Multisoliton solutions to Einstein's equations

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We discuss a multisoliton solution to Einstein's equations in vacuum. The solution is interpreted as many gravitational solitons propagating and colliding on a homogeneous cosmological background. Following a previous letter, we characterize the solitons by their localizability and by their peculiar properties under collisions. Furthermore, we define an associated frame-dependent velocity field which illustrates the solitonic character of these gravitational solitons in the classical sense.

I. INTRODUCTION

The solutions to Einstein's equations in vacuum obtained by the inverse scattering technique (or soliton technique) of Belinskii and Zakharov[1] are generally known as soliton solutions. The name "soliton" is applied because of the technique used rather than as a consequence of specific properties of these solutions. In fact, Belinskii and Zakharov succeeded in a generalization of the inverse scattering method to general relativity. That method is used to solve certain nonlinear equations, such as the Korteweg-de Vries equation, for example, Ref. 2, and it is known to give solitonlike (solitary wave) solutions. These classical solitons are characterized by their localizability, by their peculiar behavior under collisions, and by having an associated velocity of propagation.

The soliton solutions in general relativity, however, generally have no resemblance to the classical solitons. All that is required to obtain such solutions is to have vacuum metrics with two commuting Killing vectors. For example, one can assume a stationary axisymmetric background metric and obtain a new solution as a soliton solution, but the stationarity of this solution prevents any comparison with a classical soliton.

However, taking two spacelike Killing vectors, one can obtain nonstationary solutions. Some of these solutions have properties similar to the classical solitons. Belinskii and Fargion[3] studied the first of such "solitonlike" soliton solutions. They found a family of nondiagonal metrics which could be interpreted as two inhomogeneities propagating on a homogeneous cosmological background. The soliton structure in this case was revealed in the localizability of the metric components. They also defined a velocity of propagation for such inhomogeneities, but this being greater than the velocity of light, for some solutions, could not be interpreted as a velocity of physical propagation.

Ibañez and Verdaguer[4] studied a simple diagonal metric in detail. It contained four solitons, two of which collided, so that their behavior under collisions could be examined. It was suggested that the gravitational solitons (gravisolitons) have many similarities to the classical solitons because of their localization and their properties under collision. The analysis was based on intrinsic properties of the solution (deduced from the Riemann tensor) rather than on properties of the metric components.

In this paper we continue such an analysis and extend it in two ways. On the one hand, we generalize the results to a multisoliton solution. It is a rather straightforward generalization of the four-soliton results: the qualitative behavior of the multisoliton metric is similar to that of the four-soliton one. In the general case, however, the solitons collide several times and we can confirm some of the features which appear in the single-collision case.

On the other hand, we define a velocity field associated with the solitons. Although such a definition is frame dependent, the velocity is never greater than 1 and this suggests that the solitons propagate curvature and energy in some sense.

Finally we should mention that the multisoliton solutions are inhomogeneous cosmological models. They represent highly inhomogeneous universes which evolve towards homogeneous Kasner models with gravitational waves. They are an example of the creation of a background of gravitational waves as a consequence of initial inhomogeneities, in the line of the work of Adams *et al.*[5]

The plan of the paper is as follows. In Sec. II the $4m$-soliton solution is given and the properties of the metric components are analyzed in the different asymptotic regions of the space-time. The main conclusion is that the solutions represent $4m$ inhomogeneities propagating and colliding several times on a cosmological Kasner background.

In Sec. III the Riemann tensor of the metric is studied. The scalar invariants (two in our case) are given, and the components of the Riemann tensor are projected onto a physically meaningful null tetrad. This analysis shows that the solitons evolve towards pure gravitational waves as the background expands. This is seen from the intrinsic properties of the metric classification in the different asymptotic regions.

In Sec. IV a velocity field associated with the solitons is proposed and its properties are studied. By means of the Bel-Robinson tensor we associate a superenergy density and a superenergy flux with the solitons. The results, although valid in a peculiar frame, are suggestive and illustrate our previous interpretation that the solitons start as quasiparticles and evolve towards pure gravitational waves.

In Sec. V the optical scalars are studied. We study the shear and expansion of null rays produced by the solitons. The convergence of null rays produced by the solitons and the shear induced by them resemble that produced by gravitational plane waves in cosmological backgrounds.[6] We should emphasize here that these solutions are very simple and are an example of plane waves that can be studied analytically.

II. THE METRIC

In this paper we study a family of exact solutions of the vacuum Einstein equations. These solutions have been obtained by the inverse scattering technique developed by Belinskii and Zakharov.[1] This technique has been reviewed in several references[7] and we shall not repeat it here.

We recall that with this technique new vacuum solutions of Einstein's equations can be generated when there are two commuting Killing vectors (we assume here two spacelike Killing vectors), namely, when the metric has the form

$$ds^2=f(t,z)(dz^2-dt^2)+g_{ab}(t,z)\,dx^a dx^b\ , \qquad a,b=1,2\ . \tag{1}$$

A new solution is obtained when a particular "seed" solution (f_0,g_0) is known.

