Soliton solutions and cosmological gravitational waves

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We examine plane-symmetric cosmological solutions to Einstein's equations which can be generated by the "soliton" technique, using the homogeneous Bianchi solutions as seeds and arbitrary numbers of real or complex poles. In some circumstances, these solutions can be interpreted as "incipient" gravitational waves on the Bianchi background. At early times they look like nonlinear inhomogeneities propagating at nearly the speed of light ("gravisolitons"), while at late times they look like cosmological gravitational waves.

I. INTRODUCTION

Attention has recently focused on the possibility that there could exist a background of gravitational waves.[1–4] Such backgrounds would almost certainly have to originate at a cosmological red-shift, so any success in detecting them could yield vital information about the early Universe. Because the waves would generally have a considerably longer period than the ones generated by the sort of burst or continuous sources expected at the present epoch, they could be observable by a variety of new detection techniques: for example, by the Doppler tracking of interplanetary spacecraft,[5–8] by scrutinizing the timing noise in pulsars,[9–12] or by monitoring perturbations to planetary orbits.[13–15]

One way in which a background of gravitational waves could arise would be as a superposition of waves from many individual bursts generated astrophysically at some time in the past.[1,16,17] This could be especially interesting if a population of pregalactic black holes formed.[2] If some fraction of the holes formed in binary systems, there could also be a background of continuous waves generated by the superposition of their quadrupole emission.[18] Another possibility is that the waves may be "primordial," in the sense that they derive from the initial conditions of the Universe. For example, even if the early Universe was of Friedmann type, there could be a thermal background of gravitons with a temperature ~ 1 K originating at the Planck time[4]; quantum effects could also generate longer-wavelength (nonthermal) gravitons provided the Friedmann equation of state deviated from $p=\rho/3$ at early times.[19] Unfortunately, these sorts of primordial backgrounds would probably be undetectable.

A more interesting possibility, and the one relevant to this paper, is that the primordial waves may reflect a purely classical irregularity in the initial structure of the Universe. In this case, the waves would not look like radiation at sufficiently early times because they would have a wavelength larger than the Universe's particle horizon. Also, their dimensionless amplitude, instead of being small, would be of order unity and thus severely distort the background spacetime.[2] Thus, classical primordial waves could arise only if the early Universe deviated considerably from the smooth structure observed today. They might have a variety of forms, depending on the type of initial irregularity: if the Universe started off completely chaotic,[20] one might expect an isotropic stochastic background with a wide range of wavelengths.[2,21,22] On the other hand, a less extreme form of irregularity might generate a more correlated anisotropic background, for example, one with plane waves all traveling in the same direction.[23]

There is no doubt that solutions to Einstein's equations which contain incipient gravitational waves in this sense ought to exist. Although the complexity of the equations makes their identification nontrivial, the study of inhomogeneous and anisotropic cosmologies has seen considerable progress during the last few years.[24,25] In particular, Adams *et al.*[26] have studied solutions which describe gravitational waves in Bianchi backgrounds. By confining attention to plane-wave solutions, which break the homogeneity of the Bianchi model only in the direction of wave propagation (z), they manage to find exact solutions which do indeed exhibit the sort of properties anticipated. At early times the Universe is highly inhomogeneous, with the anisotropy parameter depending on z, while at late times one gets waves traveling in the z direction. They thus show explicitly how chaotic behavior near the initial singularity can be transformed into gravitational waves.

In this paper we will apply the inverse scattering or "soliton" technique, developed by Belinskii and Zakharov[27] and reviewed in Sec. II, to the study of inhomogeneous cosmologies. The soliton technique provides a procedure to solve the Einstein equations in vacuum when there are two commuting Killing vectors (in our application, two spacelike Killing vectors). This limitation is not too restrictive; in particular, it includes Bianchi types I to VII and their inhomogeneous generalizations.[25] The two main ingredients of the soliton technique are the so-called "pole trajectories" and particular solutions of the Einstein equations which serve as "seeds." By analyzing solutions generated by the homogeneous Kasner (Bianchi I) seed, we will show that it is possible to construct cosmological models which look like gravitational waves at late times (in that they evolve towards homogeneity in a wavelike manner) but which behave like very inhomogeneous

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"gravisolitons" at early times. Since these gravisolitons propagate at around the speed of light and carry energy in some sense,[28] they are naturally interpreted as incipient gravitational waves. In this respect, they resemble the Adams *et al.* solutions even though the latter are not themselves solitons. (They contain many more parameters and modes and are thus, in some sense, more general than our solutions; however, they do not possess the structural persistence associated with soliton solutions.) Particular examples of cosmological soliton solutions are already known; we will here present a more comprehensive analysis of their properties.

Soliton solutions can be characterized according to the number n of real and complex poles they contain. If $n=1$, one must have a single real pole. (If a pole is complex, its complex conjugate is also a pole, so complex poles always go in pairs.) Belinskii and Zakharov[27] have studied this case; it remains the seed solution in the region $z^2<t^2$ and contains the perturbation only in the region $z^2>t^2$. Because the first derivative of the metric is discontinuous on the light cone, the solution can be interpreted as a pair of outward-propagating gravitational shock waves,[24,25] with the region behind the shocks being left unperturbed. By adding more real poles, one generates a more complicated solution with a sequence of propagating discontinuities. This is discussed in Sec. III; the form of the solution is initially highly structured, but one gets the exact seed solution at each point once all the shocks have transversed it.

A somewhat different situation arises for solutions with complex poles, which we discuss in Sec. IV. These also have the feature that they evolve to the seed solution at each point at late times, but they are only asymptotically (not exactly) seedlike and they are continuous on the light cones. Belinskii and Zakharov[27] have studied the solutions with a single complex pole ($n=2$), which can be generated by the axisymmetric Kasner seed, while Ibañez and Verdaguer[28] have studied solutions generated by the same seed with *two* complex poles ($n=4$). In these cases, the metric tends to the seed solution in both the "far region" ($z^2>t^2$, $z\to\infty$) and the "causal region" ($z^2<t^2$, $t\to\infty$). The maximum perturbation also appears near the light cones and it is this feature which leads us to interpret them as gravitational waves (viz., inhomogeneities propagating at the speed of light). We extend these analyses to the more general case with an arbitrary number of complex poles.

The soliton solutions can only be given explicitly if one makes the mathematically simplifying assumption that the metric is diagonal ($g_{xy}=0$). This corresponds to waves with a single polarization. We specialize to this situation in Sec. V. We find that the perturbations always decay like $t^{-1/2}$ in the "light-cone region" ($z^2\sim t^2$, $t\to\infty$) and like t^{-1} in the causal region, but that the solutions tend to the seed metric in the far region only with special types of seeds. In other cases, the solutions may be asymptotically flat or static in the far region or they may even have a timelike singularity at $z=\infty$; these solutions are of interest in their own right, but they cannot represent gravitational waves on a Kasner background.

In Sec. VI we consider the more general nondiagonal solutions; these can be interpreted as having two polarizations. Although one no longer has explicit analytic solutions, one can still derive asymptotic expansions. In the two-soliton case, Belinskii and Fargion[29] have already found nondiagonal solutions in which the background is contracting in the z direction. We extend their analysis to the more general Kasner background. We find that the two-soliton solutions still tend to the Kasner seed in the causal and light-cone regions (with the perturbations again falling off like t^{-1} and $t^{-1/2}$, respectively), but they do so in the far region only if the z axis is contracting. If it is expanding sufficiently fast, the metric tends to diagonality in the far region but not to the seed metric. We find that these same features pertain in the general n-soliton nondiagonal solution.

