Estimation of the parameters of the gravitational-wave signal of a coalescing binary system

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The statistical theory of signal detection and the estimation of its parameters are reviewed and applied to the case of detection of the gravitational-wave signal from a coalescing binary by a laser interferometer. The correlation integral and the covariance matrix for all possible static configurations are investigated numerically. Approximate analytic formulas are derived for the case of narrow band sensitivity configuration of the detector.

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I. INTRODUCTION

In order to detect the gravitational waves by the proposed laser interferometers [1,2] it is necessary to improve and develop the data analysis techniques. The gravity-wave signals from unknown astrophysical sources are very weak as compared with the noise in the detector and it may be crucial to employ as optimal techniques as possible.

In this paper we review a classical method of signal detection and estimation of signal parameters, the maximum likelihood estimation, and we apply it to one important source of gravitational waves: the coalescing binary system. As the two bodies in a compact binary system spiral together, they emit a periodic gravitational-wave signal, called a chirp, whose amplitude and frequency rise as coalescence approaches. For the simple model of two point masses the wave form during the nearly Newtonian phase is easily obtained from the quadrupole formula [3]. The signal depends on four parameters: amplitude, phase, time of arrival, and chirp mass.

In a previous publication [4] we derived the spectral density of noise for the possible configurations of the laser interferometer and we calculated numerically the signal-to-noise ratio. In another publication [5] the Fourier transform of the chirp was derived and the correlation integral and the covariance matrix were investigated for the standard recycling configuration of the detector.

In Sec. II we review the theory of signal detection and parameter estimation. In the case of the stationary Gaussian noise, which we assume will be the case for the gravitational wave detectors, the optimal procedure consists of filtering the data with a *linear* filter. In the case of a chirp we show that to determine time of arrival and the chirp mass we have to pass the data through two banks of linear filters and maximize a certain functional formed from the outputs of the two filters. Then the estimates of the phase and the amplitude can be explicitly calculated.

In Secs. III and IV we investigate numerically the correlation integral and the covariance matrix for all possible configurations of the detector.

In Sec. V we give the explicit analytic formulas for the correlation integral and components of the covariance matrix in the case of a resonant configuration of the detector.

II. THE STATISTICAL THEORY OF DETECTION OF SIGNALS AND ESTIMATION OF SIGNAL PARAMETERS

In the case of a gravitational wave from a coalescing binary we know the form of the signal as a function of a number of unknown parameters. Thus the problems of detection of such a signal and estimation of its parameters are closely connected. Let us first consider the problem of signal detection.

As a result of noise a datum from the detector is a value of a certain *random variable*. For example, in a gravitational wave detector based on a laser interferometer one of the main sources of noise is the photon counting noise [6]. Therefore the current measured at the photodiode is a random variable. Since we take measurements every certain interval of time the data from a detector form a sample of a certain (discrete) *stochastic process*. The presence of the signal will affect the probability distribution of the stochastic process.

Let X_i be the random variable and let $\mathbf{X}=(X_1,X_2,\ldots,X_n)$ be the stochastic process in question and let x_i and $\mathbf{x}=(x_1 x_2,\ldots,x_n)$ be a sample of the random variable and the stochastic process respectively. If there is no signal we have a joint *probability density function* (PDF) $p_0(\mathbf{x})$ and if the signal is present we have a PDF $p_1(\mathbf{x})$. To decide which is the PDF we have to

*Deceased.

devise a rule called the *test* which divides the range of values of $\mathbf{x}$ into two sets R and its complement R'.

We say $p_1(\mathbf{x})$ if $\mathbf{x}\in R$ and $p_0(\mathbf{x})$ if $\mathbf{x}\in R'$.

The detection probability $P_D(R)$ is given by

$$P_D(R)=\int_R p_1(\mathbf{x})d\mathbf{x} \tag{1}$$

and the false alarm probability $P_F(R)$ (i.e., a probability of deciding that signal is present even though it is really absent) is given by

$$P_F(R)=\int_R p_0(\mathbf{x})d\mathbf{x} \ . \tag{2}$$

The most appropriate approach to the detection of gravitational waves seems to be the Neyman-Pearson approach. In this approach we seek a test that maximizes the detection probability subject to a preassigned false alarm probability $P_F(R)=\alpha$. The solution that can be obtained using the method of Lagrange multipliers is the region R defined by

$$R=\{\mathbf{x}:\Lambda(\mathbf{x})\geq k\} \ , \tag{3}$$

where $\Lambda(\mathbf{x})$ is the likelihood ratio given by

$$\Lambda(\mathbf{x})=\frac{p_1(\mathbf{x})}{p_0(\mathbf{x})} \tag{4}$$

and k is defined by

$$P_F[\Lambda(\mathbf{x})\geq k]=\alpha \ . \tag{5}$$

Thus if $\Lambda(\mathbf{x})\geq k$ we say that the hypothesis denoted by H_1, that the signal is present, is true whereas if $\Lambda(\mathbf{x})<k$ we say that the hypothesis H_0, that the signal is absent, is true.

For theoretical considerations it is convenient to consider the continuous time stochastic processes. We assume that we make observations over a certain interval of time $[0,T]$ and that the noise $n(t)$ is additive. Thus we have

$$x(t)=n(t) \ , \quad 0\leq t\leq T \ , \text{ if } H_0 \text{ is true} \ , \tag{6}$$

$$x(t)=s(t)+n(t) \ , \quad 0\leq t\leq T \ , \text{ if } H_1 \text{ is true} \ , \tag{7}$$

where $s(t)$ is the signal for which we are searching. In the following we assume that $s(t)$ is a deterministic function of time.

We assume that the noise $N(t)$ in the detector is a zero-mean Gaussian stochastic process with the correlation function $K_N(t,u)$.

A stochastic process $X(t)$ is said to be Gaussian if every finite linear combination $Z=\sum_{i=1}^{M}\alpha_i X(t_i)$ is a Gaussian random variable, i.e., Z has a Gaussian probability density function. It can be shown that in the above case the logarithm of the likelihood ratio is given by

$$\ln\Lambda[x(t)]=\int_0^T x(t)q(t)dt-\frac{1}{2}\int_0^T s(t)q(t)dt \ , \tag{8}$$

where $q(t)$ is the solution of the integral equation

$$s(t)=\int_0^T K_N(t,u)q(u)du \ . \tag{9}$$

The likelihood ratio $\Lambda[x(t)]$ depends on $x(t)$ only through the integral

$$G=\int_0^T x(t)q(t)dt \ . \tag{10}$$

The integral G is called the *detection statistics.*

Thus the Neyman-Pearson test in the above case consists of passing the data through a linear filter $k(\tau)$ whose impulse response is

$$k(\tau)=q(T-\tau) \ , \quad 0\leq\tau\leq T \ , \tag{11}$$

$$k(\tau)=0 \ , \quad \tau<0,\tau>T \ , \tag{12}$$

where $q(u)$ is the solution of the integral equation (9) with the appropriate boundary conditions.

If the noise is *stationary* [i.e., $K_N(t,u)=K_N(t-u)$] and if the whole of the signal is included in the interval $[0,T]$ the integral equation can be solved by Fourier transform.

The Fourier transform $\hat{K}(\omega)$ of the impulse response $K(\tau)$ is given by

$$\hat{K}(\omega)=\exp[-i\omega T]\frac{\hat{S}^*(\omega)}{\Phi(\omega)} \ , \tag{13}$$

where $\hat{S}(\omega)$ is the Fourier transform of the signal and $\Phi(\omega)$ is the two-sided spectral density of noise.