The main ingredients of the generating technique are the so-called "pole trajectories" with equations

$$\mu_i{}^2-2(z_i-iw_i)\mu_i+t^2=0\ , \qquad z_i=z_i^0-z\ ,\quad i=1,\ldots,n \tag{2}$$

where n is the "soliton number" and z_i^0 and w_i are arbitrary real constants. We shall only consider the case when $w_i\neq0$, i.e., complex poles. For a complex pole trajectory μ_i, its complex conjugate is also a pole trajectory. Thus complex poles always appear in pairs. (For a discussion of real poles see Carr and Verdaguer[8] and Belinskii and Francaviglia.[7])

From the solutions of (2) and the seed metric (f_0,g_0), new solutions (f,g) can be obtained using only algebraic manipulations. Explicit solutions of (2) have been given by Carr and Verdaguer.[8] For complex poles, Eq. (2) has two different solutions:

$$\mu_i=t\sqrt{\sigma_i}\,e^{i\varphi_i}\ ,$$
$$\sigma_i^{\pm}=\frac{w_i{}^2+z_i{}^2}{t^2}+a_i\pm\sqrt{2}\left[\frac{(w_i{}^2+z_i{}^2)^2}{t^4}+\frac{w_i{}^2-z_i{}^2}{t^2}+\frac{w_i{}^2+z_i{}^2}{t^2}a_i\right]^{1/2}\ , \tag{3}$$
$$a_i=\left[1+2\frac{w_i{}^2-z_i{}^2}{t^2}+\frac{(w_i{}^2+z_i{}^2)^2}{t^4}\right]^{1/2}\ .$$

These solutions satisfy $\sigma_i^+=1/\sigma_i^-$ and $0<\sigma_i^-<1$, $1<\sigma_i^+<\infty$.

The behavior of (3) in the different asymptotic regions is

$$\sigma_i^-=1-\frac{2w_i}{t}+O(t^{-2})\ ,\qquad |z_i|\ll t\to\infty\ \text{(interaction region)}$$
$$\sigma_i^-=1-\left[\frac{(w_i{}^2+z_i^{0\,2})^{1/2}-z_i^0}{t}\right]^{1/2}+O(t^{-1})\ ,\qquad |z_i|=t\to\infty\ \text{(light-cone region)} \tag{4}$$
$$\sigma_i^-=\frac{t^2}{4z^2}[1+O(z^{-1})]\ ,\quad t\ll|z|\to\infty\ \text{(far region)}$$

[expressions inverse to (4) correspond to σ_i^+].

The first region (interaction region) is contained in the intersection of the causal cones of each z_i^0 $(t^2=z_i^{0\,2})$. The far region is the region not causally connected with any z_i^0.

As the seed metric we will use the Kasner metric, which is a Bianchi type-I solution. It can be written in the form (1) as

$$ds^2=t^{(\delta^2-1)/2}(dz^2-dt^2)+t^{1+\delta}dx^2+t^{1-\delta}dy^2\ . \tag{5}$$

$\delta=0$ corresponds to the axisymmetric Kasner metric. For $\delta>1$ the z axis is expanding and for $\delta<1$ it is contracting. Flat space is given by $\delta=1$. The Kasner metric has a volume expansion proportional to t^{-1}.[9]

All the new solutions obtained via the inverse scattering technique are inhomogeneous along the z axis and have the Kasner singularity at $t=0$. However, depending on the soliton number n, on the prescription taken for σ, σ^+ or σ^- and on the choice of the arbitrary parameters, they can have very different properties. A general analysis of these properties has been carried out by Carr and Verdaguer.[8] Here we are interested in a family of solutions which exhibit a solitonic character, i.e., a multisoliton solution which can be interpreted as physical solitons propagating on a homogeneous (Kasner) background. The simplest such multisoliton solutions are given by the diagonal metrics.[8]

$$g_{11}=t^{1+\delta}\prod_{k=1}^{2m}\sigma_k\ ,\quad g_{22}=\frac{t^2}{g_{11}}\ ,\quad n=4m$$
$$f=\frac{t^{(\delta^2-1)/2-8m^2}}{\prod_{k=1}^{2m}H_k}\prod_{k=1}^{2m}\left[\frac{\sigma_k{}^{\delta+4-4m}}{(1-\sigma_k)^2}\right]\prod_{\substack{k,l=1\\k>l}}^{2m}\left[\left[(\sigma_k+\sigma_l)t^2-\frac{8z_kz_l\sigma_k\sigma_l}{(1+\sigma_k)(1+\sigma_l)}\right]^2-\frac{64w_k{}^2w_l{}^2\sigma_k{}^2\sigma_l{}^2}{(1-\sigma_k)^2(1-\sigma_l)^2}\right]^2\ , \tag{6}$$

