\dot{t}$.

Now, one can define a unit timelike vector field on $\mathcal{V}_I$, which coincides with $\mathbf{u}$ on Σ, as follows:

$$\mathbf{u}=\frac{1}{\sqrt{A^2-B^2f^2}}\left[\frac{\partial}{\partial t}+f\frac{\partial}{\partial r}\right], \tag{12}$$

where $f(t,r)$ is an arbitrary function subject to the restrictions

$$f \stackrel{\Sigma}{=} r'(t), \tag{13}$$

$$A^2-B^2f^2>0. \tag{14}$$

The function f can be interpreted as defining the different possible motions of matter inside the star, and it is only restricted on Σ. There are many choices for f, and each of them corresponds to a different fluid (in a wide sense) with different physical magnitudes. Because given the timelike vector field $\mathbf{u}$, or equivalently the function f, the most general stress-energy tensor can be decomposed as follows:

$$T_{I\mu\nu}=(\rho+p)u_\mu u_\nu+pg_{I\mu\nu}+\Omega^2 l_\mu l_\nu+\Pi_{\mu\nu}, \tag{15}$$

where $\mathbf{u}$ can be considered as the velocity vector of the interior matter, which is comoving with Σ by construction, ρ its energy density, and p its isotropic pressure. Ω^2 is the radiating energy density, l^μ is an outgoing radial null vector field, and $\Pi_{\mu\nu}$ is a tensor of anisotropic pressures (which is trace-free and orthogonal to $\mathbf{u}$). Of course, this decomposition depends essentially on $\mathbf{u}$, that is to say, on the particular choice of f. However, it is convenient to remark here that for every $A(t,r)$ and $B(t,r)$ such that $r(t)$ exists, we can determine, via Einstein's equations, ρ, p, Ω^2, and $\Pi_{\mu\nu}$ on and inside Σ *for every choice* of $f(t,r)$ compatible with (13) and (14).

Thus, despite the fact that the fluid is comoving with the star (and matching) surface, great freedom remains with respect to the fluid in the interior. This freedom is mathematically described by the arbitrary function f.

The energy-momentum tensor (15) can then be thought of as representing a nonperfect fluid with radially directed outgoing radiation. The vector fields l^μ and u^μ are both chosen to be future direct, and then we normalize them such that $l_\mu u^\mu = -1$.

The exterior of the star is described by the Vaidya radiating metric so that the only nonvanishing component of the stress-energy tensor $T_{E\mu\nu}$ is listed in (A5). Nevertheless, it will be convenient to express $T_{E\mu\nu}$ as

$$T_{E\mu\nu}=\Phi^2 L_\mu L_\nu \; , \tag{16}$$

where L_μ is a null one-form field proportional to dT, which has been normalized such that $L_\mu u^\mu \overset{\Sigma}{=} -1$ for the same reasons given above. The energy-momentum tensor (16) is that of outgoing incoherent radiation.

It is very well known that, once the matching has been performed with the Darmois junction conditions, some relations between the energy-momentum tensors at both sides of Σ can be deduced [20]. These relations are given by

$$T_{I\mu\nu}\tau^\mu n_I^\nu \overset{\Sigma}{=} T_{E\mu\nu}\tau^\nu n_E^\nu \; , \tag{17}$$

$$T_{I\mu\nu}n_I^\mu n_I^\nu \overset{\Sigma}{=} T_{E\mu\nu}n_E^\mu n_E^\nu \; , \tag{18}$$

for any four-vector $\tau^\mu = \{u^\mu, \partial_\theta, \partial_\phi\}$ tangent to Σ. The fact that the vectors tangent to Σ are the same at both sides of Σ is a simple consequence of the continuity of the first fundamental form. In our case, the only nontrivial relations derived from (17) and (18) are

$$\Omega^2 \overset{\Sigma}{=} \Phi^2 \; , \tag{19}$$

$$p+\Omega^2+\Pi_{\mu\nu}n_I^\mu n_I^\nu \overset{\Sigma}{=} \Phi^2 \; , \tag{20}$$

where we have taken into account that $T_{I\mu\nu}$ is given by (15) and $T_{E\mu\nu}$ by (16). Combining the last two equations we obtain

$$p+\Pi_{\mu\nu}n_I^\mu n_I^\nu \overset{\Sigma}{=} 0 \; , \tag{21}$$

which is a relation without quantities of the exterior.