$\Phi(\omega)$ is the Fourier transform of the correlation function

$$\Phi(\omega)=\int_{-\infty}^{\infty}K_N(\tau)\exp(i\omega\tau)d\tau \ . \tag{14}$$

Depending on whether signal is absent or present, the detection statistics G has the probability density functions

$$p_0(G)=(2\pi d^2)^{-1/2}\exp\left[\frac{-G}{2d^2}\right] , \tag{15}$$

$$p_1(G)=(2\pi d^2)^{-1/2}\exp\left[\frac{-(G-d^2)}{2d^2}\right] , \tag{16}$$

where

$$d^2=\int_0^T s(t)q(t)dt \ . \tag{17}$$

These are Gaussian probability density functions. This is clear as G is a linear functional of a Gaussian stochastic process. The above PDF's are determined by a single parameter d.

We call this parameter the signal-to-noise ratio. If the noise is white (i.e., $\Phi(\omega)=\text{const}$) d^2 is the ratio of the powers of the filtered signal and the filtered noise.

With the filter given by formula (13) the signal-to-noise ratio can be expressed as

$$d^2=\int_{-\infty}^{\infty}\frac{|S(\omega)|^2}{\Phi(\omega)}\frac{d\omega}{2\pi} \ . \tag{18}$$

The false alarm and detection probabilities are given by

$$Q_F=\text{erfc}(G_0/d) \ , \tag{19}$$

$$Q_D=1-\text{erfc}(d-G_0/d) \ , \tag{20}$$

where erfc is the error function and G_0 is the threshold with which the statistics G is compared.

The plot of detection probability as a function of false alarm probability for a given value of d is called the *receiver operating characteristic* (ROC).

Thus the detection of a known signal buried in a Gaussian noise is completely determined by signal-to-noise ratio d.

A. Parameter estimation theory

In general we know the form of the signal as a function of a number of parameters. For example, in the case of a coalescing binary the unknown parameters are the time of arrival of the signal, masses of the members of the binary, the amplitude, and the phase of the wave.

In such a case in the process of detection of the signal we must also determine its parameters. The classical estimation method proposed here is the maximum likelihood estimation [7,8].

Let $\theta=(\theta_1,\theta_2,\ldots,\theta_m)$ be the set of unknown parameters of the signal $s(t,\theta)$. As in the case of the completely known signal we can consider two probability density functions $p_1[x(t);\theta]$ and $p_0[x(t)]$ depending on whether the signal is present or absent and we form the likelihood ratio

$$\Lambda[x(t);\theta]=p_1[x(t);\theta]/p_0[x(t)] .$$

The maximum likelihood estimators (MLE's) $\hat{\theta}$ are those values of the parameters θ that maximize the likelihood ratio $L[x(t);\theta]$. Thus the MLE's can be found by solution of the set of simultaneous equations:

$$\frac{\partial}{\partial\theta_k}\Lambda[x(t);\theta]=0 . \tag{21}$$

In the case of Gaussian noise the MLE's can be obtained by passing the data through a bank of linear filters for suitably spaced values of the parameters and each of the filters being determined by solution of the integral equation

$$s(t,\theta)=\int_0^T K_N(t,u)q(u;\theta)du . \tag{22}$$

The values of the parameters that maximize the output are the maximum likelihood estimators.

The maximum likelihood estimator $\hat{\theta}(x)$ is a random variable since it is a functional of the random variable $x(t)$ determined by the set of equations (21).

Let Γ_{ij} be the matrix whose components are given by

$$\Gamma_{ij}=-E\frac{\partial^2}{\partial\theta_i\partial\theta_k}\ln\Lambda[x(t);\theta] , \tag{23}$$

where E means the expectation value. Γ_{jk} is called the Fisher information matrix.

We say that the estimate $\hat{\theta}$ of the set of parameters θ is unbiased if the expectation value of $\hat{\theta}$ is equal to the true values of parameters, i.e.,

$$E(\hat{\theta})=\theta . \tag{24}$$

There is a general result called the Cramer-Rao inequality which provides bounds on the variances of unbiased maximum likelihood estimators:

$$E(\hat{\theta}_i-\theta_i)(\hat{\theta}_j-\theta j)\geq(\Gamma)^{-1}_{ij} . \tag{25}$$

An estimator that is unbiased and which attains the lower bound given by the right-hand side of inequality (25) is called an efficient estimate. The condition for an unbiased estimate to be efficient is that it must be a sufficient statistics; i.e., $\Lambda(x;\theta)$ can be written in the form

$$\Lambda(x;\theta)=Q[\hat{\theta}(x);\theta]r(x) \tag{26}$$

with the factor $r(x)$ independent of the parameter θ and that the following equation must hold:

$$\frac{\partial}{\partial\theta}\ln\Lambda(x;\theta)=k(\theta)[\hat{\theta}(x)-\theta] , \tag{27}$$

where $k(\theta)$ is independent of x.

It can be proven [9] that the maximum likelihood estimator is asymptotically unbiased, efficient, and has a Gaussian distribution with the mean equal to the true value of the parameter and the variance given by the right-hand side of the Cramer-Rao inequality. Asymptotically means that the number of estimates of the parameter tends to infinity.

In the Gaussian case the Fisher information matrix is given by

$$\Gamma_{ij}=\left.\frac{\partial^2 H(\theta_1,\theta_2)}{\partial\theta_1\partial\theta_2}\right|_{\theta_1=\theta_2=\theta} , \tag{28}$$

where

$$H(\theta_1,\theta_2)=\int_0^T q(t;\theta_1)s(t;\theta_2)dt \tag{29}$$

and $q(t;\theta)$ is the solution of Eq. (22).

The integral H is called the correlation integral or the ambiguity function.

With the filter given by Eq. (13) the correlation integral is given by

$$H(\theta_1,\theta)=2\int_0^\infty \mathrm{Re}\frac{S(\omega;\theta_1)S^*(\omega,\theta_2)}{\Phi(\omega)}\frac{d\omega}{2\pi} . \tag{30}$$

It can be shown that the right-hand side of the Cramer-Rao inequality is the better approximation of the covariance matrix of the estimators of parameters of the signal the higher the signal-to-noise ratio [10]. Following Helstrom we shall call the inverse of the Fisher matrix the covariance matrix. However one should remember that the inverse of the Fisher matrix provides the lower bounds on the accuracy of the estimation of the parameters and in practice the errors will always be greater. The question arises how big should the signal-to-noise ratio be in order that the inverse of the Fisher matrix approximates well the covariance matrix. In a paper by one of the authors [11], where the simulations of the detection of the gravitational wave signal from a coalescing binary and estimation of its parameters were performed, it was found that for a signal-to-noise ratio of 25 the agreement between the variances of the estimators obtained from numerical simulations and the theoretical covariance matrix is very good whereas at signal-to-noise ratio of 8 the variances of the estimators from the simulations are distinctly greater than that given by the inverse of the Fisher matrix.

Standard textbooks on the theory of signal detection are monographs by Helstrom [8], Van Trees [12], Whalen [13], and Weinstein and Zubakhov [14]. These texts are oriented towards applications to radar.

The first treatment of the subject with application to the detection of gravitational waves was a review article by Davis [15]. This article has an additional advantage of being written in the contemporary language of stochastic processes.

III. ESTIMATION OF THE PARAMETERS OF THE GRAVITATIONAL WAVE SIGNAL FROM A COALESCING BINARY

We assume that we detect the gravitational wave by means of a laser interferometer. We also assume that arms of the detector are at right angles each of length L.

