Particle and string fluid interpretation for the scalar sector of Kaluza-Klein theories

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The scalar sector of the effective low-energy six-dimensional Kaluza-Klein theory is seen to represent an anisotropic fluid composed of two perfect fluids if the extra space metric has a Euclidean signature, or a perfect fluid of geometric strings if it has an indefinite signature. The Einstein field equations with such fluids can be explicitly integrated when the four-dimensional spacetime has two commuting Killing vectors.

I. INTRODUCTION

In modern Kaluza-Klein theories[1,2] the extra space is considered a compact space of the size of the Planck length whose isometries are the gauge symmetries of some gauge theory.[3] Fourier expanding the N-dimensional metric tensor in terms of the extra coordinates one finds, at the zero mode or low-energy limit, an effective four-dimensional theory for the coupling of gravity with the vector fields (Yang-Mills gauge bosons) associated with the mixed components of the N-dimensional metric and the scalar fields (Higgs bosons) associated with the components of the extra space metric.[3,4]

The scalar fields are an essential ingredient to obtain any realistic effective low-energy theory consistent with the higher-dimensional field equations.[5] Thus, for instance, in five dimensions if we neglect the scalar field by setting $g_{55}=\text{const}$ the five-dimensional equations lead to the four-dimensional Einstein-Maxwell equations with an unacceptable restriction on the electromagnetic field: $F_{\mu\nu}F^{\mu\nu}=0$. The scalar field g_{55} or dilaton is related to the size of the extra space and may lead to dynamical compactification as a result of classical cosmologic evolution,[6] or by the effect of quantum fluctuations about the classical configuration of the dilaton which give the dilaton a mass and the size of the extra space.[7]

When the scalar field is included in five dimensions, the low-energy theory is the Jordan and Thiry four-dimensional theory[8] of coupled gravity and electromagnetism with a massless scalar field like in the Brans-Dicke theory. But unlike the Brans-Dicke theory, which contains an arbitrary constant,[9,10] here the constant is fixed, as it should be in a truly unified theory, by the general five-dimensional covariance.[4,11] However, the theory may be transformed into the source-free Brans-Dicke theory by means of a conformal transformation which involves an arbitrary parameter of the four-dimensional metric or, also, it may be transformed to the Einstein equations with a stiff fluid.[12,13] In this respect one may as well consider the use of extra dimensions as a useful auxiliary tool to obtain meaningful four-dimensional theories.

Here we investigate the low-energy limit of the six-dimensional theory and its four-dimensional interpretation. For simplicity in the interpretation we consider the scalar sector only. We allow the possibility that the extra space has an indefinite metric. The existence of a closed timelike curve with a radius the size of the Planck time causes no problem at the zero-mode approximation but implies tachyons for the massive fields associated to the higher modes. We find that when the metric of the extra space has the Euclidean signature the four-dimensional theory describes the coupling of the Brans-Dicke field with an anisotropic fluid composed of two perfect-fluid components.[14] On the other hand, when the extra space has an indefinite metric the four-dimensional theory describes the coupling of the Brans-Dicke field with a perfect fluid of geometric strings.[15] There is a clear correspondence between the description of an anisotropic distribution of pointlike objects and the description of an anisotropic distribution of stringlike objects.

Besides the interest this unified picture may have for the description of the interaction of gravity with the Brans-Dicke field and such fluids, it has also an important practical application: the integration of four-dimensional field equations.

In fact if we assume that the N-dimensional metric coefficients depend on two coordinates only, the higher-dimensional Einstein equations can be integrated by means of a generalization of the solution-generating technique of Belinsky and Zakharov.[16–19] As a consequence the four-dimensional Einstein equations with the Brans-Dicke field and the fluids described are automatically integrated.

In Sec. II we describe the four-dimensional effective low-energy theory of the scalar sector of the Kaluza-Klein theory and by means of conformal transformations it is related to other scalar theories. The fluid interpretation is given in Sec. III and it may be generalized to more dimensions. In Sec. IV the integration of the field equations when the four-dimensional metric has two commuting Killing vectors is considered.