We stress that we have chosen to emphasize soliton solutions in this paper because of their mathematical tractability. In general, however, the initial cosmological structure associated with primordial gravitational waves would be more complicated. One would not expect the homogeneity to be broken in just one direction and many other solutions could be envisaged which were not representable as solitons (cf. Adams *et al.*[26]). Nevertheless, the persistence of solitons suggest that they might be singled out as modeling the most likely form of irregularity in the early Universe and we would expect many of the features found in this paper to be valid in the more general case. In this context, it should be noted that the problem of gravitational waves in inhomogeneous cosmological models can be investigated using numerical techniques[30–34] and approximation methods,[35–38] as well as by studying exact solutions.[39–45] It seems likely that progress will be made by combining all of these different approaches. Finally, we emphasize that the solutions derived here may also be of interest in a different context if one transforms the coordinates (z,t) to (ρ,t) or (ρ,θ). We study these (noncosmological) solutions in a separate paper.[46]

II. SOLITON SOLUTIONS

We first summarize the Belinskii-Zakharov[27,47] technique for generating solutions of the Einstein equations in vacuum. These solutions have the form

$$ds^2=f(dz^2-dt^2)+g_{ab}dx^a dx^b \quad (a,b=1,2)\ , \tag{2.1}$$

where the metric coefficient f and the two-dimensional matrix $\underline{g}$ are functions of t and z alone. This metric includes the homogeneous Bianchi models of type I to VII, all of which have two commuting Killing vectors, as well as their generalizations to models with the homogeneity broken in the z direction (cf. Adams *et al.*[26]).

Because of their cosmological relevance, we shall restrict our attention to metrics for which

$$\det\underline{g}=t^2\ . \tag{2.2}$$

This does not, in fact, involve a loss of generality because we can take any other value for $\det\underline{g}$, provided its square root is a solution of the wave equation in t and z (as required by the Einstein equations), by performing a coordinate transformation

$$z'=a(z+t)+b(z-t)\ ,$$
$$t'=a(z+t)-b(z-t)\ , \tag{2.3}$$

with arbitrary functions a and b. This leaves the metric (2.1) invariant, while changing the coefficient f according to

$$f(dz^2-dt^2)=\frac{f(dz'^2-dt'^2)}{r(a_{,z}+a_{,t})(b_{,z}-b_{,t})}\ .$$

Thus, for a given det$\underline{g}$, all the results that follow can be generalized by performing a transformation of the form (2.3).

The Einstein equations in vacuum take the form

$$\underline{U}_{,t}-\underline{V}_{,z}=0\ , \tag{2.4}$$

$$(\ln f)_{,t}=-\frac{1}{t}+\frac{1}{4t}\mathrm{Tr}\,(\underline{U}^2+\underline{V}^2)\ ,$$
$$(\ln f)_{,z}=\frac{1}{2t}\mathrm{Tr}\,(\underline{U}\,\underline{V})\ , \tag{2.5}$$

where

$$\underline{U}\equiv t\,\underline{g}_{,t}\underline{g}^{-1},\quad \underline{V}\equiv t\,\underline{g}_{,z}\underline{g}^{-1}\ . \tag{2.6}$$

These equations, and many of the results of this section, can be obtained from the Belinskii-Zakharov[47] equations for an axisymmetric metric by changing their cylindrical coordinate ρ into it. The Belinskii-Zakharov technique is based on the inverse scattering transform and provides a procedure for generating (soliton) solutions to the nonlinear equation (2.4) for the matrix $\underline{g}$ when a particular seed solution $\underline{g}_0$ is known. For each solution $\underline{g}$, Eqs. (2.5) can be integrated explicitly. In this technique, one associates a linear "eigenvalue" problem with Eq. (2.4):

$$\left[\partial_z-\frac{2\lambda^2}{\lambda^2-t^2}\partial_\lambda\right]\underline{\psi}=-\frac{(t\underline{V}+\lambda\underline{U})}{\lambda^2-t^2}\underline{\psi}\ ,$$
$$\left[\partial_t-\frac{2\lambda t}{\lambda^2-t^2}\partial_\lambda\right]\underline{\psi}=-\frac{(t\underline{U}+\lambda\underline{V})}{\lambda^2-t^2}\underline{\psi}\ , \tag{2.7}$$

where λ is a complex "spectral" parameter and $\underline{\psi}(\lambda,t,z)$ is a two-dimensional matrix that satisfies

$$\underline{g}(t,z)=\underline{\psi}(0,t,z)\ . \tag{2.8}$$

Given a particular solution $\underline{g}_0$ of Eq. (2.4), Eqs. (2.7) must be integrated to find the corresponding solution $\underline{\psi}_0(\lambda,t,z)$. This integration can be done easily for diagonal metrics, as shown by Jantzen,[48] and even for some nondiagonal metrics, like Bianchi type II, as shown by Belinskii and Francaviglia.[49] Once $\underline{\psi}_0$ has been found, a solution $\underline{\psi}$ can be generated by purely algebraic operations if one assumes that $\underline{\psi}$ is the product of a two-dimensional matrix, with n simple (nondegenerate) poles in the complex λ plane, and ψ_0. Equation (2.8) shows that an n-soliton solution for $\underline{g}(t,z)$ can then be found.

The explicit procedure is as follows.

(1) One starts by choosing the number n and by specifying whether the "pole trajectories," defined by

$$\mu_k=u_k-z\pm[(u_k-z)^2-t^2]^{1/2}\quad (k=1,\ldots,n)\ , \tag{2.9a}$$

are real or complex. Here the μ_k are solutions of the equations

$$\mu_{k,z}=\frac{2\mu_k{}^2}{t^2-\mu_k{}^2},\quad \mu_{k,t}=\frac{2t\mu_k}{t^2-\mu_k{}^2}\ , \tag{2.9b}$$

and the u_k are arbitrary (real or complex) constants. From the $\underline{\psi}_0$ matrix associated with a given seed metric $\underline{g}_0$, one then constructs the vectors

$$m_a^{(k)}=(m_0)_c^{(k)}[\psi_0{}^{-1}(\mu_k,t,z)]_{ca}\ , \tag{2.10a}$$

where $(m_0)_c^{(k)}$ are arbitrary real or complex parameters. One should note that, if one starts with real-pole trajectories, then the parameters $(m_0)_c^{(k)}$ also have to be real. On the other hand, if one starts with a complex trajectory μ_k, its complex conjugate is also a trajectory. Thus, complex trajectories always go in pairs, and we can put $\mu_{k+n/2}=\bar{\mu}_k$. The complex parameters $(m_0)_c^{(k)}$ will then satisfy $(m_0)_c^{(k+n/2)}=(\bar{m}_0)_c^{(k)}$.

(2) The next step is the construction of the $n\times n$ matrix

$$\Gamma_{kl}=\frac{m_c^{(k)}(g_0)_{cb}m_b^{(l)}}{\mu_k\mu_l-t^2} \tag{2.10b}$$

and its inverse, $D_{kl}=(\Gamma_{kl})^{-1}$, from which one can derive a matrix $\underline{g}'$:

$$g'_{ab}=(g_0)_{ab}-\sum_{k,l}^{n}\frac{D_{kl}m_c^{(k)}(g_0)_{ca}m_d^{(l)}(g_0)_{db}}{\mu_k\mu_l}\ . \tag{2.11}$$

This is a solution of Eq. (2.4) and, while it does not satisfy condition (2.2), the matrix

$$\underline{g}=t^{-n}\left[\prod_{k=1}^{n}\mu_k\right]\underline{g}' \tag{2.12a}$$

does and so is the required n-soliton solution of the Einstein equations.

(3) After the solution for $\underline{g}$ has been found, we can calculate the metric coefficient f, defined by Eqs. (2.5). It is a remarkable fact that these equations can be integrated explicitly for the n-soliton solution (2.12a). This has been proved by Belinskii and Zakharov[47] for axisymmetric metrics, but it is also true in the cosmological context. The proof is inductive and uses the fact that the n-soliton solution can be obtained step by step, starting with the one-soliton solution. The final result is

$$f=f_0t^{-n^2/2}\frac{\left[\prod_{k=1}^{n}\mu_k\right]^{n+1}}{\left[\prod_{\substack{k,l=1\\k>l}}^{n}(\mu_k-\mu_l)^2\right]}\det\Gamma_{kl}\ , \tag{2.12b}$$

where the term in square brackets is 1 for $n=1$ and f_0 is the value of f for the seed metric.

Although the solutions given by (2.12) have been obtained for nondegenerate poles, solutions with multiple poles can be obtained from them by a limiting procedure in which $u_k\to u_l$. Thus, in the axisymmetric version of metric (2.1), taking the Minkowski metric as seed and using $2n$ real poles, the n-Kerr metric is generated[47]; by approaching the same pole trajectory (giving n double poles), the Tomimatsu-Sato ($\delta=n$) solution[50] is generated.[51,52] We shall discuss some cosmological examples of double-pole solutions later.