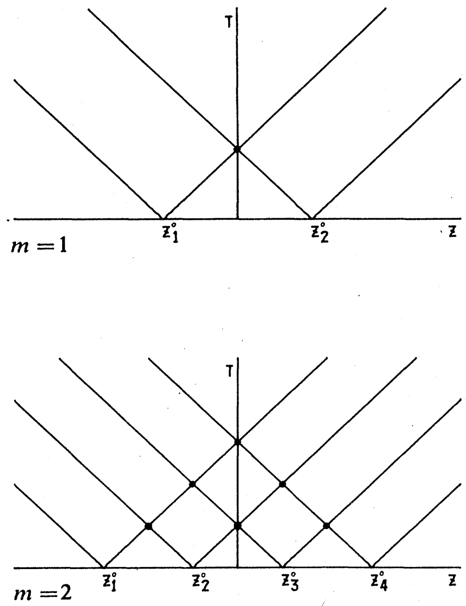

FIG. 1. The t-z diagrams for $m=1$ and $m=2$ solutions. The solitons propagate near the light cones. Soliton collisions are represented by black dots.

where

$$H_k=(1-\sigma_k)^2+\frac{16{w_k}^2{\sigma_k}^2}{(1-\sigma_k)^2t^2} \tag{7}$$

and where we take σ_i^+ for i even and σ_i^- for i odd, so that from (6) and the limits on σ given above, the metric (6) tends to the seed metric in the interaction region and in the far regions. The inhomogeneities are thus localized on the "light-cone" regions ($|z_i|\sim t$).

The parameters z_i^0 represent the "origins" of the solitons and w_i their "widths." Since one can take one of the z_i^0 as zero the relevant parameters of the solution (6) are $4m$ ($2m-1$ z_i^0's, $2m$ w_i's, and the seed parameter δ).

In order to interpret these solutions, it is convenient to draw the t-z diagrams of Fig. 1. In these diagrams the light cones from the z_i^0 are shown. The solitons propagate roughly along the lines of the light cones so that we have in general $4m$ solitons and $m(2m-1)$ collisions, the latter represented by black dots.

A qualitative picture of the multisoliton solutions can be easily given by studying the metric components in the asymptotic regions. With the asymptotic values for σ given in (4) we obtain the following limits for the metric (hereafter the K index on a quantity refers to the value of that quantity for the Kasner seed solution):

$$\begin{aligned} g_{11}&\sim g_{11}^K[1+O(z^{-1})]\ , \quad \text{far region}\\ g_{11}&\sim g_{11}^K[1+O(t^{-1})]\ , \quad \text{causal region}\\ g_{11}&\sim g_{11}^K[1+O(t^{-1/2})]\ , \quad \text{light-cone region}\end{aligned} \tag{8}$$

g_{22} behaves like g_{11}. For the coefficient f we obtain

$$\begin{aligned} f&\sim f^K[1+O(z^{-1})]\ , \quad \text{far region}\\ f&\sim f^K\frac{4^{m(4m-6)}}{\prod_{k=1}^{2m}{w_k}^2}\\ &\quad\times\prod_{\substack{k,l=1\\k>l}}^{2m}[2({w_k}^2+{w_l}^2)-(w_k-\epsilon_l w_l)^2-{z_{kl}}^2]^2\\ &\quad\times[1+O(t^{-1})]\ , \quad \text{causal region}\end{aligned} \tag{9}$$

where

$$\epsilon_i=+1 \text{ for } i \text{ even}\ , \quad \epsilon_i=-1 \text{ for } i \text{ odd}\ . \tag{10}$$

In the light-cone region we must distinguish the different light cones. For the light cone with origin z_i^0 we obtain

$$f\sim f^K\frac{4^{m(4m-6)}}{\prod_{k=1}^{2m}({x_{ki}}^2+{w_k}^2)}\prod_{\substack{k,l=1\\k>l}}^{2m}\left\{[({w_k}^2+{z_{ki}}^2)^{1/2}+({w_l}^2+{z_{li}}^2)^{1/2}-\epsilon_k\epsilon_l(x_{ki}x_{li})^{1/2}]^2-\frac{{w_k}^2{w_l}^2}{x_{ki}x_{kl}}\right\}^2[1+O(t^{-1/2})]\ , \tag{11}$$

where

$$z_{ki}=z_k^0-z_i^0 \tag{12}$$

and where $x_{ki}=({w_k}^2+{z_{ki}}^2)^{1/2}-z_{ki}$ for solitons traveling in the positive z direction ("to the right") and $x_{ki}=({w_k}^2+{z_{ki}}^2)^{1/2}+z_{ki}$ for solitons traveling in the opposite direction ("to the left").