Expressions (19)–(21) are the main physical equations relating quantities at both sides of Σ, and they are closely related to the matching conditions as we shall presently see. Equation (19) expresses the fact that the energy density of the radiation must be continuous on the matching surface. This relation can be obtained by differentiating the matching condition (9), that is to say, by differentiating on Σ the equality between the total gravitational masses contained inside Σ. Of course, this is a very natural result. On the other hand, (21) informs us that the total normal pressure must vanish on the matching surface, which again is a very satisfactory result and generalizes similar properties in the static case. In fact, (21) is strictly equivalent to the matching condition (10). In order to prove this, let us note that (21) is generated by

$$T_{I\mu\nu}(u^\mu n_I^\nu + n_I^\mu n_I^\nu) \overset{\Sigma}{=} 0 \; , \tag{22}$$

which via Einstein's equations can be rewritten in terms of the Einstein tensor $G_{I\mu\nu}$ as

$$G_{I\mu\nu}(u^\mu n_I^\nu + n_I^\mu n_I^\nu) \overset{\Sigma}{=} 0 \; . \tag{23}$$

Now, by using (A8), (11), and (A1)–(A4), it is straightforward to show that this last relation, and therefore (21), leads exactly to the matching condition (10).

Therefore we can reformulate the results of this and the previous sections in the form of the following fundamental theorem.

Every spherically symmetric metric can be locally matched to an exterior Vaidya solution provided that there exists a timelike surface such that the corresponding total normal pressure vanishes on it. In that case, this surface is the matching surface.

This theorem is the natural generalization of similar results previously obtained by Bel and Hamoui [4] and Misner and Sharp [3] for the case of a Schwarzschild exterior and no radiation. It is also a generalization of more recent results obtained by Santos [14,17], as will become clear in the next section.

IV. DISCUSSION

The main result proven in this paper is that, in general, every spherically symmetric space-time (apart from very singular cases) can be matched to an exterior Vaidya metric. It should be stressed that this can be deduced using geometrical arguments only, because it is a direct consequence of the matching conditions (7)–(10), as was proven in Sec. II. However, the fact that two metrics can be matched smoothly *must* have a satisfactory interpretation from the physical point of view, and this interpretation has been given in Sec. III. This is not a trivial result, because there are metrics which are usually thought of as nonradiating, and therefore one might think that they should not be matchable to a Vaidya exterior. The fact that they are matchable indicates that there are alternative interpretations for the interior metrics in which they do radiate, and we have identified them as those in which the energy-momentum tensor is decomposed using a velocity vector comoving with the matching surface and such that the total normal pressure vanishes on it. Of course, this velocity vector may not coincide with those fixed *a priori* for other reasons. As an illustrative example, let us note that the matching of a general Robertson-Walker geometry and a Vaidya exterior has been successfully performed recently by the authors [21]. It is usual to think that the Robertson-Walker geometry is generated by a perfect fluid without radiation, but the fact that it can be matched to a Vaidya exterior geometrically means that there are other possible sources for Robertson-Walker geometry in which the matching surface is comoving with the matter and there exists radiation. The discussion of this matter can be found in Ref. [21].

As is obvious, this paper studies the most general case of collapse with a Vaidya exterior because we have not restricted the interior in any way whatsoever. This is an important point, for in previous work on the subject it was thought that interior *solutions* to Einstein's equations for some kind of prefixed matter were needed (see below for a discussion of the work of other authors). We have shown that this is not the case, and that any reasonable spherically symmetric metric can be a good interior for a Vaidya exterior. Thus, one only has to choose the functions A and B of the interior solution arbitrarily (possibly using some reasoning to describe the desired models) and match it to the appropriate Vaidya exterior by using the matching conditions (7)–(10). Then, a coherent physical interpretation always exists along the lines of the results presented in Sec. III and summarized in the theorem proven there.