In the cosmological context, the technique has been applied to generate one- and two-soliton solutions from Bianchi type-I (Ref. 27) and Bianchi type-II (Ref. 49) seed metrics. These solutions are inhomogeneous in the z direction. For simplicity, and also to allow comparison with other studies (like those of Adams *et al.*,[26] which find inhomogeneous solutions with many more parameters by breaking the homogeneity of the Kasner metric in the z direction), we shall here study the n-soliton solutions which can be obtained using the Kasner metric as seed. The Kasner metric in the form (2.1) is

$$ds^2=t^{(\Delta^2-1)/2}(dz^2-dt^2)+t^{1+\Delta}dx^2+t^{1-\Delta}dy^2\ . \quad (2.13)$$

This is related to the standard Kasner form,

$$ds^2=-dt'^2+t'^{2p_1}dx^2+t'^{2p_2}dy^2+t'^{2p_3}dz^2$$
$$(p_1+p_2+p_3=p_1{}^2+p_2{}^2+p_3{}^2=1)\ ,$$

by a time transformation and

$$p_1=\frac{2(1+\Delta)}{\Delta^2+3},\quad p_2=\frac{2(1-\Delta)}{\Delta^2+3},\quad p_3=\frac{\Delta^2-1}{\Delta^2+3}\ .$$

The real parameter Δ is arbitrary, but we shall assume that we always have $\Delta\geq 0$ or $\Delta\leq 0$ since one can obtain one from the other by interchanging x and y. $\Delta=0$ corresponds to the axisymmetric Kasner solution, while $|\Delta|=1$ corresponds to Minkowski space; the z axis is expanding for $|\Delta|>1$ and contracting for $|\Delta|<1$.

III. REAL-POLE TRAJECTORIES

The simplest case is the one-soliton solution. The pole trajectory (2.9) with $u_1\equiv z_1^0$ (real) will be

$$\mu_1=z_1^0-z\pm[(z_1^0-z)^2-t^2]^{1/2}\ , \quad (3.1)$$

with either sign being allowed. We shall call z_1^0 the "origin" of the soliton. Given a seed metric, Eqs. (2.12) generate a solution only in the region $(z_1^0-z)^2\geq t^2$, where the pole is real. In $(z_1^0-z)^2<t^2$, the solution remains the seed metric.[27] The spacetime is thus divided into two regions: inside the light cone $(z_1^0-z)^2=t^2$, the solution is unperturbed; outside the light cone, one has an inhomogeneous one-soliton solution depending on just two parameters. The overall metric is continuous, but it has discontinuities in its first derivatives on the light cone itself. Plane-symmetric cosmological metrics with discontinuous first derivatives have been discussed by Wainwright[24] and Carmeli *et al.*[25] In the soliton context, Belinskii and Francaviglia[49] describe these solutions as gravitational shock waves.

The one-soliton solution generated from the Kasner seed was given by Belinskii and Zakharov.[27] The axisymmetric version of their metric was studied by Verdaguer[52]; in the latter context there are no discontinuities since the pole is

$$\mu_1=z_1^0-z\pm[(z_1^0-z)^2+\rho^2]^{1/2}\ ,$$

and therefore real everywhere. We can now imagine the following sequence of solutions. The two-soliton solution generated from the one-soliton solution will contain two light cones: $(z_2^0-z)^2=t^2$ and $(z_1^0-z)^2=t^2$. In the region contained in *both* light cones, the solution will still be the seed metric. Inside each light cone, excluding the intersection region, it will be of the "one-soliton" form; in the remaining regions, it will be of the "two-soliton" form and be described by four parameters. The overall metric will have discontinuous first derivatives along both light cones. From this one can generate the three-soliton solution and so on.

The common feature of these metrics, for an observer at finite z, is that they start very complicated but evolve towards the seed metric as the inhomogeneities propagate outwards at the speed of light. If the soliton origins $z=(z_1^0,z_2^0,\ldots,z_n^0)$ are equally spaced, the inhomogeneity will be reduced in steps at regular intervals. As mentioned in Sec. II, it is also possible that one could have multiple poles. Note, however, that double or even-multiple poles can be obtained as the limiting case in which two or more complex-conjugate poles become real. We shall study an example of this when dealing with complex-pole trajectories in the next section.

IV. COMPLEX-POLE TRAJECTORIES

Complex poles differ from real poles in that they do not have discontinuous derivatives. The parameters u_k in Eq. (2.9) are now complex,

$$u_k\equiv z_k^0-iw_k\ , \quad (4.1)$$

and the complex-conjugate poles may be written as

$$\mu_k=\sqrt{\sigma_k}\,te^{i\phi_k},\quad \mu_{k+n/2}=\bar{\mu}_k\quad (k=1,2,\ldots,n/2)\ . \quad (4.2)$$

The functions $\sigma_k(z,t)$ and $\phi_k(z,t)$ are given by

$$\cos\phi_k=\frac{2z_k\sqrt{\sigma_k}}{t(1+\sigma_k)},\quad \sin\phi_k=\frac{2w_k\sqrt{\sigma_k}}{t(1-\sigma_k)}\quad (z_k\equiv z_k^0-z)\ . \quad (4.3)$$

Two explicit solutions for $\sigma_k(z,t)$ can be found from these equations by solving a fourth-degree equation. This gives two real solutions, $\sigma_k^{(-)}$ and $\sigma_k^{(+)}=1/\sigma_k^{(-)}$, where

$$\sigma_k^{(-)}=\frac{w_k{}^2+z_k{}^2}{t^2}+a_k-\sqrt{2}\left[\frac{w_k{}^2-z_k{}^2}{t^2}+\frac{(w_k{}^2+z_k{}^2)^2}{t^4}+\frac{(w_k{}^2+z_k{}^2)}{t^2}a_k\right]^{1/2},$$
$$a_k=\left[1+\frac{2(w_k{}^2-z_k{}^2)}{t^2}+\frac{(w_k{}^2+z_k{}^2)^2}{t^4}\right]^{1/2}\ . \quad (4.4)$$

These satisfy $0<\sigma_k^{(-)}<1$ and $1<\sigma_k^{(+)}<\infty$.

It is convenient to write down the main asymptotic values for $\sigma_k^{(-)}$. We shall distinquish four asymptotic regions: (i) the causal region ($|z_k|\ll t\to\infty$ for all k), which is contained in the intersection of the set of light cones with origins at $z=(z_1^0,z_2^0,\ldots,z_{n/2}^0)$, (ii) the light-cone region ($|z_k|\sim|z|=t\to\infty$), (iii) the far region ($t\ll|z_k|\sim|z|\to\infty$), and (iv) the "initial region" ($|z_k|,|w_k|\gg t\to 0$). The respective limiting values of $\sigma_k^{(-)}$ are

$$\begin{aligned}
\sigma_k^{(-)}&=1-\frac{2w_k}{t}+O(t^{-2})\ \ \text{(causal region)}\ ,\\
\sigma_k^{(-)}&=1-2\left[\frac{[w_k{}^2+(z_k^0)^2]^{1/2}-z_k^0}{t}\right]^{1/2}\\
&\quad+O(t^{-1})\ \ \text{(light-cone region)}\ ,\\
\sigma_k^{(-)}&=\frac{t^2}{4z_k{}^2}[1+O(z^{-2})]\ \ \text{(far region)}\ ,\\
\sigma_k^{(-)}&=\frac{t^2}{4(w_k{}^2+z_k{}^2)}[1+O(t^2)]\ \ \text{(initial region)}\ .
\end{aligned}\tag{4.5}$$

Belinskii and Zakharov[27] have shown that one recovers the seed solution in the limit $\sigma_k\to 1$. Inspection of Eqs. (4.5) shows that this is always the case in the causal and light-cone regions, the perturbations decaying as t^{-1} and $t^{-1/2}$, respectively. Therefore, for an observer at finite z, the n-soliton solution with complex poles will always evolve towards the seed metric. The situation differs from the real-pole case, however, in that one never gets the *exact* seed solution. Nevertheless, the asymptotic property of the cosmological soliton solution can guide one in choosing an appropriate seed metric. The behavior of the soliton solutions in the far region will generally be quite different from the seed metric. However, we will later see particular examples of seeds for which the far region also tends to the seed metric asymptotically. In such cases, the soliton solution can be regarded as a "perturbation" on the seed background.