The expression (11) has different limits on the different light cones. To illustrate this, we shall take $w_i\sim w\ll|z_{ki}|$. This case is particularly interesting since for small w the solitions are very localized so that one can follow them almost from the start. For the soliton traveling to the right, starting at z_m^0 and never colliding, one has $f\sim f^K/w^2$. For the soliton traveling to the right, starting at z_{m-1}^0 and undergoing one collision, one has $f\sim f^K/w^4$, and so on. For the soliton traveling to the right starting at z_1^0, and undergoing $2m-1$ collisions, one has $f\sim f^K/w^{4m}$, which agrees with the asymptotic value in the causal region. Thus it appears that each collision increases the value of the coefficient f relative to its value in the corresponding Kasner background. After each collision the solitons have a greater longitudinal expansion along their direction of propagation.

Thus the multisoliton metric tends to the Kasner seed metric in the far region, and in the causal region as well, but with a different expansion coefficient f. The difference is related to the number of collisions. In the light-cone region it differs in $O(1/\sqrt{t})$ from that of the background, which is typical of gravitational waves in expand-

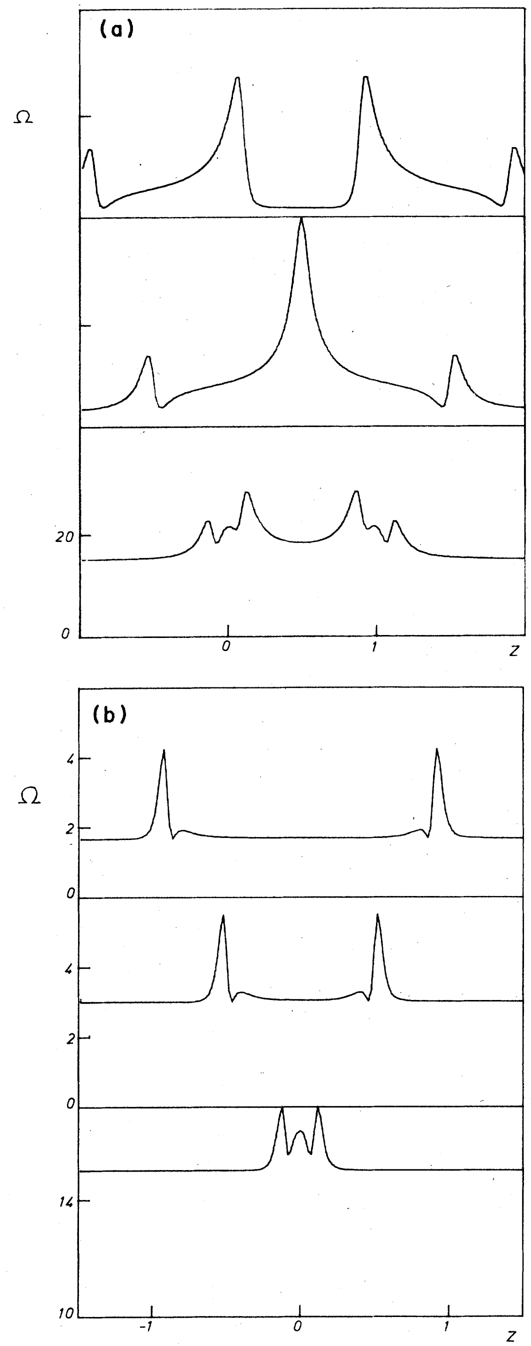

FIG. 2. (a) The volume expansion Ω is graphed against the propagation axis z for $m=1$. The Kasner parameter is $\delta=0$. The widths and origin of the solitons are $w_1=w_2=0.05$ and $z_1^0=0$ and $z_2^0=1$. The different curves from bottom to top correspond to the time sequence $t=0.1$ (before collision), $t=0.5$ (collision time), and $t=0.9$ (after collision). (b) This is the same as (a), but taking the widths and origins of the solitons to have the values: $w_1=0.1$, $w_2=0.05$, and $z_1^0=z_2^0=0$.

ing cosmological backgrounds. One should not interpret this to mean that in the light-cone region the multisoliton gravitational field tends to that of the Kasner seed. In fact, as we shall see from the Riemann tensor, the metric in that region tends to a Petrov type-N (pure radiation) metric, which is completely different from the Kasner seed (Petrov type-I metric).

To illustrate the inflation produced by the passage of a soliton on a volume element, we represent in Fig. 2(a) the volume expansion $\Omega=(1/V)dV/dt$ for a metric with $m=1$, where $V=fg_{11}\,g_{22}$. We also see in this figure the solitonic character of the metric: the shape of the solitons is unchanged after the collision.

An interesting case occurs when we consider $z_i^0=0$ for all i, but $w_i \neq w_j$ (otherwise the metric is just the seed solution). In this case the collision takes place exactly at $t=0$. When $m=1$ this solution represents only two solitons, which appear at $z=0$ and propagate along the z axis in opposite directions. However, this solution is completely different from the diagonal two-pole complex solution ($m=\frac{1}{2}$), which differs completely from the Kasner seed in the far region.[8] Figure 2(b) represents the volume expansion with $m=1$ for this case. Only two solitons appear with characteristics similar to those of Fig. 2(a).