We are now going to compare our results with some work of other authors who have treated similar problems. We shall identify the restrictions used in these works and we shall discuss how their results can be obtained by particularizing our general framework to the cases they study. This analysis will also allow us to present alternative, but equivalent, versions of our general results.

(a) We start with a series of relevant papers by Santos *et al.* [14–16] in which radiative collapse has been treated. Their results were put together in a final and interesting report [17] in which the work can be found.

In Santos's approach [14], isotropic coordinates are used to describe $\mathcal{V}_I$ and radiative coordinates for the Vaidya metric, just as in our work. However, in order to find the surface equations using the Darmois junction conditions, he imposes the condition that the isotropic interior coordinates are comoving with the interior fluid and the matching surface. This is exactly equivalent to assuming that the velocity vector field of the fluid is shear-free. In our general framework, their results can be recovered by simply restricting the equation of Σ to $r=$const and setting the function $f(t,r)$ equal to zero, which is the shear-free case. With these restrictions our matching conditions specialize to exactly those of Santos.

On the other hand, Santos assumes that the energy-momentum tensor of the interior matter is that of comoving perfect fluid with heat conduction, and the main result of his work is that the pressure must be proportional to the heat flux on Σ. Here there is an inexplicitly stated assumption in the sense that the tensor of anisotropic pressures is set equal to zero.

In fact, Santos's results can be straightforwardly generalized in the following way. Given the velocity vector (12) subject to (13) and (14), we can decompose the energy-momentum tensor either as in (15) or as

$$T_{I\mu\nu}=(\tilde{\rho}+\tilde{p})u_\mu u_\nu+\tilde{p}g_{I\mu\nu}+q_\mu u_\nu+q_\nu u_\mu+\tilde{\Pi}_{\mu\nu} , \qquad (24)$$

where now $\mathbf{q}$ is the vector of heat flow, which must be orthogonal to $\mathbf{u}$, and $\tilde{\rho},\tilde{p}$, and $\tilde{\Pi}_{\mu\nu}$ (which again is trace-free and orthogonal to $\mathbf{u}$) have the standard interpretation. The forms (15) and (24) are strictly equivalent. They both represent the most general energy-momentum tensor associated with $\mathbf{u}$, and they simply are alternative possible decompositions. The relations between the physical quantities of (24) and those of (15) can be obtained straightforwardly. Therefore, as in the previous decomposition, for every $A(t,r)$ and $B(t,r)$ such that $r(t)$ exists, we can determine, via Einstein's equations, $\tilde{\rho},\tilde{p},\mathbf{q}$, and $\tilde{\Pi}_{\mu\nu}$ on and inside Σ for every choice of $f(t,r)$ compatible with (13) and (14). In terms of the new quantities, we can obtain the relations deduced from Eqs. (17) and (18), which are now

$$q \overset{\Sigma}{=} \Phi^2 , \qquad (25)$$

$$q \overset{\Sigma}{=} \tilde{p}+\tilde{\Pi}_{\mu\nu}n_I^\mu n_I^\nu , \qquad (26)$$

where $q=q^\mu n_{I\mu}$. These relations are equivalent to (19) and (21), respectively. We can thus reformulate our theorem of Sec. III as follows.

Every spherically symmetric metric can be locally matched to an exterior Vaidya solution provided that there exists a timelike surface such that the corresponding total radial pressure $\tilde{p}+\tilde{\Pi}_{\mu\nu}n_I^\mu n_I^\nu$ and heat flux q are equal on it. In that case, this surface is the matching surface.

Equation (26) is the generalization of Santos's main relation to arbitrary fluids, shear-free or not, with a nonvanishing tensor of anisotropic pressures. As is obvious, this generalization reduces to his main relation when $\Pi_{\mu\nu}n_I^\mu n_I^\nu \overset{\Sigma}{=} 0$.

(b) In studying radiative collapse, radiative coordinates at both sides of the matching surface have been commonly used by several authors [7–12]. In order to include their results in our general framework, it is convenient to rewrite our matching conditions in terms of the mentioned coordinates. To that end, let us describe the general spherically symmetric interior metric as

$$ds_I^2=-e^{4\beta}\left[1-\frac{2\tilde{m}}{r}\right]dt^2-2e^{2\beta}dt\,dr+r^2d\Omega^2 , \qquad (27)$$

where β and $\tilde{m}$ are functions of the new radiative coordinates t,r, which should not be confused with the coordinates t,r in (1).