In Fig. 1, the time evolution of $\sigma_k^{(-)}(t,z)$ is represented for $w_k=0.2$. The slope of the curves between small and large values of z is governed by the parameter w_k; the smaller w_k, the steeper the slope. Since the solitons can be associated with the z derivatives of σ_k, the parameter w_k reflects the "width" of the solitons. All the metric-dependent quantities can be obtained from the pole equations (2.9b). In terms of σ_k, these become

$$\begin{aligned}
\sigma_{k,z}&=\frac{8z_k\sigma_k{}^2(1-\sigma_k)}{H_k(1+\sigma_k)t^2}\ ,\\
\sigma_{k,t}&=\frac{2\sigma_k(1-\sigma_k{}^2)}{H_kt}\ ,\\
H_k&\equiv(1-\sigma_k)^2+\frac{16w_k{}^2\sigma_k{}^2}{(1-\sigma_k)^2t^2}\ .
\end{aligned}\tag{4.6}$$

These equations give a "recursion" relation, enabling one to find, for instance, the Riemann tensor for the n-soliton solution with complex poles. It is easy to see, using Eqs. (4.5), that $\sigma_{k,z}$ has a maximum in the light-cone region, indicating that the corresponding soliton solution contains inhomogeneities propagating at the speed of light as $t\to\infty$.

In the limit $w_k\to 0$, the two complex-conjugate poles become a double real pole, with $\sigma_k=1$ inside the light cone $z_k{}^2=t^2$ and $\sigma_k=\mu_k{}^2/t^2$ outside it. Since $\sigma_{k,z}$ has a δ-function discontinuity on the light cone, this suggests a gravitational shock wave propagating at the speed of light (cf. the one-soliton case). Therefore, double real-pole solutions can be studied as limiting cases of complex-pole solutions.

As mentioned earlier, we shall restrict attention to the soliton solutions generated from homogeneous Kasner seeds. It is clear that the key point will be the behavior of the metric in the far region. Providing the metric approaches the seed metric in that region, the solution can be interpreted as consisting of inhomogeneities moving on the corresponding Kasner background. If the metric in the far region is different from the seed metric, the propagating solitons are simply connecting two regions of spacetime with different features.

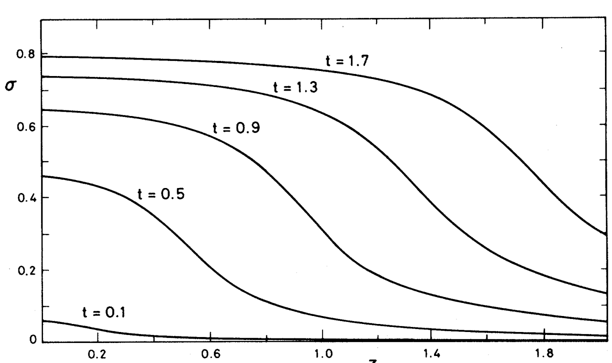

FIG. 1. This shows the function $\sigma^{(-)}(z,t)$ for different values of t, as defined by Eq. (4.4). The origin is $z=0$ and the width parameter is $w=0.2$. The function is unchanged if one reverses the sign of z. When σ approaches 1, the corresponding soliton solution approaches the seed metric.

V. DIAGONAL METRICS

We shall first study the diagonal (one-polarization) n-soliton metrics. There are two reasons for this: (1) they are simpler than the nondiagonal ones—in particular, asymptotic calculations of the Riemann tensor can be carried out explicitly; and (2) they can be used as a paradigm to understand the general nondiagonal (two-polarization) solutions.

Equation (2.11) implies that diagonal metrics can be obtained from a diagonal seed (such as the Kasner solution) by taking one of the arbitrary constants $(m_0)_c^{(k)}$ in Eq. (2.10a) to be zero. We therefore assume

$$(m_0)_1^{(k)}=0\ .\tag{5.1}$$

The general expression for the metric coefficients g_{11} and g_{22} can be obtained for n-solitons by adding solitons one at a time. The result is

$$\begin{aligned}
g_{11}&=t^{-n}\left[\prod_{k=1}^{n}\mu_k\right](g_0)_{11}\ ,\\
g_{22}&=t^2/g_{11}\ .
\end{aligned}\tag{5.2a}$$

When $\underline{g}_0$ depends on t only, the coefficient f is best found by integrating Eqs. (2.5) directly, rather than using Eq. (2.12b). For the Kasner seed we get

$$f=f_0 t^{n(n-2\Delta)/2}\left[\prod_{k=1}^{n}\mu_k\right]^{(2+\Delta-n)}\left[\frac{\prod_{\substack{k,l=1\\k>l}}^{n}(\mu_k-\mu_l)^2}{\prod_{k=1}^{n}(\mu_k{}^2-t^2)}\right]. \tag{5.2b}$$

When $\Delta=1$ (corresponding to the Minkowski seed), this expression can be interpreted as the cosmological version of the axisymmetric static Belinskii-Zakharov[47] solutions.

The solutions (5.2) are valid for real and/or complex poles. If one only has complex poles, they can be expressed explicitly in terms of $\sigma_k(z,t)$. With the Kasner seed we obtain

$$\underline{g}=\begin{bmatrix} t^{1+\Delta}\prod_{k=1}^{n/2}\sigma_k & 0\\ 0 & t^{1-\Delta}\prod_{k=1}^{n/2}\sigma_k{}^{-1}\end{bmatrix}, \tag{5.3a}$$

and

$$f=\frac{t^{(\Delta^2-1-n^2)/2}}{\prod_{k=1}^{n/2}H_k}\prod_{k=1}^{n/2}\left[\frac{\sigma_k^{(\Delta-n+4)}}{(1-\sigma_k)^2}\right]\prod_{\substack{k,l=1\\k>l}}^{n/2}\left\{\left[(\sigma_k+\sigma_l)t^2-\frac{8z_kz_l\sigma_k\sigma_l}{(1+\sigma_k)(1+\sigma_l)}\right]^2-\frac{64w_k{}^2w_l{}^2\sigma_k{}^2\sigma_l{}^2}{(1-\sigma_k)^2(1-\sigma_l)^2}\right\}^2. \tag{5.3b}$$

As discussed earlier, the metric in the causal region represents a Kasner background on which is superposed some inhomogeneities with "amplitude" decreasing as t^{-1}. On the other hand, the amplitude decreases in the light-cone region as $t^{-1/2}$, which is typical of linear gravitational waves on a cosmological background. It remains to study the behavior of the metric in the far region; this is the only asymptotic region in which it may deviate appreciably from the seed solution.

We shall distinguish several possibilities: (a) n solitons with $\sigma_k^{(-)}$; (b) n solitons with $\sigma_k^{(+)}$; and (c) r solitons with $\sigma_k^{(-)}$ and $n-r$ solitons with $\sigma_k^{(+)}$. Since $\sigma_k^{(+)}=1/\sigma_k^{(-)}$, case (b) reduces to case (a) if one interchanges x and y in the seed and then interchanges them again in the soliton solution. Similarly, in case (c), we only need consider $r\geq n-r$; the asymptotic structure in the far region will then be similar to case (a) with $2r-n$ solitons, although the structure in the regions $z_k\sim t$ will be very different since we still have n solitons there. From the point of view of solitons propagating in a homogeneous background, the most interesting case is (c) with $r=n/2$, since in the far region we then have

$$\underline{g}\to\underline{g}_0[1+O(z^{-1})]\ .$$

(If the number of $\sigma_k^{(-)}$ and $\sigma_k^{(+)}$ solitons are different, either $g_{11}\to\infty$ or $g_{22}\to\infty$ as $z\to\infty$.) These models can therefore be interpreted as $n/2$ pairs of solitons with origins at $z=(z_1^0,z_2^0,\ldots,z_{n/2}^0)$ propagating on a Kasner background.[28] As $t\to\infty$, they propagate near the two branches of the light cones $z_k{}^2=t^2$, evolving towards gravitational waves.