Then the dimensionless response $\delta L/L$ of the laser interferometer to the coalescing binary gravitational wave signal where δL is the difference in the changes of L in each arm is given by [16] (in units with $c=1$ and $G=1$)

$$\frac{\delta L}{L}=\frac{4\mathcal{M}^{5/3}}{R}(\pi f)^{2/3}\left[F_{+}\frac{1+\cos^2\alpha}{2}\cos\left[2\pi\int_{t_i}^{t}f(t')dt'+\phi'\right]+F_x\cos\alpha\sin\left[2\pi\int_{t_i}^{t}f(t')dt'+\phi'\right]\right], \tag{31}$$

where

$$F_{+}=\cos2\phi\cos\theta\sin2\psi+\tfrac{1}{2}\sin2\phi(1+\cos^2\theta)\cos2\psi , \tag{32}$$

$$F_x=\cos2\phi\cos\theta\cos2\psi-\tfrac{1}{2}\sin2\phi(1+\cos^2\theta)\sin2\psi \tag{33}$$

and ϕ,θ,ψ are the three Eulerian angles where ϕ and θ describe the incoming direction of the wave and ψ is the angle between one semiaxis of the ellipse of polarization and the node direction. The angle α is the angle between the line of sight and the vector normal to the orbit of the binary, ϕ' is an arbitrary phase of the wave form. $\mathcal{M}$ is the "chirp" mass defined by $\mathcal{M}^{5/3}=\mu m^{2/3}$ where $m=m_1+m_2$ is the total mass of the components of the binary, $\mu=m_1m_2/m$ is the reduced mass and $f(t)$ is the frequency of the gravitational wave (twice the orbital frequency) given by

$$f(t)=\frac{1}{\pi}\frac{5}{256}\frac{1}{\mathcal{M}^{5/3}}\frac{1}{(t_0-t)^{3/8}} , \tag{34}$$

where t_0 is the time at which coalescence occurs, R is the distance from the binary, and t_i is an arbitrarily chosen initial time.

It is convenient to choose the time t_i as the time when the gravitational wave has a certain frequency f_i. Thus by the above formula t_0 and t_i are not independent and t_0 can be expressed in terms of t_i and the arbitrarily chosen frequency f_i.

The above formulas were obtained using the quadruple approximation and assuming that the orbit of the binary is circular.

The angles θ, ϕ, ψ, and α can only be estimated if we have a network of three or more detectors [17]. Here we restrict ourselves to estimation by only one detector, therefore we introduce a root mean square response $\langle h\rangle$ defined as

$$\langle h\rangle^2=\frac{1}{16\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{2\pi}\int_0^{\pi}(\delta L/L)^2\sin\theta\, d\theta\, d\phi\, d\psi\sin\alpha\, d\alpha \tag{35}$$

and consider a characteristic response of the detector

$$(\delta L/L_c)=\langle h\rangle\sin\left[2\pi\int_{t_i}^{t}f(t')dt'+\phi\right] . \tag{36}$$

It can be shown that $\langle h\rangle=8\mathcal{M}^{5/3}(\pi f)^{2/3}/5R$.

Thus in this case the gravitational wave signal $s(t)$ can be written in the form

$$s(t)=Ag(t;\mathcal{M},t_i)\sin[F(t;\mathcal{M},t_i)+\phi] . \tag{37}$$

We shall show that for the above signal maximum likelihood estimates of amplitude A and phase ϕ can be calculated explicitly and we only need a two-parameter bank of filters to estimate the chirp mass and the initial time.

A. Estimation of A, ϕ, t_i and $\mathcal{M}$

Let us write the signal $s(t)$ as $s(t;A)=Ah(t)$. Then $q(t;A)=Ar(t)$, where $r(t)$ is the solution of the integral equation

$$h(t)=\int_0^T K_N(t,u)r(u)du . \tag{38}$$

The logarithmic likelihood ratio [Eq. (8)] can now be expressed as

$$\ln\Lambda[x(t);A]=A\int_0^T r(t)x(t)dt-\frac{1}{2}A^2\int_0^T r(t)h(t)dt . \tag{39}$$

It is maximized by the linear estimate

$$\hat{A}=\frac{\int_0^T r(t)x(t)dt}{\int_0^T r(t)h(t)dt} . \tag{40}$$

Since $E[x(t)]=Ah(t)$ we have that $E[\hat{A}]=A$ and thus the above estimate of A is unbiased.

Let us substitute the estimate $\hat{A}$ for A in the likelihood ratio. We get

$$\ln\Lambda[x(t);\hat{A}]=\frac{1}{2}\frac{\left[\int_0^T r(t)x(t)dt\right]^2}{\int_0^T r(t)h(t)dt} . \tag{41}$$

The estimates of the remaining parameters of the chirp (phase, time of arrival, and chirp mass) are found by maximizing the above functional. We shall next show that an explicit formula can be found for the phase ϕ.

Let us write $h(t)$ as

$$h(t)=g(t)\sin[F(t)]\cos\phi+g(t)\cos[F(t)]\sin\phi \tag{42}$$

and let h_c and h_s be solutions of integral equations

$$g(t)\cos[F(t)]=\int_0^T K_N(t,u)h_c(u)du , \tag{43}$$

$$g(t)\sin[F(t)]=\int_0^T K_N(t,u)h_s(u)du . \tag{44}$$

Consequently $r(t)$ can be written as

$$r(t)=h_c(t)\sin\phi+h_s(t)\cos\phi . \tag{45}$$

Let us first consider the denominator of Eq. (41). It can be written as

$$\left[\int_0^T g\cos F\, h_c\, dt\right]\sin^2\phi+\left[\int_0^T g\,\sin u\, h_s\, dt\right]\cos^2\phi+\left[\int_0^T g\cos u\, h_s+\int_0^T g\,\sin u\, h_c\right]\sin\phi\cos\phi . \tag{46}$$

The Fourier transforms of $C:=Ag\cos F$ and $S:=Ag\sin F$ are given to a very good accuracy by [5]

$$\hat{C}=K'f^{-7/6}\exp\left[-2i\pi f t_i-i\frac{1}{\mathcal{M}^{5/3}}a(f)+i\frac{\pi}{4}\right] , \tag{47}$$

$$\hat{S}=K'f^{-7/6}\exp\left[-2i\pi f t_i-i\frac{1}{\mathcal{M}^{5/3}}a(t)-i\frac{\pi}{4}\right] , \tag{48}$$

where

$$K'=\frac{1}{(30)^{1/2}}\frac{1}{(\pi)^{2/3}}\frac{\mathcal{M}^{5/6}}{R} \tag{49}$$

and

$$a(f)=\frac{1}{2^7}\left[\frac{3}{(\pi f)^{5/3}}+\frac{5\pi f}{(\pi f_i)^{8/3}}-\frac{8}{(\pi f_i)^{5/3}}\right] . \tag{50}$$

Fourier transforms of h_c and h_s are given by formulas analogous to Eq. (13):

$$\hat{h}_c(\omega)=\frac{\exp[-i\omega T]}{A}\frac{\hat{C}^*(\omega)}{\Phi(\omega)} , \tag{51}$$

$$\hat{h}_s(\omega)=\frac{\exp[i\omega T]}{A}\frac{\hat{S}^*(\omega)}{\Phi(\omega)} . \tag{52}$$

Hence by a formula similar to Eq. (30) it follows that

$$\int_0^T g\cos F\, h_c dt=\int_0^T g\sin F\, h_s dt , \tag{53}$$

$$\int_0^T g\cos F\, h_c dt=\int_0^T g\sin F\, h_s dt=0 . \tag{54}$$

Consequently the solution of the equation

$$\frac{\partial}{\partial\phi}\ln\Lambda[x(t);\phi]=0 \tag{55}$$

that maximizes the likelihood ratio is given by

$$\hat{\phi}=\arctan\left[\frac{\int_0^T h_c(t)x(t)dt}{\int_0^T h_s(t)x(t)dt}\right] . \tag{56}$$

This is the maximum likelihood estimate of the phase.