The Darmois junction conditions (3)–(4) lead now to

$$r \overset{\Sigma}{=} R , \qquad (28)$$

$$\frac{\dot{t}}{\dot{T}} \overset{\Sigma}{=} e^{-2\beta} , \qquad (29)$$

$$\tilde{m} \overset{\Sigma}{=} m , \qquad (30)$$

$$\frac{\partial\beta}{\partial r}\left[1-\frac{2\tilde{m}}{r}\right]-\frac{\partial\tilde{m}}{\partial r}\frac{1}{2r}+\frac{\partial\beta}{\partial r}\frac{\dot{r}}{\dot{t}}e^{-2\beta} \overset{\Sigma}{=} 0 . \qquad (31)$$

These are the matching conditions expressed in the new radiative coordinates, and they are fully equivalent to (7)–(10). As before, we see that for all smooth functions β and $\tilde{m}$, (31) will have a solution for $r(t)$. When this solution defines a timelike surface Σ the matching is possible. For the sake of simplicity we will choose $\lambda=T$ from now on. Then Σ will be defined by $R \overset{\Sigma}{=} R(T)$ in $\mathcal{V}_E$ and $r \overset{\Sigma}{=} r(T)$, $t \overset{\Sigma}{=} t(T)$ in $\mathcal{V}_I$.

Suppose now that we have already matched some given interior with a Vaidya exterior; i.e., that $\beta,\tilde{m},m$ and $r(T)$, $t(T)$, $R(T)$ are known functions. Then we can perform a coordinate transformation in $\mathcal{V}_I$, $t,r\rightarrow\hat{t},\hat{r}$, in such a way that the new coordinates verify that $d\hat{t}$ is timelike, $d\hat{r}$ is null, $\hat{t} \overset{\Sigma}{=} T$, $\hat{r} \overset{\Sigma}{=} R$. In these coordinates the metric takes the same form (27) but with new functions β and $\tilde{m}$ (which we call $\hat{\beta}$ and $\hat{m}$). If such a coordinate transformation is possible then we shall say that both sides of Σ are described with the same coordinate system. According to this, $\hat{t}$ and $\hat{r}$ will be renamed as T and R, respectively (i.e., the coordinates in $\mathcal{V}_E$).

It is easy to prove, taking into account that (28)–(31) hold, that the coordinate transformation

$$t = t(T), \quad r = R \tag{32}$$

satisfies the properties mentioned above, where $t(T)$ is the solution found by solving the matching conditions. This coordinate transformation implies that the new functions $\hat{m}$ and $\hat{\beta}$ are related to the old ones by

$$\hat{m} = \tilde{m}, \quad e^{2\hat{\beta}} = e^{2\beta}\dot{t} \ . \tag{33}$$

We can now rewrite the matching conditions (28)–(31) in terms of the new coordinates T,R and the new functions $\hat{\beta}$, $\hat{m}$ (considered now as functions of T and R), and we get

$$\hat{\beta} \overset{\Sigma}{=} 0 \ , \tag{34}$$

$$\hat{m} \overset{\Sigma}{=} m \ , \tag{35}$$

$$\frac{\partial \hat{\beta}}{\partial R}\left[1 - \frac{2\hat{m}}{R}\right] - \frac{\partial \hat{m}}{\partial R}\frac{1}{2R} - \frac{\partial \hat{\beta}}{\partial T} \overset{\Sigma}{=} 0 \ . \tag{36}$$

From a historical point of view the matching conditions (35), (36) were first found by Bondi [7], and he used them to study systems of collapsing radiating stars. The full set of conditions (34)–(36) were obtained later by Herrera and collaborators [8] and successfully used in the series of papers about gravitational collapse referred to before. As we have just seen, for each matching performed by means of (28)–(31) there exists a geometrically equivalent matching performed by means of (34)–(36). We can say then that both systems of matching conditions are equivalent. However, there are some important features of the first system (28)–(31) that make it preferable, as we shall see below.