More understanding of the metrics can be obtained by examining the Riemann tensor. If the metric is diagonal, this has a very simple form. It can be written as the 6×6 matrix

$$R^{(\alpha\beta)}{}_{(\gamma\delta)}=\begin{bmatrix}\underline{E} & \underline{B}\\ -\underline{B} & \underline{E}\end{bmatrix}, \tag{5.4a}$$

where the indices are (01), (02), (03), (23), (31), and (12). $\underline{E}$ and $\underline{B}$ are 3×3 matrices whose nonzero components are

$$E_{11}=e_1=-(e_2+e_3),\quad E_{22}=e_2\ ,$$

$$E_{33}=e_3,\quad B_{12}=B_{21}=b\ ,$$

with

$$e_2=\frac{1}{2f}\left[\frac{g_{11}''}{g_{11}}-\frac{1}{2}\left[\frac{g_{11}'}{g_{11}}\right]^2-\frac{1}{2}\frac{\dot g_{11}}{g_{11}}\frac{\dot f}{f}-\frac{1}{2}\frac{g_{11}'}{g_{11}}\frac{f'}{f}\right],$$

$$e_3=\frac{1}{2f}\left[\frac{1}{2}\frac{g_{11}'}{g_{11}}\frac{g_{22}'}{g_{22}}-\frac{1}{2}\frac{\dot g_{11}}{g_{11}}\frac{\dot g_{22}}{g_{22}}\right], \tag{5.4b}$$

$$b=\frac{1}{2f}\left[\frac{\dot g_{11}'}{g_{11}}-\frac{1}{2}\frac{\dot g_{11}}{g_{11}}\frac{g_{11}'}{g_{11}}-\frac{1}{2}\frac{\dot g_{11}}{g_{11}}\frac{f'}{f}-\frac{1}{2}\frac{g_{11}'}{g_{11}}\frac{\dot f}{f}\right].$$

Here a dot denotes ∂_t and a prime ∂_z. The Riemann components can be calculated explicitly, using Eq. (4.6) for $\dot\sigma_k/\sigma_k$ and σ_k'/σ_k, and Eq. (2.5) for $\dot f/f$ and f'/f. The final result is expressed in terms of the functions $\sigma_k(t,z)$.

A. n solitons with $\sigma_k^{(-)}$

We first discuss case (a) for the situation with $\Delta\leq0$. The asymptotic expression for the coefficient f in the far zone is then

$$f=t^{(\Delta+n+1)(\Delta+n-1)/2}z^{-n(\Delta+n)}[1+O(z^{-2})]\ , \tag{5.5a}$$

and the Riemann components are

$$e_1=\frac{1}{4f}\left[\frac{(\Delta+n+1)(\Delta+n-1)^2}{2t^2}+O(z^{-2})\right],$$

$$e_2=\frac{1}{4f}\left[-\frac{(\Delta+n+1)^2(\Delta+n-1)}{2t^2}+O(z^{-2})\right], \tag{5.5b}$$

$$e_3=\frac{1}{4f}\left[\frac{(\Delta+n+1)(\Delta+n-1)}{t^2}+O(z^{-2})\right],$$

$$b=\frac{3n(\Delta+n-1)(\Delta+2)}{8zf}\left[\frac{1}{t}+O(z^{-2})\right].$$

From this we can define a critical value

$$\Delta_c=-n\ , \tag{5.6}$$

for which

$$\underline{E} \to t^{-3/2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad \underline{B} \to 0$$

as $z \to \infty$. The metric thus becomes Petrov type D as $z \to \infty$. Note that the asymptotic value does not depend on n. This is very different from the behavior in the causal region, where the solution tends to the corresponding Kasner background. The Kasner solution is Petrov type I unless $\Delta = -1$ (flat space) or $\Delta = 0$ (axisymmetric Kasner), in which cases it is also Petrov type D.

For $\Delta < \Delta_c$, the Riemann tensor satisfies

$$\underline{E} \to 0, \quad \underline{B} \to 0$$

as $z \to \infty$, so the metric is asymptotically flat in the far region. The interpretation of these metrics is that they represent inhomogeneities on a flat background which evolve towards the Kasner solution in the causal region due to soliton propagation along the z axis. Note that the metric can be made asymptotically Minkowski explicitly by the coordinate change $t' = t \cosh y$, $y' = t \sinh y$.

For $0 \geq \Delta > \Delta_c$ and $n > 2$, the metric is singular in the far zone ($\underline{E} \to \infty$) and so it has a space singularity as well as a cosmological singularity. The physical interpretation of this is not clear. It is also interesting that $\underline{E} \sim z^{n-2}$ for $\Delta = -n+1$. Therefore, for $n = 2$ (corresponding to Minkowski space), we have

$$\underline{E} \to \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \underline{B} \to 0$$

as $z \to \infty$, which is a time-independent Petrov type-D metric. This metric can be interpreted as solitons propagating on a static type-D background; as the solitons propagate, they leave a causal region which tends towards flat space.

B. The real-pole limit

We now discuss the limit in which $w_k \to 0$, that is, the limit in which the complex poles become real. The n-soliton solution then contains $n/2$ degenerate double real poles. As an illustrative example, we shall consider the $n = 2$ case. The complex two-soliton solution is

$$g_{11} = t^{1+\Delta}\sigma, \quad g_{22} = t^{1-\Delta}\sigma^{-1},$$
$$f = t^{(\Delta^2-5)/2} \frac{\sigma^{(\Delta+2)}}{(1-\sigma)^2 H}, \tag{5.7}$$

where we have dropped the index 1. If we now take the limit $w \to 0$ and set $z_1^0 = 0$, we obtain

$$g_{11} = t^{1+\Delta}\left[\frac{\mu}{t}\right]^2, \quad g_{22} = t^{1-\Delta}\left[\frac{t}{\mu}\right]^2,$$
$$f = \frac{t^{(\Delta^2+3)/2}}{(z^2-t^2)^2}\left[\frac{\mu}{t}\right]^{2\Delta}, \tag{5.8}$$

where μ is the real pole:

$$\mu = -z \pm (z^2 - t^2)^{1/2};$$

we take the + or − sign, depending on the choice of solution for σ [cf. Eq. (4.4)]. This is the double real soliton solution. It can be compared with the diagonal limit of the one-soliton solution of Belinskii and Zakharov[27]:

$$g_{11} = t^{1+\Delta}\frac{\mu}{t}, \quad g_{22} = t^{1-\Delta}\frac{t}{\mu},$$
$$f = \frac{t^{\Delta^2/2}}{(z^2-t^2)^{1/2}}\left[\frac{\mu}{t}\right]^{\Delta}. \tag{5.9}$$

The main difference is the exponent of (μ/t).

C. Degenerate complex poles

Solutions with degenerate complex poles can be easily obtained from Eq. (5.3) by taking the limits $z_k^0 \to z_l^0$, $w_k \to w_l$ for all k,l. This leads to a cosmological version of the Tomimatsu-Sato axisymmetric solutions:

$$g_{11} = t^{1+\Delta}\sigma^{n/2}, \quad g_{22} = t^{1-\Delta}\sigma^{-n/2},$$
$$f = \frac{t^{(\Delta^2-n^2-1)/2}\sigma^{n(\Delta+n)/2}}{H^{n/2}(1-\sigma)^{n^2/2}} \times \left[\frac{z^2(1-\sigma)^2 + w^2(1+\sigma)^2}{z^2(1-\sigma)^4 + w^2(1+\sigma)^4}\right]^{n(n-2)/4}, \tag{5.10}$$
$$H = (1-\sigma)^2 + \frac{16w^2\sigma^2}{(1-\sigma)^2 t^2}.$$

As usual, the metric tends to the Kasner seed in the causal region, with the perturbation decaying as t^{-1}. In the far region, $\sigma \to t^2/4z^2$ and so f has the asymptotic form given by Eq. (5.5a). Likewise, the Riemann components asymptotically look like Eq. (5.5b). Thus, the behavior is similar to that of case (A).

VI. NONDIAGONAL METRICS

In the nondiagonal case, one can no longer give explicit expressions for the n-soliton metric elements. This is largely due to the problem of inverting the $n \times n$ matrix Γ_{kl}, defined by Eq. (2.10b), which is required in the evaluation of solution (2.12). However, information can still be obtained from asymptotic expressions which can be calculated in the limit in which the nondiagonal metric tends towards diagonality. There is thus a connection between the nondiagonal solutions and those of the last section. We will first deal with the nondiagonal two-soliton solutions because some of their features apply for general n. In contrast to the diagonal case, these can be considered as solitons with two polarizations.