III. RIEMANN TENSOR

The intrinsic properties of the space-time are described by the Riemann tensor and its invariants. Thus, in order to see the solitonic character of the solution (6) one must investigate the Riemann tensor. The nonvanishing components of the Riemann tensor are given by Carr and Verdaguer[8] as

$$R^{01}{}_{01}=e_1=-\frac{1}{2f}\left[\frac{g_{22}''}{g_{22}}-\frac{1}{2}\left[\frac{g_{22}'}{g_{22}}\right]^2 -\frac{1}{2}\frac{\dot g_{22}}{g_{22}}\frac{\dot f}{f}-\frac{1}{2}\frac{g_{22}'}{g_{22}}\frac{f'}{f}\right], \tag{13a}$$

$$R^{02}{}_{02}=e_2=-\frac{1}{2f}\left[\frac{g_{11}''}{g_{11}}-\frac{1}{2}\left[\frac{g_{11}'}{g_{11}}\right]^2 -\frac{1}{2}\frac{\dot g_{11}}{g_{11}}\frac{\dot f}{f}-\frac{1}{2}\frac{g_{11}'}{g_{11}}\frac{f'}{f}\right], \tag{13b}$$

$$R^{03}{}_{03}=e_3=-e_1-e_2\,, \tag{13c}$$

$$R^{31}{}_{01}=b=-\frac{1}{2f}\left[\frac{\dot g_{11}'}{g_{11}}-\frac{1}{2}\frac{g_{11}'}{g_{11}}\frac{\dot g_{11}}{g_{11}} -\frac{1}{2}\frac{\dot g_{11}}{g_{11}}\frac{f'}{f}-\frac{1}{2}\frac{g_{11}'}{g_{11}}\frac{\dot f}{f}\right] \tag{13d}$$

(dots and primes mean ∂_t and ∂_z, respectively). e_1, e_2, and e_3 are the "electric" components of the gravitational field and b is the "magnetic" component.

Using the fact that the solutions of the pole equations satisfy the equations[8]

$$\dot\sigma_i=\frac{2\sigma_i(1-\sigma_i^2)}{tH_i}\,,\quad \sigma_i'=\frac{8z_i\sigma_i^2(1-\sigma_i)}{t^2H_i(1+\sigma_i)}\,, \tag{14}$$

the expressions (13) are easily evaluated. Therefore the derivatives of the metric (6) can be written as a function of σ_i.

In the far region the Riemann tensor tends to the Kasner Riemann tensor. The same thing occurs in the interaction region, although as we mentioned above, the longitudinal expansion is rescaled by a certain factor.

In the light-cone regions we must distinguish the right-directed solitons from the left-directed solitons. Thus for the right-directed solitons,

$$e_1 \sim -e_2 \;, \quad e_3 \sim 0 \;, \quad b \sim e_1 \;, \tag{15a}$$

and for the left-directed solitons,

$$e_1 \sim -e_2 \;, \quad e_3 \sim 0 \;, \quad b \sim -e_1 \;. \tag{15b}$$

The asymptotic values for the different solitons have a behavior similar to that of the coefficient f, as discussed in Sec. II. Thus for the solitons which never collide, $|e_1| \sim (1/f^K)\sqrt{w}/\sqrt{t}$. For the solitons which experience one collision, $|e_1| \sim (1/f^K)w^2\sqrt{w}/\sqrt{t}$, and so on. (We have assumed $w_i \sim w \ll |z_{ik}|$.) Hence the value of e_1 decreases with each collision. The sign of e_1 reverses in passing from right-directed to left-directed solitons.

The metric (6) is of Petrov type I, as is the Kasner metric, except in the light-cone regions where it is of Petrov type N, which is typical of pure gravitational waves. Thus, the metric (6) represents inhomogeneities propagating on a Kasner background which evolve towards pure gravitational waves. In this sense, the metric (6) shows how primordial gravitational waves can be produced as a consequence of irregularities near the cosmological singularity.