In the cited works of Bondi and Herrera *et al.*, has been usual to start with some knowledge of the structure of $\hat{\beta}$ and $\hat{m}$, leaving some undetermined functions for which the matching conditions must be satisfied. In addition, they consider some prefixed form of the energy-momentum tensor. Then, from the whole set of equations they try to determine the unknown functions together with Σ. This type of program may run into difficulties because of two different reasons. First, because of the prefixed form of the energy-momentum tensor, which usually does not take into account the tensor of anisotropic pressures, it could happen that an interior solution is matchable to a Vaidya exterior but does not have a satisfactory interpretation in terms of the physical quantities. This would be the case if the prefixed velocity vector of the fluid were eventually not comoving with the matching surface. This is closely related to the freedom in choosing the function f, a freedom which must be maintained. In other words, the function f must not be determined *a priori* by means of the Einstein equations for the previously given energy-momentum tensor.

The second problem arises when the set of equations they solve does not have a solution, which can happen because of the fixed functional form of $\hat{\beta}$ and $\hat{m}$. For instance, it might occur that condition (34) is not verified, and if so, its solution will be incompatible with condition (36). In any case, this is a spurious and irrelevant problem which arises due to the peculiar way in which the subject is treated in the cited references. For, given any $\hat{\beta}$ and $\hat{m}$, one has to bear in mind that these functions are only a particular choice among the infinite number of possible $\hat{\beta}$'s and $\hat{m}$'s which give rise the *same interior.* Thus, if the chosen $\hat{\beta}$ does not satisfy the necessary conditions one must find another one, which always exists, that does satisfy the conditions. Usually, this amounts to taking a new function $\hat{\beta}$ by subtracting $\hat{\beta}_\Sigma$ from the initial one. In this way, (36) is not an algebraic relation for $R(T)$ and becomes a differential equation for the same variable, as is the initial matching condition (31).

Therefore, it becomes obvious that the natural way to treat this problem is to start from (28)–(31) without prefixing anything and carrying out the program that we have presented. Somehow, this method chooses the specific form of the function $\hat{\beta}$ naturally. We also believe that this should be the way to do the matching from a theoretical point of view, because in principle nothing can be said about the relation between the coordinates at both sides of Σ.

Finally, we should like to note that the matching conditions (7)–(10) hold also in the case in which the interior part of the space-time is interchanged with the exterior one. Thus, our work can be applied to the study of a Vaidya radiating cavity surrounded by a general spherically symmetric metric. This can be of interest in the case of local inhomogeneities in a regular universe, along the lines of Ref. [21].

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APPENDIX

In this appendix we shall present the lengthy formulas used in the main text.

The nonvanishing components of the Einstein tensor $G_{I\mu\nu}$ for the line element (1) are

$$G_{I00} = -\frac{A^2}{B^2}\left[\frac{2}{B}\frac{\partial^2 B}{\partial r^2} - \frac{1}{B^2}\left[\frac{\partial B}{\partial r}\right]^2 + \frac{4}{rB}\frac{\partial B}{\partial r}\right] + \frac{3}{B^2}\left[\frac{\partial B}{\partial t}\right]^2 , \tag{A1}$$

$$G_{I11}=\frac{1}{B^2}\left[\frac{\partial B}{\partial r}\right]^2+\frac{2}{rB}\frac{\partial B}{\partial r}+\frac{2}{AB}\frac{\partial A}{\partial r}\frac{\partial B}{\partial r}+\frac{2}{rA}\frac{\partial A}{\partial r}+\frac{B^2}{A^2}\left[-\frac{2}{B}\frac{\partial^2 B}{\partial t^2}-\frac{1}{B^2}\left[\frac{\partial B}{\partial t}\right]^2+\frac{2}{AB}\frac{\partial A}{\partial t}\frac{\partial B}{\partial t}\right], \tag{A2}$$