Substituting the estimate of phase into the likelihood ratio we get

$$\ln\Lambda[x(t);t_i,\mathcal{M}]=\frac{1}{2}\frac{1}{\int_0^T h_c(t)g(t)\cos[F(t)]dt}\left[\left[\int_0^T h_c(t)x(t)dt\right]^2+\left[\int_0^T h_s(t)x(t)dt\right]^2\right] . \tag{57}$$

The above analysis determines the optimal analysis of the data to find the maximum likelihood estimates of the parameters of the chirp.

First one passes the data through two banks of linear filters: $h_c(t;t_i,\mathcal{M})$ and $h_s(t;t_i,\mathcal{M})$ for suitably spaced values of parameters t_i and $\mathcal{M}$. The maximum likelihood estimates of time of arrival $\hat{t}_i$ and chirp mass $\hat{\mathcal{M}}$ are those values of t_i and $\mathcal{M}$ that maximize the functional

$$\mathcal{F}(t_i,\mathcal{M})=\frac{\left[\int_0^T h_c(t)x(t)dt\right]^2+\left[\int_0^T h_s(t)x(t)dt\right]^2}{\int_0^T h_c(t)g(t)\cos[F(t)]dt} . \tag{58}$$

Once maximum likelihood estimates of t_i and $\mathcal{M}$ are found by linear filtering we calculate the maximum likelihood estimates of the phase and the amplitude from formulas (40), (45), and (56) with $t_i = \hat{t}_i$ and $\mathcal{M} = \hat{\mathcal{M}}$.

In practice one can perform the correlations in the above functional using the fast Fourier transforms. Then one shall need two banks of only *one*-parameter filters parametrized by chirp mass $\mathcal{M}$. The position of the maximum of the functional $\mathcal{F}$ will determine the time of arrival.

Next we shall derive convenient formulas for the calculation of the covariance matrix.

B. The covariance matrix

Let us choose the amplitude parameter of the chirp in the following way:

$$A = \frac{1}{(30)^{1/2}} \frac{1}{(\pi)^{2/3}} \frac{\mathcal{M}^{5/6}}{R} . \tag{59}$$

Instead of the chirp mass itself it is convenient to choose the parameter

$$k = \frac{1}{\mathcal{M}^{5/6}} . \tag{60}$$

Then from Eq. (30) and the formulas for Fourier transform of the chirp and the optimal filter, the ambiguity function (correlation integral) is given by

$$H(A_1, A_2, \Delta t, \Delta k, \Delta\phi) = 2A_1 A_2 \int_0^\infty f^{-7/3} \frac{\cos[2\pi f \Delta t + a(f)\Delta k - \Delta\phi]}{S_h(f)} df , \tag{61}$$

where

$$a(f) = \frac{1}{2^7} \left[\frac{3}{(\pi f)^{5/3}} + \frac{5\pi f}{(\pi f_i)^{8/3}} - \frac{8}{(\pi f_i)^{5/3}} \right] , \tag{62}$$

$$\Delta t = t_{i2} - t_{i1} , \tag{63}$$

$$\Delta k = k_2 - k_1 , \tag{64}$$

$$\Delta\phi = \phi_2 - \phi_1 \tag{65}$$

and $S_h(f)$ is the spectral density of noise in the interferometer referred to as the dimensionless response $\delta L / L$.

The signal-to-noise ratio d is given by

$$d^2 = H_{A_1 = A_2, \Delta t = 0, \Delta k = 0, \Delta\phi = 0} = 2A^2 \int_0^\infty \frac{1}{f^{7/3}} \frac{df}{S_h(f)} . \tag{66}$$

With our choice of parameters the components of the Fisher information matrix can be computed from the reduced ambiguity function H' defined by $H' := H/(A_1 A_2)$

$$H'(\Delta t, \Delta k, \Delta\phi) = 2\int_0^\infty \frac{1}{f^{7/3}} \frac{\cos[2\pi f \Delta t + a(f)\Delta k - \delta\phi]}{S_h(f)} df . \tag{67}$$

The components of the Fisher information matrix are given by the integrals

$$\Gamma_{AA} = 2\int_0^\infty \frac{1}{f^{7/3}} \frac{df}{S_h(f)} , \tag{68}$$

$$\Gamma_{At} = \Gamma_{Ak} = \Gamma_{A\phi} = 0 , \tag{69}$$

$$\Gamma_{tt} = 2A^2 \int_0^\infty \frac{(2\pi f)^2}{f^{7/3}} \frac{df}{S_h(f)} , \tag{70}$$

$$\Gamma_{tk} = 2A^2 \int_0^\infty \frac{2\pi f a(f)}{f^{7/3}} \frac{df}{S_h(f)} , \tag{71}$$

$$\Gamma_{t\phi} = 2A^2 \int_0^\infty \frac{2\pi f}{f^{7/3}} \frac{df}{S_h(f)} , \tag{72}$$

$$\Gamma_{kk} = 2A^2 \int_0^\infty \frac{a(f)^2}{f^{7/3}} \frac{df}{S_h(f)} , \tag{73}$$

$$\Gamma_{k\phi} = 2A^2 \int_0^\infty \frac{a(f)}{f^{7/3}} \frac{df}{S_h(f)} , \tag{74}$$

$$\Gamma_{\phi\phi} = 2A^2 \int_0^\infty \frac{1}{f^{7/3}} \frac{df}{S_h(f)} . \tag{75}$$

The covariance matrix C is the inverse of the Γ matrix.

We shall show that the time of arrival and the chirp mass components of the covariance matrix can be obtained from a suitable two-dimensional matrix. Let $H^{(2)} := \sqrt{H_c^2 + H_s^2}$ where

$$\begin{aligned} H_c &= E\left[\int_0^T h_c(t) x(t) dt \right] = A_2 \sin(\phi_2) \int_0^T h_c(t; t_{i1}, k_1) C(t; t_{i2}, k_2) dt \\ &= 2A_2 \sin(\phi_2) \int_0^\infty \frac{1}{f^{7/3}} \frac{\cos[2\pi f \Delta t + a(f)\Delta k]}{S_h(f)} , \end{aligned} \tag{76}$$

$$\begin{aligned} H_s &= E\left[\int_0^T h_s(t) x(t) dt \right] \\ &= A_2 \cos(\phi_2) \int_0^T h_s(t; t_{i1}, k_1) S(t; t_{i2}, k_2) dt = 2A_2 \cos(\phi_2) \int_0^\infty \frac{1}{f^{7/3}} \frac{\sin[2\pi f \Delta t + a(f)\Delta k]}{S_h(f)} \end{aligned} \tag{77}$$

is called the two-dimensional ambiguity function.

One easily finds that $H^{(2)}|_{t_1=t_2,k_1=k_2}=d^2$, where d is the signal-to-noise ratio.