This interpretation is supported by studying the components of the Riemann tensor in the null tetrad consisting of the vectors $\vec{\mathrm{n}}$, $\vec{\mathrm{l}}$, and $\vec{\mathrm{m}}$:

$$\begin{aligned}
&\vec{\mathrm{n}}=\frac{1}{\sqrt{2f}}(\partial_t+\partial_z)\;, \quad \vec{\mathrm{l}}=\frac{1}{\sqrt{2f}}(\partial_t-\partial_z) \\
&\vec{\mathrm{m}}=\frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{g_{11}}}\partial_x+i\frac{1}{\sqrt{g_{22}}}\partial_y\right], \\
&\vec{\mathrm{m}}^*=\frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{g_{11}}}\partial_x-i\frac{1}{\sqrt{g_{22}}}\partial_y\right].
\end{aligned} \tag{16}$$

The nonvanishing components of the Riemann tensor in this tetrad are

$$\begin{aligned}
&\psi_0=R_{\mu\nu\alpha\beta}n^\mu m^\nu n^\alpha m^\beta=\tfrac{1}{2}(e_2-e_1)+b\;, \\
&\psi_2=\tfrac{1}{2}R_{\mu\nu\alpha\beta}n^\mu l^\nu(n^\alpha l^\beta-m^\alpha m^{*\beta})=-\tfrac{1}{2}e_3\;, \\
&\psi_4=R_{\mu\nu\alpha\beta}l^\mu m^{*\nu}l^\alpha m^{*\beta}=\tfrac{1}{2}(e_2-e_1)-b\;.
\end{aligned} \tag{17}$$

ψ_0 and ψ_4 represent the radiative part of the field, whereas ψ_2 contains the Coulomb part. ψ_0 gives the radiative component along the left-directed waves and ψ_4 along the right-directed waves.

In the light-cone regions $\psi_2 \sim 0$, so that only the radiative part remains. For the right-directed solitions $\psi_0 \sim 0$ and $\psi_4 \sim -2e_1$ while for the left-directed solitons $\psi_0 \sim -2e_1$ and $\psi_4 \sim 0$. Figure 3 shows the component ψ_0. First, only left-directed solitons appear. Afterwards, only the solitons that do not collide appear because the "intensity" of the left-directed solitons which collide is much smaller.

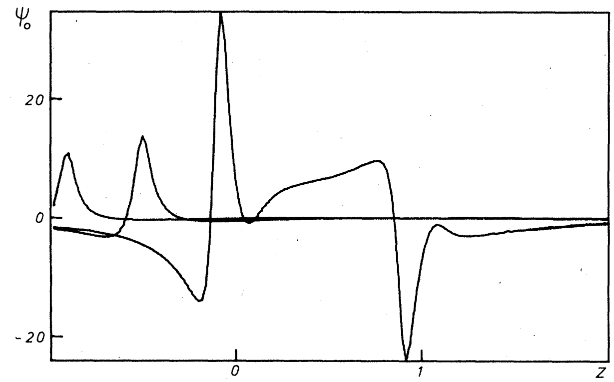

FIG. 3. This shows the time evolution of the Riemann component ψ_0 for $m=1$. The parameters are the same as in Fig. 2(a), but with the times increasing from right to left.

Finally one must study the scalar invariants of the Riemann tensor. For this metric there are only two invariants different from zero:

$$\begin{aligned}
&I_1=\tfrac{1}{2}(e_1{}^2+e_2{}^2+e_3{}^2-2b^2)\;, \\
&I_2=\tfrac{1}{6}(e_1{}^3+e_2{}^3+e_3{}^3+3b^2e_3)\;.
\end{aligned} \tag{18}$$

The evolution in time of one of these invariants has been given by Ibañez and Verdaguer[4] for $m=1$, and it clearly shows a solitonic behavior. The qualitative behavior of the other invariant is similar.

We can summarize this section by saying that the metric (6) represents intrinsic inhomogeneities propagating on a Kasner background which behave as classical solitons (i.e., they are localized and have shapes which are not changed by collisions as seen in Fig. 3) and evolve towards gravitational waves.

At this point a question naturally arises: do the solitons carry some kind of energy and can we define a velocity of propagation?

IV. THE VELOCITY

In general relativity we do not have a local definition for the energy of the gravitational field. Several quantities have been proposed for the energy density or energy flux of the gravitational field. One of these is the Bel-Robinson tensor $T^{\mu\nu\alpha\beta}$ which is defined as the gravitational analog of the electromagnetic stress-energy tensor. It therefore represents the energy density of local relative acceleration. However, the Bel-Robinson tensor does not have dimensions of an energy density, but of the square of an energy density. The Bel-Robinson tensor is given by

$$T^{\mu\nu\alpha\beta}=R^{\mu\rho\alpha\sigma}R^{\nu}{}_{\rho}{}^{\beta}{}_{\sigma}+{}^*R^{\mu\rho\alpha\sigma*}R^{\nu}{}_{\rho}{}^{\beta}{}_{\sigma}\;, \tag{19}$$

where ${}^*R^{\mu\nu\alpha\beta}$ is the dual tensor of $R^{\mu\nu\alpha\beta}$:

$${}^*R^{\mu\nu\alpha\beta}=\eta^{\mu\nu\rho\sigma}R_{\rho\sigma}{}^{\alpha\beta}\;. \tag{20}$$

$\epsilon=T^{\mu\nu\alpha\beta}u_\mu u_\nu u_\alpha u_\beta$ represents the superenergy density as seen by an observer with velocity u^μ ($u^\mu u_\mu=-1$), and

$$P^\alpha=-(\delta^\alpha_\mu+u^\alpha u_\mu)T^{\mu\nu\rho\sigma}u_\nu u_\rho u_\sigma \tag{21}$$

is the corresponding Poynting vector.