A. Two-soliton solutions

The two-soliton solutions have been studied in some detail by Belinskii and Fargion[29] for Kasner metrics with range $1 \geq \Delta \geq 0$ (i.e., the space is either contracting in the z direction or it is Minkowski). These solutions are the simplest nondiagonal metrics with complex poles that one can evaluate and their analytical expressions are relatively simple. We shall here study the asymptotic behavior of the solutions which can be obtained from all Kasner seeds, as well as their diagonal limit. Following the procedure in Secs. II and IV, the two-soliton solution can be shown to be (see also Belinskii and Fargion[29])

$$\begin{aligned} g_{11} &= \frac{t^{1+\Delta}}{D}\{(\sigma+\sigma^{-1}-2)\sin^2(\phi+\delta) \\ &\quad +[L_0{}^2\sigma^{-(1+\Delta)}+L_0{}^{-2}\sigma^{(1+\Delta)}+2]\sin^2\phi\}\ , \\ g_{22} &= \frac{t^{1-\Delta}}{D}\{(\sigma+\sigma^{-1}-2)\sin^2(\phi-\delta) \\ &\quad +[L_0{}^2\sigma^{(1-\Delta)}+L_0{}^{-2}\sigma^{-(1-\Delta)}+2]\sin^2\phi\}\ , \\ g_{12} &= \frac{2w}{D}\{L_0\sigma^{-(1+\Delta)/2}[\sin(\phi+\delta)+\sigma\sin(\phi-\delta)] \\ &\quad +L_0{}^{-1}\sigma^{-(1-\Delta)/2}[\sin(\phi-\delta)+\sigma\sin(\phi+\delta)]\}\ , \\ f &= Ct^{(\Delta^2-5)/2}\sigma^2 DH^{-1}(1-\sigma)^2(\sin\phi)^{-2}\ , \end{aligned} \tag{6.1}$$

where the notation of Sec. IV is used (although we drop the index 1) and

$$\begin{aligned} D &\equiv (\sigma+\sigma^{-1}-2)\sin^2\delta+(L_0{}^2\sigma^{-\Delta}+L_0{}^{-2}\sigma^{\Delta}+2)\sin^2\phi\ , \\ \delta &\equiv \Delta\phi+\delta_0\ . \end{aligned} \tag{6.2}$$

C is an arbitrary real constant; L_0 and δ_0 are also real constants, related to the definitions of Sec. II in that the vector $m_a^{(1)}$ of Eq. (2.10a) is given by

$$m_a^{(1)}=(d_1 e^{i\gamma_1}\mu^{-(1+\Delta)/2},\ d_2 e^{i\gamma_2}\mu^{-(1-\Delta)/2})\ , \tag{6.3a}$$

where d_a and γ_a are arbitrary real constants and

$$L_0\equiv d_1/d_2,\quad \delta_0\equiv\gamma_2-\gamma_1\ . \tag{6.3b}$$

We can determine the connection with the diagonal (one-polarization) two-soliton solution by taking the limits

$$L_0\to 0,\quad CL_0{}^{-2}\to C'(\text{finite})\ . \tag{6.4a}$$

The dominant term in D, from Eq. (6.2), is

$$D\sim L_0{}^{-2}\sigma^{\Delta}\sin^2\phi\ ,$$

and so the metric (6.1) becomes

$$\begin{aligned} & g_{11}=t^{1+\Delta}\sigma+O(L_0{}^2),\quad g_{22}=t^{1-\Delta}\sigma^{-1}+O(L_0{}^2)\ , \\ & g_{12}=O(L_0)\ , \\ & f=C't^{(\Delta^2-5)/2}H^{-1}\sigma^{(\Delta+2)}(1-\sigma)^{-2}+O(L_0{}^2)\ . \end{aligned} \tag{6.4b}$$

Comparing with Eq. (5.7), we see that the two-soliton solution (6.1) can be regarded as a generalization of the diagonal metric (5.7). The constant d_a (or L_0) can be interpreted as a "polarization" parameter, since one can reduce the two-polarization metric to a one-polarization metric by taking one of the parameters to be zero ($d_1=0$).

All the diagonal metrics, whatever the Kasner seed, had the common feature that they evolved towards the seed in the asymptotic causal and light-cone regions. The same holds for the nondiagonal two-soliton solution. In fact, in the causal region, taking the limits (4.5), we obtain

$$\begin{aligned} g_{11} &= t^{1+\Delta}\left[1-\frac{2w}{t}\frac{1-L_0{}^2}{1+L_0{}^2}\right]\ , \\ g_{22} &= t^{1-\Delta}\left[1+\frac{2w}{t}\frac{1-L_0{}^2}{1+L_0{}^2}\right]\ , \\ g_{12} &= \frac{4zw\Delta}{t}\frac{1}{L_0+L_0{}^{-1}}[1+O(t^{-1})]\ , \\ f &= C\frac{(L_0+L_0{}^{-1})^2}{16w^2}t^{(\Delta^2-1)/2}[1+O(t^{-1})]\ , \end{aligned} \tag{6.5}$$

where we have imposed on δ_0 the Belinskii and Fargion[29] condition

$$\delta_0=\pi(1-\Delta)/2\ . \tag{6.6}$$

Otherwise, the first-order term of g_{12} is a constant. Thus, the two-soliton solution in the causal region can be seen as a perturbation on the corresponding Kasner background whose amplitude decreases as t^{-1}. We can interpret Eq. (6.5) in another way: it is the diagonal two-soliton solution (5.7) in the causal region with a modified width parameter

$$w'=w\frac{1-L_0{}^2}{1+L_0{}^2}\ . \tag{6.7}$$

Note that the $O(t^{-1})$ part of g_{12} is negligible compared to the expressions for g_{11} and g_{22} in Eq. (6.5) since

$$t^2=\det\underline{g}=g_{11}g_{22}-g_{12}{}^2\ .$$

The g_{12} terms therefore give the nondiagonal correction to higher-order terms in the diagonal metric.

The solution in the asymptotic light-cone region can be obtained in a similar way, using Eq. (4.5). The results are

$$\begin{aligned} g_{11} &= t^{1+\Delta}\left[1-2\left[\frac{w}{t}\right]^{1/2}\frac{(L_0{}^{-2}-L_0{}^2-2\sin\delta_0)}{(L_0+L_0{}^{-1})^2+4\sin^2\delta_0}\right]\ , \\ g_{22} &= t^{1-\Delta}\left[1+2\left[\frac{w}{t}\right]^{1/2}\frac{(L_0{}^{-2}-L_0{}^2-2\sin\delta_0)}{(L_0+L_0{}^{-1})^2+4\sin^2\delta_0}\right]\ , \\ g_{12} &= 4\sqrt{wt}\,\frac{[(L_0+L_0{}^{-1})\cos\delta_0+(L_0-L_0{}^{-1})\sin\delta_0]}{(L_0+L_0{}^{-1})^2+4\sin^2\delta_0}\ , \\ f &= C\frac{[4\sin^2\delta_0+(L_0+L_0{}^{-1})^2]}{32w^2}t^{(\Delta^2-1)/2}[1+O(t^{-1/2})]\ , \end{aligned} \tag{6.8}$$

the amplitude of the perturbation decreasing as $t^{-1/2}$. This equation can be regarded as giving the first-order terms of the diagonal two-soliton metric (5.7) with the width parameter w changed in a manner analogous to Eq. (6.7). As before, the metric coefficient g_{12} represents a higher-order nondiagonal deviation from the g_{11} and g_{22} terms given by Eq. (6.8). One should not deduce from Eq. (6.8) that the Riemann tensor behaves the same way in the light-cone region as in the seed solution, because the z dependence in Eq. (6.8) has been hidden in the assumption $|z|\sim t$. The light-cone region has to be seen as a boundary between the causal and far regions which contains the propagating solitons.