Let the two-dimensional Γ matrix be defined by

$$\Gamma^{(2)}_{ij}=\frac{\partial^2}{\partial\theta^1_i\partial\theta^2_j}H^{(2)}\bigg|_{\theta^1_i=\theta^2_i} , \tag{78}$$

where $\theta_1=t_i$ and $\theta_2=k$. Then one finds that

$$\Gamma^{(2)}_{ij}=\left[\frac{\partial^2}{\partial\theta^1_i\partial\theta^2_j}H_c+\frac{1}{d^2}\frac{\partial H_s}{\partial\theta^1_i}\frac{\partial H_c}{\partial\theta^2_j}\right] . \tag{79}$$

Hence we get

$$\Gamma^{(2)}_{ij}=\left[\Gamma_{ij}-\frac{\Gamma_{\phi i}\Gamma_{\phi j}}{\Gamma_{\phi\phi}}\right] , \tag{80}$$

and one can verify that components $C^{(2)}_{ij}$ of the inverse of the two-dimensional Γ matrix are equal to the respective components of the four-dimensional Γ matrix, i.e.,

$$C^{(2)}_{tt}=C_{tt} , \tag{81}$$

$$C^{(2)}_{kk}=C_{kk} , \tag{82}$$

$$C^{(2)}_{tk}=C_{tk} . \tag{83}$$

C. A new set of parameters

Let us introduce a set of new parameters A_c and A_s given by

$$A_c=A\cos\phi , \tag{84}$$

$$A_s=A\sin\phi . \tag{85}$$

We shall show that this set of parameters possesses nice statistical properties.

The logarithmic likelihood ratio is given by

$$\ln\Lambda[x(t);A_c,A_s]=A_c\int_0^T h_s(t)x(t)dt+A_s\int_0^T h_c(t)x(t)dt-\frac{1}{2}A_c^2\int_0^T h_s(t)S(t)dt-\frac{1}{2}A_s^2\int_0^T h_c(t)C(t)dt , \tag{86}$$

where h_c and h_s are solutions of the integral equations (43) and (44), respectively.

The maximum likelihood estimates of A_c and A_s are easily found:

$$\hat{A}_c=\frac{\int_0^T h_s(t)x(t)dt}{\int_0^T h_s(t)S(t)dt} , \tag{87}$$

$$\hat{A}_s=\frac{\int_0^T h_c(t)x(t)dt}{\int_0^T h_c(t)C(t)dt} . \tag{88}$$

One immediately finds that $E[\hat{A}_c]=A_c$ and $E[\hat{A}_s]=A_s$ and therefore the above estimates are unbiased. Using Eqs. (87) and (88) the logarithmic likelihood ratio can be written as

$$\ln\Lambda=\left[A_c\hat{A}_c-\frac{1}{2}A_c^2\right]\int_0^T q_1(t)S(t)dt+\left[A_s\hat{A}_s-\frac{1}{2}A_s^2\right]\int_0^T h_c(t)C(t)dt . \tag{89}$$

Thus the likelihood ratio has the form

$$\Lambda[x(t);A_c,A_s]=g(\hat{A}_c,\hat{A}_s,A_c,A_s)h[x(t)] \tag{90}$$

and hence [18] the above estimates $\hat{A}_c$ and $\hat{A}_s$ are jointly sufficient estimates of A_c and A_s.

From Eqs. (87) and (88) one finds that standard deviations and correlation coefficient of estimates $\hat{A}_c$ and $\hat{A}_s$ are given by

$$\sigma_c=\mathrm{Var}\,\hat{A}_c=\left[\int_0^T h_s(t)S(t)dt\right]^{-1} , \tag{91}$$

$$\sigma_s=\mathrm{Var}\,\hat{A}_s=\left[\int_0^T h_c(t)C(t)dt\right]^{-1} , \tag{92}$$

$$\rho=\mathrm{Cov}\,\hat{A}_c\hat{A}_s=0 . \tag{93}$$

From Eq. (89) we have

$$\frac{\partial}{\partial A_c}\ln\Lambda[x(t);A_c,A_s]=\int_0^T h_s(t)S(t)[\hat{A}_c-A_c] ,$$
$$\frac{\partial}{\partial A_s}\ln\Lambda[x(t);A_c,A_s]=\int_0^T h_c(t)C(t)[\hat{A}_s-A_s] . \tag{94}$$

Using the above equations one can verify that the Cramer-Rao inequality [19] in our case becomes an equality and consequently $\hat{A}_c$ and $\hat{A}_s$ are jointly efficient estimates of A_c and A_s.

The Fisher information matrix Γ with the new variables A_c and A_s instead of A and ϕ is no longer diagonal. One easily finds the relations

$$\Gamma_{A_cA_c}=\Gamma_{AA},\Gamma_{A_ct_i}=\frac{A_c}{A^s}\Gamma_{\phi t_i}\Gamma_{A_s\mathcal{M}}=\frac{A_s}{A^2}\Gamma_{\phi\mathcal{M}} ,$$
$$\Gamma_{A_sA_s}=\Gamma_{AA},\Gamma_{A_st_i}=-\frac{A_s}{A^s}\Gamma_{\phi t_i},\Gamma_{A_s\mathcal{M}}=\frac{A_c}{A^2}\Gamma_{\phi\mathcal{M}} . \tag{95}$$

The remaining components of Γ matrix are the same in both coordinates.

IV. NUMERICAL INVESTIGATION OF THE TWO-DIMENSIONAL AMBIGUITY FUNCTION

In the previous section we have shown that to determine the maximum likelihood estimates of the chirp we have to pass the data through two banks of the two-parameter linear filters and for each set of the parameters form the statistics defined by Eq. (58). The accuracy of the determination of the time of arrival and the chirp mass is determined by the curvature at the maximum of the two-dimensional ambiguity function $H^{(2)}$. In Figs. 1–12 we give both the perspective and the contour plots of this function for all possible static configurations of the

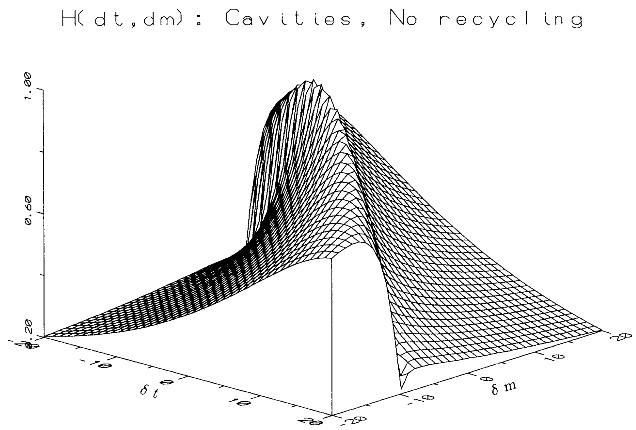

FIG. 1. Delay lines in the arms of the interferometer and no recycling configuration. The two-dimensional ambiguity function $H^{(2)}$: the contour plot.

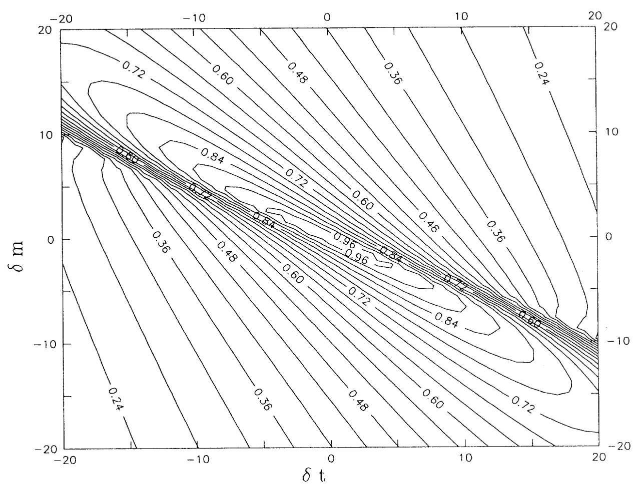

FIG. 2. Delay lines in the arms of the interferometer and no recycling configuration. The two-dimensional ambiguity function $H^{(2)}$: the perspective plot.