Now we define an orthonormal tetrad given by

$$\vec{e}_0=\frac{1}{\sqrt{f}}\partial_t\ ,\quad \vec{e}_1=\frac{1}{\sqrt{g_{11}}}\partial_x\ ,\quad \vec{e}_2=\frac{1}{\sqrt{g_{22}}}\partial_y\ ,$$
$$\vec{e}_3=\frac{1}{\sqrt{f}}\partial_z\ . \tag{22}$$

By projecting the Riemann tensor and its dual onto this tetrad, we can calculate the superenergy density and the Poynting vector, as seen by an observer at rest in this frame:

$$\epsilon=e_1{}^2+e_2{}^2+e_3{}^2+2b^2=\vec{E}^2+\vec{B}^2\ ,$$
$$P^0=P^1=P^2=0\ , \tag{23}$$
$$P^3=2(e_1-e_2)b=2(\vec{E}\times\vec{B})\ .$$

From (23) the analogy with electromagnetism is clear. There is only an energy flux along the z direction. For the Kasner metric, $P^3=0$ (since the magnetic component b vanishes). The effect of the solitons on the superenergy density is to diminish it, due to the expansion of the volume of the space produced by the passage of the solitons.

An interesting characteristic quantity associated with the solitons is their velocity of propagation. Belinskii and Fargion[3] defined this velocity as the velocity of the world line of the peak of the soliton field of the metric. In some instances the velocity defined in this way was greater than the velocity of light, and hence could not represent a physical velocity.

From (23), and bearing in mind the interpretation of the Bel-Robinson tensor as a quantity that in some sense describes the energy carried by the solitons, we define a velocity field associated with them. This velocity field describes, at least qualitatively, the velocity of the solitons and is defined as the ratio of the superenergy tensor flux (Poynting vector) to the superenergy density:

$$v(t,z)=\frac{P^3}{\epsilon}\ . \tag{24}$$

It is important to note that the definition (24) is not covariant, and that it is a velocity measured with respect to the orthonormal frame (22). Nevertheless, we think it is representative of the solitonic behavior of these inhomogeneities. From (23) we obtain

$$v=\frac{2(e_1-e_2)b}{e_1{}^2+e_2{}^2+e_3{}^2+2b^2}\ . \tag{25}$$

Initially at $t\to 0$, the velocity field satisfies $|v|\sim 0$ everywhere. On the light-cone regions $|v|\to 1-O(1/t)$, and in the far region and interaction region $|v|\sim 0$. This can be interpreted as having a flux of radiation localized mainly on the light cones from the origin. First, at the origin, the two competing fluxes, right and left directed, almost cancel. Later, as the right- (left-) directed radiation flux gets far from the origin, the canceling effect of the left- (right-) directed flux is gradually reduced.

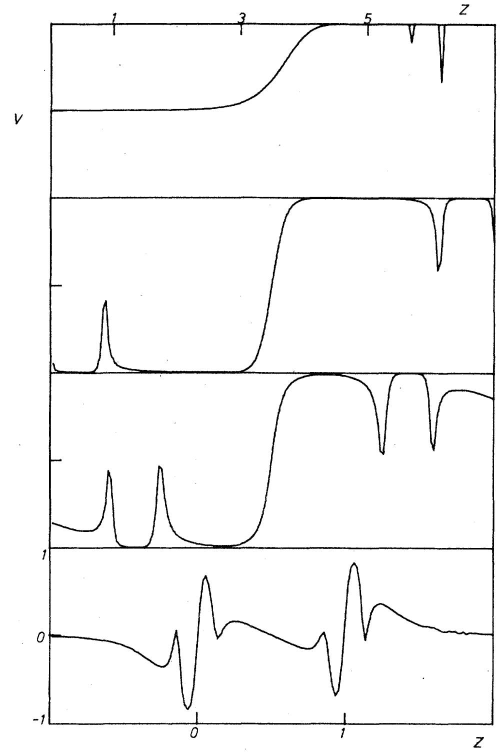

FIG. 4. Time evolution of the soliton velocity field v for $m=1$. The parameters and time sequence are the same as in Fig. 2(a), but $w_1=w_2=0.08$. The top curve shows the velocity field at $t=5$.