$$G_{I22}=\frac{1}{\sin^2\theta}G_{I33}=r^2\left[\frac{1}{B}\frac{\partial^2 B}{\partial r^2}-\frac{1}{B^2}\left[\frac{\partial B}{\partial r}\right]^2+\frac{1}{rB}\frac{\partial B}{\partial r}+\frac{1}{A}\frac{\partial^2 A}{\partial r^2}+\frac{1}{rA}\frac{\partial A}{\partial r}\right]$$
$$+r^2\frac{B^2}{A^2}\left[-\frac{2}{B}\frac{\partial^2 B}{\partial t^2}-\frac{1}{B^2}\left[\frac{\partial B}{\partial t}\right]^2+\frac{2}{AB}\frac{\partial A}{\partial t}\frac{\partial B}{\partial t}\right], \tag{A3}$$

$$G_{I01}=-\frac{2}{B}\frac{\partial^2 B}{\partial r\,\partial t}+\frac{2}{B^2}\frac{\partial B}{\partial r}\frac{\partial B}{\partial t}+\frac{2}{AB}\frac{\partial A}{\partial r}\frac{\partial B}{\partial t}. \tag{A4}$$

On the other hand, the only nonvanishing component of $G_{E\mu\nu}$ related to the metric (2) is

$$G_{E00}=-\frac{2}{R^2}\frac{dm}{dT}. \tag{A5}$$

The explicit expressions for the first fundamental forms of Σ are given by

$$ds_I^2|_\Sigma \overset{\Sigma}{=} -(A^2\dot t^2-B^2\dot r^2)d\lambda^2+B^2r^2d\Omega^2, \tag{A6}$$

$$ds_E^2|_\Sigma \overset{\Sigma}{=} -(\chi\dot T+2\dot R)\dot T\,d\lambda^2+R^2d\Omega^2, \tag{A7}$$

and the unit outward one-forms normal to Σ are

$$\mathbf{n}_I=\left|\frac{AB}{(A^2\dot t^2-B^2\dot r^2)^{1/2}}\right|_\Sigma(-\dot r\,dt+\dot t\,dr), \tag{A8}$$

$$\mathbf{n}_E=\left|\frac{1}{(\chi\dot T^2-2\dot R\dot T)^{1/2}}\right|_\Sigma(-\dot R\,dT+\dot T\,dR). \tag{A9}$$

Finally, the explicit expressions for the nonvanishing components of the extrinsic curvatures of Σ are

$$K_{I00}\overset{\Sigma}{=}\frac{AB}{(A^2\dot t^2-B^2\dot r^2)^{1/2}}\left[\left[\frac{\dot r}{A}\frac{\partial A}{\partial t}-\frac{A}{B^2}\frac{\partial A}{\partial r}\dot t\right]\dot t^2-2\dot r\frac{1}{B}\frac{\partial B}{\partial t}\dot t^2+\dot r^2\dot t\left[\frac{2}{A}\frac{\partial A}{\partial r}-\frac{1}{B}\frac{\partial B}{\partial r}\right]+\dot r^3\frac{B}{A^2}\frac{\partial B}{\partial t}-\ddot r\dot t+\dot r\ddot t\right], \tag{A10}$$

$$K_{I22}\overset{\Sigma}{=}\frac{1}{\sin^2\theta K_{I33}}\overset{\Sigma}{=}\frac{Ar}{(A^2\dot t^2-B^2\dot r^2)^{1/2}}\left[\frac{B^2r\dot r}{A^2}\frac{\partial B}{\partial t}+\dot t\frac{\partial(Br)}{\partial r}\right], \tag{A11}$$

$$K_{E00}\overset{\Sigma}{=}\frac{\dot T}{(\chi\dot T^2+2\dot R\dot T)^{1/2}}\left[-\frac{\dot T^2}{R}\left[\frac{m\chi}{R}-\frac{dm}{dT}\right]+\dot R\left[\frac{\ddot T}{\dot T}-3\frac{\dot T m}{R^2}\right]-\ddot R\right], \tag{A12}$$

$$K_{E22}\overset{\Sigma}{=}\frac{1}{\sin^2\theta}K_{E33}\overset{\Sigma}{=}\frac{\dot T}{(\chi\dot T^2+2\dot R\dot T)^{1/2}}\left[\frac{\dot R R}{\dot T}+2m-R\right]. \tag{A13}$$

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