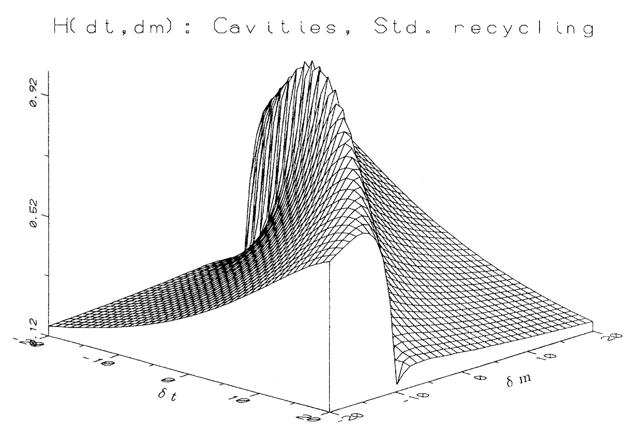

FIG. 3. Fabry-Perot cavities in the arms of the interferometer and no recycling configuration. The two-dimensional ambiguity function $H^{(2)}$: the contour plot.

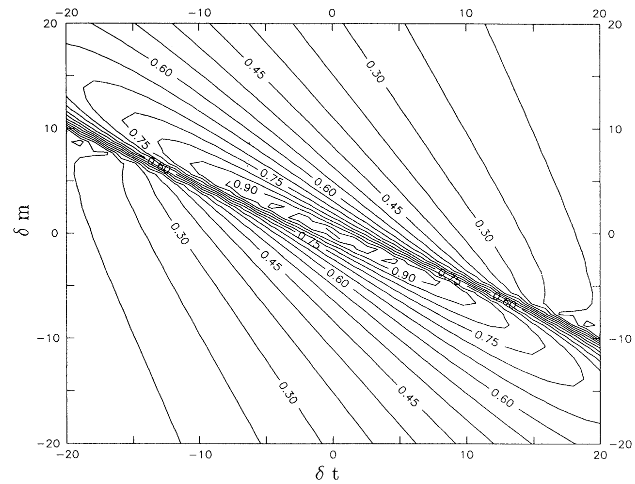

FIG. 4. Fabry-Perot cavities in the arms of the interferometer and no recycling configuration. The two-dimensional ambiguity function $H^{(2)}$: the perspective plot.

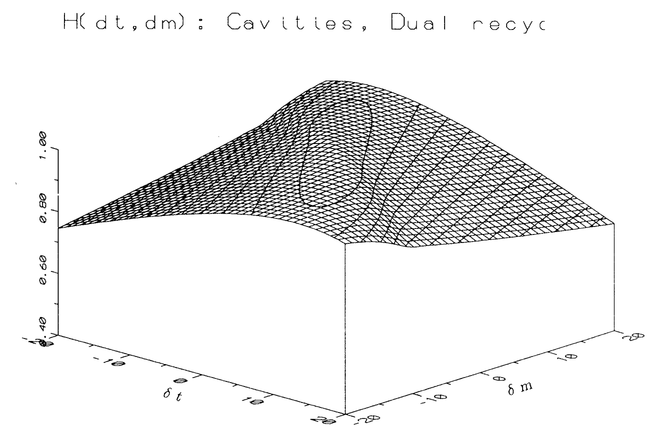

FIG. 5. Delay lines in the arms of the interferometer and the standard recycling configuration. The two-dimensional ambiguity function $H^{(2)}$: the contour plot.

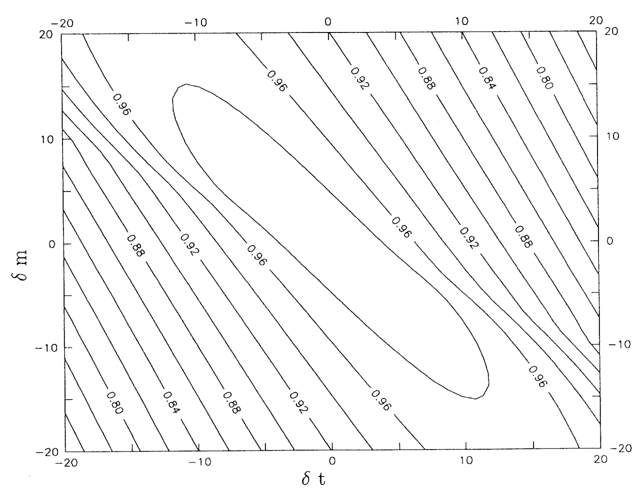

FIG. 6. Delay lines in the arms of the interferometer and the standard recycling configuration. The two-dimensional ambiguity function $H^{(2)}$: the perspective plot.

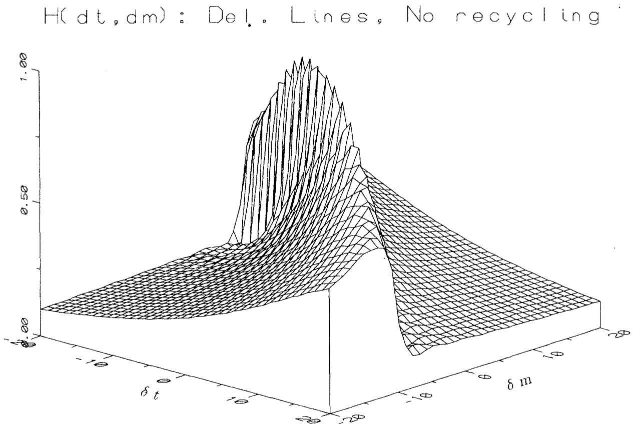

FIG. 7. Fabry-Perot cavities in the arms of the interferometer and the standard recycling configuration. The two-dimensional ambiguity function $H^{(2)}$: the contour plot.

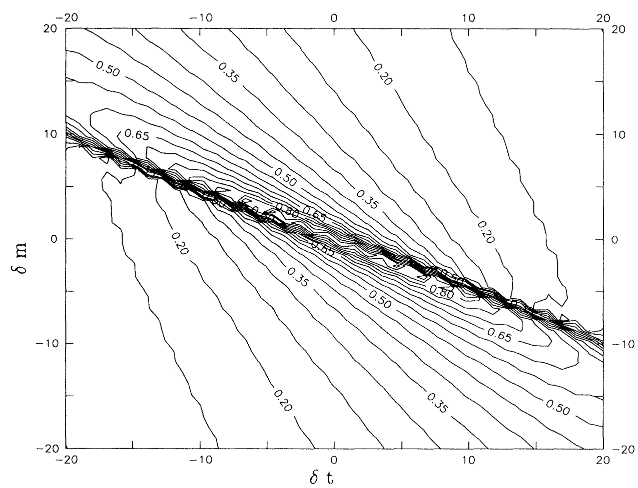

FIG. 8. Fabry-Perot cavities in the arms of the interferometer and the standard recycling configuration. The two-dimensional ambiguity function $H^{(2)}$: the perspective plot.