In Fig. 4 the time evolution of the velocity field of the metric is shown for the case $m=1$. Initially one can see the four solitons, with some tails, moving in opposite directions from their points of origin. The two inner solitons collide, while the two outer solitons become bounded by a step decreasing in the velocity modulus. In the interaction region for the model considered, we soon have many points moving nearly at the speed of light. In Fig. 4 this velocity field at a much later time is shown. The irregular tails present at earlier times have disappeared and only a smooth and localized soliton propagating at the speed of light remains. An interpretation of this fact could be that the gravitational solitons represent the resulting net flux of localized energy. Initially their motion is slow and particlelike, but later they approach the speed of light leaving behind their slow tails and becoming pure gravitational waves.

Although this velocity is given in a peculiar frame, it is never greater than one which shows that one can interpret it as the velocity propagation of the gravitational field (and this qualitative result is independent of the reference frame).

V. OPTICAL SCALARS

For a metric of the type being considered here, there is a preferred null geodesic congruence and two optical scalars, the expansion and shear of this null congruence, which intrinsically characterize the metric. Therefore it is of interest to study the effect produced by the solitons on these scalars.

Another reason for their study, as emphasized in a previous paper,[4] is that several authors (Kahn and Penrose,[10] Szekeres,[11] and Nutku and Halil[12]) have given examples of colliding plane gravitational waves on flat space, in which a singularity inevitably appears. Later Tipler[13] proved that the collision of any plane waves, gravitational or electromagnetic, requires a singularity, either in the past or in the future of the interaction. Centrella and Matzner[6] studied the collision of plane gravitational waves in an expanding vacuum cosmology and found that the expansion avoids the singularity. However, Tipler's theorem still holds because a singularity occurs in the past.

Here we see an explicit example of this by studying the focusing effect of the solitons upon a null congruence defined by a null vector n defined by the conditions $n^0=n^3=n$ and $n^1=n^2=0$. The geodesic equation for such a null vector is simply

$$(\partial_t+\partial_z)(nf)=0 . \tag{26}$$

We choose the normalization

$$n=\frac{1}{\sqrt{2f}} , \quad \vec{n}=\frac{1}{\sqrt{2f}}(\partial_t+\partial_z) . \tag{27}$$

From (27) we define a null tetrad:

$$\vec{n}=\frac{1}{\sqrt{2f}}(\partial_t+\partial_z) , \quad \vec{l}=\frac{1}{\sqrt{2}}(\partial_t-\partial_z) ,$$
$$\vec{m}=\frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{g_{11}}}\partial_x+\frac{i}{\sqrt{g_{22}}}\partial_y\right] , \tag{28}$$
$$\vec{m}^*=\frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{g_{11}}}\partial_x-\frac{i}{\sqrt{g_{22}}}\partial_y\right] .$$

The vectors $\vec{l}$, $\vec{m}$, and $\vec{m}^*$ are parallel transported along the null congruence defined by $\vec{n}$.

The expansion of this congruence is

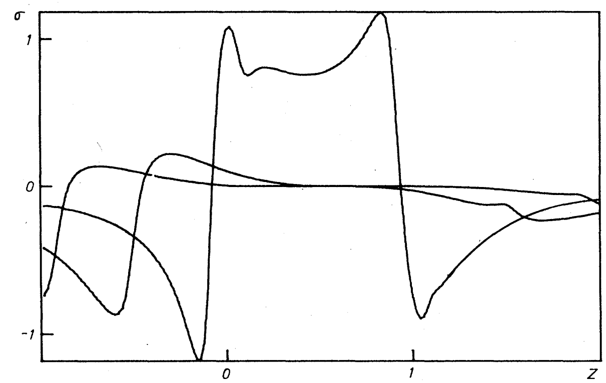

FIG. 5. Time evolution of the shear σ produced by the solitons on the null geodesic congruence defined by the vector $\vec{n}$. The parameters and time sequence are the same as in Fig. 2(a).

$$\theta=\tfrac{1}{2}n^{\mu}{}_{;\mu}=\frac{1}{2\sqrt{2}}\frac{1}{ft} . \tag{29}$$

Unless $t\rightarrow\infty$, θ cannot go to zero, and always remains positive ($\theta>0$). Therefore these null rays cannot converge to a singularity in the future.[4] For an $m=1$ metric, the expansion was given in Ref. 4.

The shear of the congruence is given by

$$\sigma=-n_{\mu;\nu}m^{\mu}m^{\nu}=\frac{1}{2\sqrt{2}}\frac{1}{f}\left[\frac{1}{t}-\frac{\dot{g}_{11}}{g_{11}}-\frac{g'_{11}}{g_{11}}\right] . \tag{30}$$

Figure 5 shows the effect of the solitons upon the shear of the congruence n^{μ} (in the case considered in this figure, $\sigma^K=0$). The shear suffers a jump as the congruence crosses a soliton, so the left-directed solitons appear more clearly than the right-directed solitons. Compare this figure with the Riemann component ψ_0 in Fig. 3. After the passage of the solitons the shear slowly approaches the Kasner shear. This behavior is very similar to the collision of plane gravitational waves in an expanding cosmology.[6]

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