We now have to study the asymptotic behavior in the far region. This depends crucially on the seed metric. For some choices, the metric will evolve towards the seed; for others, it will evolve to a completely different spacetime. In the first case, one can interpret Eq. (6.1) as two solitons propagating on a Kasner background with a velocity which tends to the speed of light. If we choose the $\sigma^{(-)}$ pole, the far region solution is

$$\frac{g_{11}}{t^{1+\Delta}}=\begin{cases}\frac{4z^2}{t^2}[1+O(z^{-2})] \quad (\Delta>2)\,,\\ \frac{4z^2}{t^2}\frac{4w^2t^{-2}L_0{}^2}{\sin^2\delta_0+4w^2t^{-2}L_0{}^2}[1+O(z^{-2})] \quad (\Delta=2)\,,\\ \left[\frac{4z^2}{t^2}\right]^{\Delta-1}\frac{4w^2t^{-2}L_0{}^2}{\sin^2\delta_0}[1+O(z^{-2})] \quad (2>\Delta>1)\,,\\ \frac{\sin^2\delta_0+4w^2t^{-2}L_0{}^2}{\sin^2\delta_0}[1+O(z^{-1})] \quad (\Delta=1)\,,\\ 1+O(z^{-1}) \quad (1>\Delta\geq 0)\,,\end{cases} \tag{6.9a}$$

$$\frac{g_{22}}{t^{1-\Delta}}=\begin{cases}\frac{t^2}{4z^2}[1+O(z^{-2})] \quad (\Delta>3)\,,\\ \frac{t^2}{4z^2}\frac{\sin^2\delta_0+4w^2t^{-2}L_0{}^2}{4w^2t^{-2}L_0{}^2}[1+O(z^{-2})] \quad (\Delta=3)\,,\\ \left[\frac{t^2}{4z^2}\right]^{\Delta-2}\frac{\sin^2\delta_0}{4w^2t^{-2}L_0{}^2}[1+O(z^{-2})] \quad (3>\Delta>2)\,,\\ \frac{\sin^2\delta_0}{\sin^2\delta_0+4w^2t^{-2}L_0{}^2}[1+O(z^{-1})] \quad (\Delta=2)\,,\\ 1+O(z^{-1}) \quad (2>\Delta\geq 0)\,,\end{cases} \tag{6.9b}$$

$$\frac{g_{12}}{2w}=\begin{cases}\left[\frac{t^2}{4z^2}\right]^{(\Delta-3)/2}\frac{\sin\delta_0}{4w^2t^{-2}L_0}[1+O(z^{-2})] \quad (\Delta>2)\,,\\ \left[\frac{t^2}{4z^2}\right]^{1/2}\frac{L_0\sin\delta_0}{\sin^2\delta_0+4w^2t^{-2}L_0}[1+O(z^{-2})] \quad (\Delta=2),\\ \left[\frac{t^2}{4z^2}\right]^{(1-\Delta)/2)}\frac{L_0}{\sin\delta_0}[1+O(z^{-2})] \quad (2>\Delta>0)\,,\\ \left[\frac{t^2}{4z^2}\right]^{1/2}\frac{L_0-L_0{}^{-1}}{\sin\delta_0}[1+O(z^{-1})] \quad (\Delta=0)\,.\end{cases} \tag{6.9c}$$

The $\sigma^{(+)}$ solutions introduce nothing essentially new since the change $\sigma\to\sigma^{-1}$ in Eq. (6.1) is equivalent to the change $L_0\to L_0{}^{-1}$, $\delta_0\to-\delta_0$, and $g_{12}\to -g_{12}$.

The important points to emphasize, because they will be relevant in the general n-soliton solution, are as follows: (i) for seeds with $\Delta>3$, all two-soliton metrics became diagonal in the far region (cf. Sec. V); and (ii) for seeds with $1>\Delta\geq 0$, all the two-soliton metrics become the seed metric, as already shown by Belinskii and Fargion.[29] Note that in case (ii) the propagation axis is contracting. Thus, when the background is contracting in the z direction, the two-soliton solutions can be interpreted as two perturbations propagating along the z axis. Since the solitons have their maximum amplitude on the light cone for large t, they move in opposite directions with a speed asymptotically approaching the speed of light, i.e., they become gravitational waves at large t.

As an example, we present in Fig. 2 the evolution of the two-soliton solution when the background is the axisymmetric Kasner ($\Delta=0$) seed. We take the width of the soliton to be relatively small ($w=0.01$) and the parameters in Eq. (6.3b) to be $L_0=1$ and $\cos\delta_0=(1.01)^{-1/2}$. In the representation of Fig. 2, the x and y axes have been rotated through $\pi/4$. It is clear that the two-soliton solution tends to Kasner in the causal and far regions if the propagation axis is contracting. However, we should note that, if we take $C\equiv w^2/\sin^2\delta_0$, the f coefficient in the far region becomes the corresponding f for the Kasner background (2.13) whereas, in the causal and light-cone regions, it will include a different constant [see Eqs. (6.5) and (6.3)]. Thus, in those asymptotic regions, the existence of solitons modifies the "longitudinal expansion" with respect to the Kasner background.

B. n-soliton solutions

We shall now discuss the general n-soliton solution with complex poles (n even) in the asymptotic regions. We will find that, for some seeds, it shares many of the asymptotic properties of the two-soliton and diagonal n-soliton solutions. In particular, as in all the solutions considered so far, the n-soliton solution evolves towards the Kasner seed, with the perturbation decreasing as t^{-1} in the asymptotic causal region. This is a consequence of the value of σ_k in that region, given by Eq. (4.5), and it would be true for any seed.

This can be seen explicitly using the soliton-generating

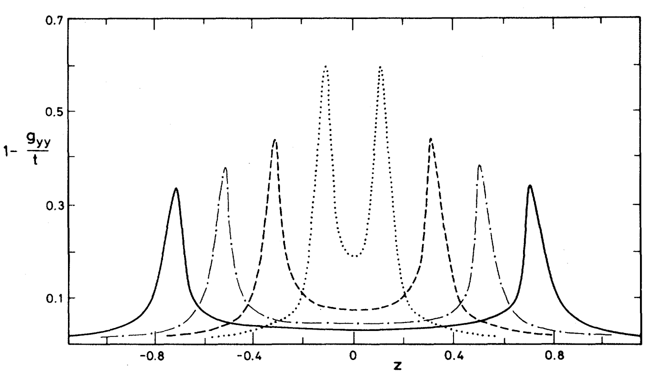

FIG. 2. This shows the time evolution of $g_{yy}(t,z)$ for the two-soliton solution (6.1) generated from the axisymmetric Kasner seed ($\Delta=0$). The Kasner background has been subtracted and "normalized" by the factor t; we have also made a $\pi/4$ rotation of the x and y axes. The soliton characteristics are as follows: origin $z=0$, width $w=0.01$, polarization parameters $d_1=d_2=1$, and $\cos\gamma_1=1$, $\cos\gamma_2=(1.01)^{-1/2}$. The dotted line is $t=0.1$, the dashed line $t=0.3$, the dot-dashed line $t=0.45$, and the continuous line $t=0.7$.

technique of Sec. II. For the Kasner seeds the vector $m_a^{(k)}$ can be written as

$$m_a^{(k)}=(d_1^{(k)}e^{i\gamma_1^{(k)}}\mu_k^{-(1+\Delta)/2},d_2^{(k)}e^{i\gamma_2^{(k)}}\mu_k^{-(1-\Delta)/2})\ , \tag{6.10}$$

with $d_a^{(k)}$ and $\gamma_a^{(k)}$ being arbitrary real parameters. The $n\times n$ matrix Γ_{kl}, defined by Eq. (2.10b), can be written in the causal region as

$$\Gamma_{kl}=\frac{1}{t}\left[A_{kl}+\frac{1}{t}B_{kl}\right]\ , \tag{6.11}$$

where the $n\times n$ matrices $A_{kl}(d_a^{(r)},\gamma_a^{(r)},w_r)$ and $B_{kl}(d_a^{(r)},\gamma_a^{(r)},w_r,\Delta)$ are complex constants. It is not difficult to see that $\det A_{kl}\neq 0$ for nondegenerate poles, so

$$D_{kl}=t[(A^{-1})_{kl}+O(t^{-1})]\ .$$

Using Eqs. (2.11) and (2.12), we now obtain

$$\begin{aligned} g_{11}&=t^{1+\Delta}\left[1-\frac{A}{t}\right]\ ,\\ g_{22}&=t^{1-\Delta}\left[1+\frac{A}{t}\right]\ ,\\ g_{12}&=B+O(t^{-1})\ , \end{aligned} \tag{6.12}$$

where

$$\begin{aligned} A&=2\sum_{k=1}^{n/2}w_k+\sum_{k,l=1}^{n}d_1^{(k)}d_1^{(l)}\exp[i(\gamma_1^{(k)}+\gamma_1^{(l)})](A^{-1})_{kl}\ ,\\ B&=-\sum_{k,l=1}^{n}d_1^{(k)}d_2^{(l)}\exp[i(\gamma_1^{(k)}+\gamma_2^{(l)})](A^{-1})_{kl}\ . \end{aligned} \tag{6.13}$$