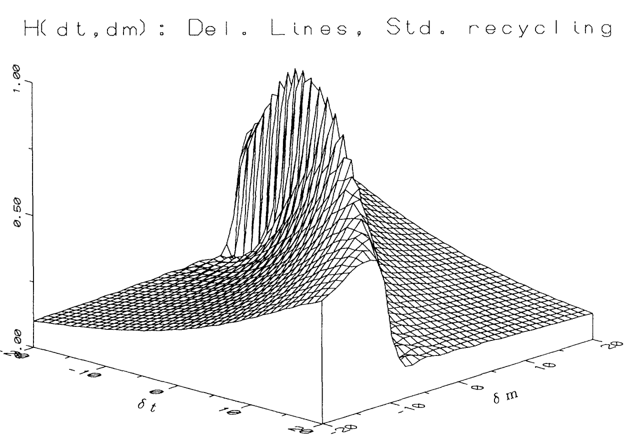

FIG. 9. Delay lines in the arms of the interferometer and the dual recycling configuration. The two-dimensional ambiguity function $H^{(2)}$: the contour plot.

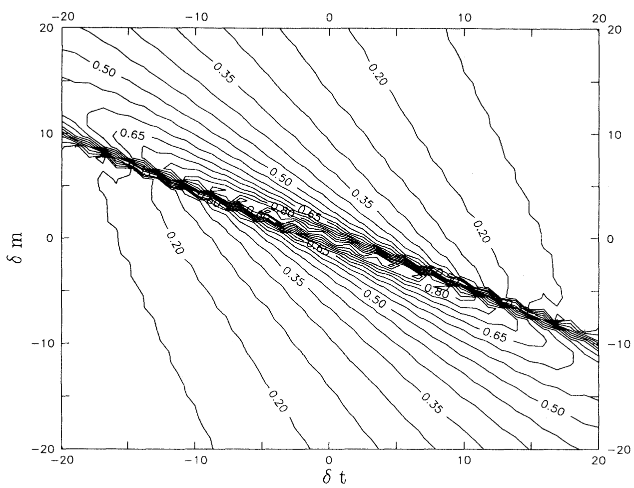

FIG. 10. Delay lines in the arms of the interferometer and the dual recycling configuration. The two-dimensional ambiguity function $H^{(2)}$: the perspective plot.

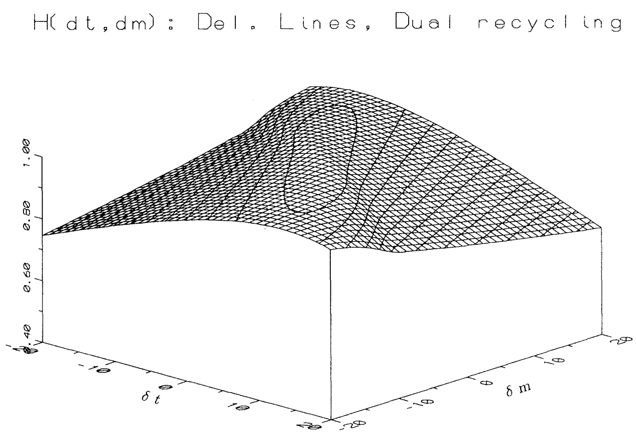

FIG. 11. Fabry-Perot cavities in the arms of the interferometer and the dual recycling configuration. The two-dimensional ambiguity function $H^{(2)}$: the contour plot.

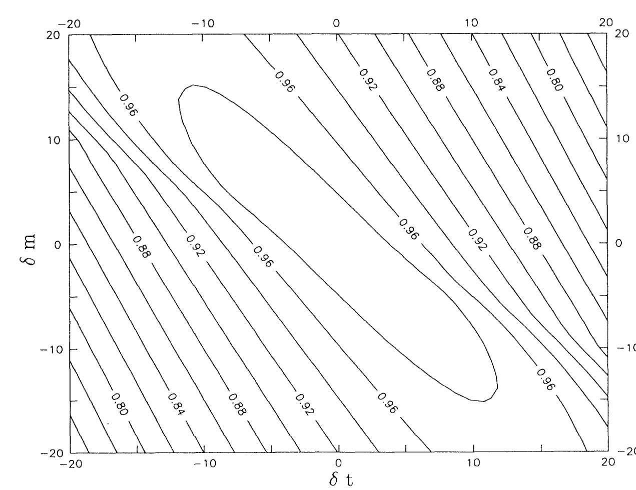

FIG. 12. Fabry-Perot cavities in the arms of the interferometer and the dual recycling configuration. The two-dimensional ambiguity function $H^{(2)}$: the perspective plot.

laser interferometer: the no-recycling configuration, standard recycling configuration, and dual-recycling configuration for both delay lines and Fabry-Perot cavities in the arms of the interferometers.

Some plots are extremely bumpy and ragged looking. This is due to limitation in the computer procedures which have been used to implement the numerics, and so they do not reflect any odd behavior of the plotted functions, which can be shown to be perfectly smooth in the domains shown.

The spectral density of noise and signal-to-noise ratios for all the configurations were investigated by these authors in a previous publication [4]. The contour plots are given for those parameters of the interferometer that were found to maximize the signal-to-noise ratio.

All the plots exhibit a ridge that runs diagonally across the t-$\mathcal{M}$ plane. This is characteristic of the signal with a frequency modulation [20] like the signal from a coalescing binary considered here. The sharper the ridge the better the accuracy of the estimation of the parameters that can be achieved. In the narrow band cases (Figs. 5, 6, 11, and 12) the ambiguity function is flatter than in the broad band cases, consequently the accuracy of the determination of the parameters gets worse. The actual accuracy in the real data analysis will be worse (see discussion at the end of Sec. II).

V. NUMERICAL INVESTIGATION OF THE COVARIANCE MATRIX

In the following six tables we give the numerical values of the components of the covariance matrix for all the configurations of the laser interferometer.

We take into account only the photon-counting noise and assume a seismic cutoff at 100 Hz. The parameters of the laser interferometer like the reflectivities of the mirrors are those that maximize the signal-to-noise ratios [4].

We assume that the gravitational signal comes from a binary at a distance of 100 Mpc. The members of the binary have the same mass equal to 1.4 solar masses, this corresponds to the chirp mass of 1.22 solar masses. The amplitude of the signal is given by Eq. (59).

In the tables we give the corresponding signal-to-noise ratios, the timing error (in milliseconds), the mass parameter error (in milliunits), and the error in phase (in radians). The "milliunit" (mu) is 10^{-3} of solar mass. The rows of the covariance matrix are consecutively the time, mass, and phase components.

TABLE I. Fabry-Perot, no recycling.

Covariance matrix	$\begin{pmatrix} 178 & -99.3 & -129 \\ -99.3 & 57.0 & 71.4 \\ -129 & 71.5 & 94.3 \end{pmatrix}$
Signal-to-noise	1.2
Timing error	13.4 ms
Mass error	7.6 mu
Phase error	9.7 r

TABLE II. Fabry-Perot, standard recycling.

Covariance matrix	$\begin{pmatrix} 3.52 & -1.90 & -2.61 \\ -1.90 & 1.05 & 1.40 \\ 2.61 & 1.40 & 1.95 \end{pmatrix}$
Signal-to-noise	8.0
Timing error	1.9 ms
Mass error	1.0 mu
Phase error	1.4 r

TABLE III. Fabry-Perot, dual recycling.