Note that the leading terms in the symmetric matrix $(A^{-1})_{kl}$ are those with $0<k\leq n/2$ and $n/2<l\leq n$. For the $n=2$ case we recover the previous results: dropping the pole index 1, we get

$$A=2w\left[\frac{d_2{}^2-d_1{}^2}{d_1{}^2+d_2{}^2}\right]\ .$$

This agrees with Eq. (6.5) if we use the definition of L given by Eq. (6.3b). We also get

$$B=4w\frac{d_1d_2}{d_1{}^2+d_2{}^2}\sin\left[\frac{\pi(1-\Delta)}{2}+\gamma_1-\gamma_2\right]\ ,$$

and using the condition (6.6), the g_{12} coefficient is $O(t^{-1})$. Note that one can always extract the constant parameter in the coefficient g_{12} by imposing the condition

$$B=0\ ; \tag{6.14}$$

this restricts the value of one of the parameters of the soliton solution.

The f coefficient can be easily found using Eq. (2.12b). It is clear from this equation that we will obtain

$$f=C\det A_{kl}t^{(\Delta^2-1)/2}[1+O(t^{-1})] \tag{6.15}$$

in the causal limit. Thus, the longitudinal expansion of the Kasner seed, given by Eq. (2.13), will be modified in the causal region by a factor $\det A_{kl}$. While this does not affect the Kasner characteristics towards which the n-soliton solution evolves, it does indicate that the presence of the solitons has an important influence on the longitudinal expansion of the background, i.e., the solitons do not behave as linear waves superposed on the Kasner seed.

In the asymptotic light-cone region we will obtain similar results to those of the two-soliton solution. We assume that the distance between solitons is finite, so that we can make the approximation

$$z_l\simeq -z\left[1-\frac{z_l{}^0}{t}\right]\ .$$

This means that we will not see the fine structure in the light-cone region, but only the gross features of the soliton solution. Proceeding as in the causal-region limit, we find

$$\Gamma_{kl}=\frac{1}{t^{3/2}}\left[E_{kl}+\frac{1}{\sqrt{t}}F_{kl}\right]\ ,$$

where E_{kl} and F_{kl} are constant $n\times n$ matrices depending on the various parameters of the solution. We obtain

$$\begin{aligned} g_{11}&=t^{1+\Delta}[1+A't^{-1/2}+O(t^{-1})]\ ,\\ g_{22}&=t^{1-\Delta}[1-A't^{-1/2}+O(t^{-1})]\ ,\\ g_{12}&=t^{1/2}[B'+O(t^{-1/2})]\ , \end{aligned} \tag{6.16}$$

where A' and B' are given by expressions similar to those for A and B in Eq. (6.13). The f coefficient will be modified by the factor $\det E_{kl}$, as before. Clearly the gross features of the general n-soliton solution in the asymptotic causal and light-cone regions are similar to those of the nondiagonal two-soliton solution. They are therefore also similar to those of the diagonal metrics, with the w_k parameters being modified by the polarization constants $d_a^{(k)}$ and $\gamma_a^{(k)}$.

In the far region, the $n\times n$ matrix Γ_{kl} cannot be treated as such, so we can no longer find approximate expressions for the general n-soliton solutions. However, some of their features can be deduced from the diagonal and nondiagonal two-soliton metrics. In fact, from the asymptotic expressions for the two-soliton solution, one can infer that, for seed metrics with $1>\Delta\geq 0$, the n-soliton solution will always tend to the seed metric in the far region. The reason is that one can find the n-soliton solution step by step, using the $(n-2)$-soliton solution as seed, etc. It is clear that at each step we will recover the seed metric in the far region. Consequently, the general n-soliton solution can be considered as n solitons propagating on a Kasner background which is contracting along the propagation axis. Since the speed of the solitons asymptotically approaches the speed of light, they evolve towards gravitational waves with two polarizations. If we take $|z_k^0-z_{k-1}^0|=d$ for all k, so that the solitons are equally spaced, the wave period will be d.

As an example, we show in Fig. 3 the g_{yy} coefficient of the four-soliton solution for the axisymmetric Kasner seed ($\Delta=0$), the Kasner background itself being subtracted. The soliton parameters are those of Fig. 2 with two pairs of equivalent solitons. The structure of four solitons propagating on a background is clear. One can also ob-

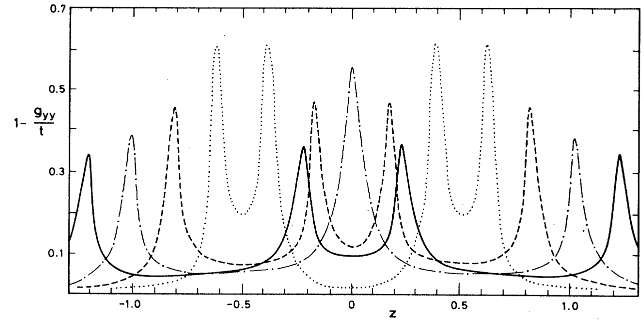

FIG. 3. This shows the time evolution of the $g_{yy}(t,z)$ component for the four-soliton solution generated by the axisymmetric Kasner seed ($\Delta=0$). The same conventions are used as in Fig. 2. The two pairs of solitons have the same parameters: $w_1=w_2=0.01$, $d_1^{(1)}=d_2^{(1)}=d_1^{(2)}=d_2^{(2)}=1$, $\cos\gamma_1^{(1)}=\cos\gamma_2^{(1)}=1$, and $\cos\gamma_1^{(2)}=\cos\gamma_2^{(2)}=(1.01)^{-1/2}$. The separation of the origins is $|z_1^0-z_2^0|=1$. The dot-dashed line corresponds to the time $t=0.45$ during the collision of the inner solitons. After the collision, they move unperturbed. For large t, the gross features of this solution are those of the two-soliton solution.

serve the collision of the two inner solitons. The amplitude of the colliding pair is much greater, implying larger curvature, than that of the other pair, but the two solitons leave the collision unmodified (as is typical of solitons in hydrodynamics). There is a "hierarchy" effect, in that at large t the gross features of the four-soliton solution will resemble those of the two-soliton solution.

For seed metrics with $\Delta>3$, the general n-soliton solution still tends asymptotically to the diagonal n-soliton solution. We have already seen that this is the case in the two-soliton solution, and one can prove the general result by induction. The reason can be seen by considering the vector $m_a^{(k)}$ of Eq. (6.10). For $\Delta>3$ and $|z|\to\infty$, the $m_1^{(k)}$ component is negligible compared to the other component and this is equivalent to taking $d_1^{(k)}\to 0$ [which, from Eq. (5.1), is the way of generating diagonal metrics]. Thus, the general n-soliton solution for the $\Delta>3$ Kasner seeds behaves asymptotically as the diagonal n-soliton solution and the results of Sec. V can be applied. This can be interpreted as the loss of one of the polarizations in the asymptotic regions.

Of course, the form of the general n-soliton solutions in the "near regions" is very different from the two-soliton and diagonal n-soliton solutions. They have a much richer fine structure, with two polarizations and many extra parameters. The four-soliton solution of Fig. 3 gives a hint of what the n-soliton solution will look like. The solutions with $1\le\Delta\le 3$ seeds, which include the Minkowski seed, have yet to be studied in detail. They are more complicated because, from Eqs. (6.9), they do not tend towards either diagonality or the seed metric as $z\to\infty$.

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