Covariance matrix	$\begin{pmatrix} 11.6 & -7.64 & -7.81 \\ -7.64 & 5.69 & 5.11 \\ -7.81 & 5.11 & 5.24 \end{pmatrix}$
Signal-to-noise	12.6
Timing error	3.4 ms
Mass error	2.4 mu
Phase error	2.3 r

TABLE IV. Delay lines, no recycling.

Covariance matrix	$\begin{pmatrix} 232 & -130 & -167 \\ -130 & 76.4 & 93.1 \\ -76.4 & 93.1 & 121 \end{pmatrix}$
Signal-to-noise	1.1
Timing error	15.2 ms
Mass error	8.7 mu
Phase error	11.0 r

TABLE V. Delay line, standard recycling.

Covariance matrix	$\begin{pmatrix} 7.81 & -3.95 & -6.03 \\ -3.95 & 2.01 & 3.03 \\ -2.01 & 3.03 & 4.73 \end{pmatrix}$
Signal-to-noise	4.6
Timing error	2.8 ms
Mass error	1.4 mu
Phase error	2.2 r

TABLE VI. Delay lines, dual recycling.

Covariance matrix	$\begin{pmatrix} 143 & -95.6 & -95.7 \\ -95.6 & 73.9 & 63.7 \\ -95.7 & 63.7 & 63.9 \end{pmatrix}$
Signal-to-noise	7.4
Timing error	12.0 ms
Mass error	8.6 mu
Phase error	8.0 r

VI. APPROXIMATE ANALYTICAL FORMULAS FOR THE NARROW BAND CASE

In the previous section the Fisher information matrix and the covariance matrix were calculated numerically for all the configurations of the laser interferometer.

In this section we shall show that for the case of narrow band noise one can derive the approximate analytical formulas for the correlation integral and the components of the covariance matrix.

The correlation integral H has the form [see Eq. (30)].

$$H(\Delta t,\Delta k,\Delta\phi)=H_0\int_{f_i}^{\infty}df f^{-7/3}\frac{\cos[2\pi f\Delta t+a(f)\Delta k+\Delta\phi]}{S_h(f)} . \tag{96}$$

For the narrow band noise, i.e., for the case of sensitivity $1/S_h(f)$ peaked at a frequency f_0 with bandwidth B (see [4], Eq. (3.7), for the case of dual recycling), we have

$$\frac{1}{S_h(f)}\simeq\frac{1}{S_0\{1+4[(f-f_0)/B]^2\}}\simeq S_0^{-1}\exp\left[-\frac{4(f-f_0)^2}{B^2}\right] . \tag{97}$$

We also introduce the approximations

$$f^{-7/3}=\exp(-\tfrac{7}{3}\ln f)=\exp[-\tfrac{7}{3}\ln f_0(1+(f-f_0)/f_0)]\simeq\exp(-\tfrac{7}{3}\ln f_0)\exp\left[-\frac{7}{3}\frac{(f-f_0)^2}{f_0}\right] , \tag{98}$$

$$a(f)\simeq a(f_0)+a'(f_0)(f-f_0)+a''(f_0)\frac{(f-f_0)^2}{2} , \tag{99}$$

$$a'(f_0)=\frac{1}{2^7}\frac{5}{\pi^{5/3}}\left[\frac{1}{f_i^{8/3}}-\frac{1}{f_0^{8/3}}\right] , \tag{100}$$

$$a''(f_0)=\frac{1}{2^7}\frac{40}{3\pi^{5/3}}\frac{1}{f_0^{11/3}} . \tag{101}$$

We define a new variable

$$x=\frac{2(f-f_0)}{B} . \tag{102}$$

The lower bound of the integral is then $x_i=2(f_i-f_0)/B$ and it is typically around -1. However to a good approximation we can set the lower bound to be $-\infty$. Consequently our integral can be written in the form

$$H\simeq I=H_0'\int_{-\infty}^{\infty}\exp[-(x^2+bx)]\cos(px^2+2qx+r)dx , \tag{103}$$

where

$$H_0'=\frac{H_0\exp(-\tfrac{7}{3}\ln f_0)}{S_0} , \tag{104}$$

$$b=\frac{-7}{6}\frac{B}{f_0} , \tag{105}$$

$$p=\frac{a''(f_0)B^2\Delta k}{8} , \tag{106}$$

$$q=\frac{1}{2}\left[a'(f_0)\frac{B}{2}\Delta k+\pi B\Delta t\right] , \tag{107}$$

$$r=a(f_0)\Delta k+2\pi f_0\Delta t+\Delta\phi . \tag{108}$$

The above integration can be performed analytically:

$$I=H_0'\frac{\sqrt{\pi}}{(1+p^2)^{1/4}}\exp\left[\frac{b^2}{1+p^2}\right]\exp\left[-\frac{q(q-2bp)}{1+p^2}\right]\cos\left[\frac{1}{2}\arctan(p+r)-\frac{pq^2-b^2p+2bq}{1+p^2}\right] . \tag{109}$$

When $\Delta t = \Delta k = \Delta\phi = 0$ then H is equal to the signal-to-noise ratio d raised to the power 2. Thus we have

$$d^2 \simeq \sqrt{\pi} H_0' \exp(b^2) \ . \tag{110}$$

One easily finds that the two-dimensional ambiguity function $H^{(2)}$ is given by

$$H^{(2)} = \frac{d^2}{(1+p^2)^{1/4}} \exp\left[-\frac{(q-bp)^2}{1+p^2}\right] . \tag{111}$$

One can then calculate the components of the two-dimensional Γ matrix in a straightforward manner:

$$\Gamma_{tt} = \frac{(\pi B d)^2}{2} , \tag{112}$$

$$\Gamma_{tk} = \frac{\pi (Bd)^2 a'}{4}\left[1 - b\frac{a'' B}{2a'}\right] , \tag{113}$$

$$\Gamma_{kk} = \frac{1}{2}\left[\frac{a'' B^2 d}{8}\right]^2 \left[1 + 4\left[b - \frac{2a'}{Ba''}\right]^2\right] . \tag{114}$$

The covariance matrix components are given by

$$C_{tt} = \frac{1}{d^2(\pi B)^2}\left[1 + 4\left[b - \frac{2a'}{Ba''}\right]^2\right] , \tag{115}$$

$$C_{tk} = -\frac{1}{\pi d^2}\frac{32}{a' B^2}\left[b - \frac{2a'}{Ba''}\right] , \tag{116}$$

$$C_{kk} = \frac{2}{d^2}\left[\frac{8}{a'' B^2}\right]^2 . \tag{117}$$

The above formulas are rather complicated. One useful approximation for the accuracy dt of the determination of the time of arrival can be obtained when resonance frequency f_0 is much larger than the bandwidth. Then to first-order approximation $dt = \sqrt{C_{tt}}$ is given by

$$dt \simeq \frac{1}{dB} . \tag{118}$$

The above formula agrees with what intuitively one would expect for narrow band systems.

VII. CONCLUSIONS

The fact that we only need two filters to estimate the phase was already obtained by other methods by Dhurandhar [21].

An important problem to be solved next is to determine how to space optimally the filters to determine the time of arrival and the chirp mass. It seems that it does not make sense to space them more finely than the accuracy of the determination of the parameters given by diagonal components of the covariance matrix. This problem is actively pursued by Dhurandhar and collaborators [21].

Recently it was found by the Caltech group [22] that the post-Newtonian parameters are detectable with the matched filtering technique. This does not change the theory presented above, however the numerical values of the components of the covariance matrix will change since the post-Newtonian parameters are correlated with the Newtonian ones. Numerical investigation of the post-Newtonian signal is left for a future investigation.

ACKNOWLEDGMENT

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