II. EFFECTIVE FOUR-DIMENSIONAL THEORY

An N-dimensional Kaluza-Klein theory is based on a N-dimensional metric γ_{AB}

$$ds^2=\gamma_{AB}dx^Adx^B=\gamma_{\mu\nu}dx^\mu dx^\nu+2\gamma_{\mu b}dx^\mu dx^b+\gamma_{ab}dx^a dx^b\ , \tag{1}$$

which we split in the usual four coordinates and $N-4$ extra coordinates with $A,B=1,\ldots,N$, $\mu,\nu=0,1,2,3$, and $a,b=5,\ldots,N-4$. The general N-dimensional covariance leads to the N-dimensional Einstein field equations in a vacuum; in the simplest Kaluza-Klein theories no extra fields are assumed; that is,

$$R_{AB}=0\ . \tag{2}$$

If we assume that the extra space is a compact space of small size, usually the size of the Planck length, the metric components γ_{AB} can be Fourier expanded in terms of the extra coordinates. The zero mode or low-energy limit is obtained by making the restrictions (a) that the extra space-time has the topology of a torus and (b) that γ_{AB} has no dependence on the extra coordinates, i.e.,

$$\gamma_{AB,a}=0\ . \tag{3}$$

The theory can be written in terms of an effective four-dimensional metric

$$g_{\mu\nu}=\gamma_{\mu\nu}-\gamma_{ab}A^a{}_\mu A^b{}_\nu\ , \tag{4a}$$

where

$$A^a{}_\mu\equiv\gamma_{b\mu}\hat{\gamma}^{ba}\quad(\hat{\gamma}^{ab}\gamma_{bc}\equiv\delta^a{}_c)\ . \tag{4b}$$

It is verified that

$$g^{\mu\nu}=\gamma^{\mu\nu},\quad \gamma^{a\mu}=-A^{a\mu},\quad \gamma^{ab}=\hat{\gamma}^{ab}+A^{a\mu}A^b{}_\mu\ .$$

When the restrictions (a) and (b) are imposed the coefficients $A^a{}_\mu$ represent $N-4$ vector fields on the space of metric $g_{\mu\nu}$ associated with the Yang-Mills vector bosons and the coefficients γ_{ab} are $(N-4)\times(N-3)/2$ scalar fields associated with the Higgs bosons.[3] Note that, in general, there is no matching between the number of massless vector fields and the number of extra dimensions and that the zero modes are not independent of the extra coordinates.

We shall single out the scalar field ϕ defined in terms of the Higgs fields by

$$\det(\gamma_{ab})=\epsilon\phi^2,\quad \epsilon=\pm1\ , \tag{5}$$

which is obviously related to the size of the extra space (dilaton field).[4]

Here we admit the possibility of having extra timelike coordinates ($\epsilon=-1$). Note that if we have, for instance, a five-dimensional theory with a closed timelike curve, say τ, in a circle of radius T of the order of the Planck time, this causes no problem for the effective low-energy theory; however, this theory contains an infinite tower of massive particles with imaginary masses (tachyons). In fact, Fourier expanding the metric tensor

$$\gamma_{AB}(x^\mu,s)=\sum_{\mu=-\infty}^{\infty}\gamma_{AB}(x^\mu)e^{in\tau/T}\ ,$$

and substituting into the five-dimensional field equations, the extra modes admit a four-dimensional interpretation as massive particles with masses $m^2=-n^2/T^2$ (tachyons). For the $n=0$ modes we have massless particles and no tachyon problem. For this reason and in order to allow a greater generality we shall assume that ϵ is arbitrary. On the other hand, if one wishes to take the viewpoint of considering the extra dimensions simply as a useful device to obtain meaningful four-dimensional theories, then this possibility must be considered.

We wish to consider the scalar sector of the theory only; i.e., we now assume that

$$A^a{}_\mu=0\quad(\gamma_{a\mu}=0)\ , \tag{6}$$

which is compatible with the field equations. Now the field equations (2) in terms of the four-dimensional metric $g_{\mu\nu}$ can be written as

$$\bar{R}_{\mu\nu}=-\phi^{-1}\phi_{,\mu;\nu}+\phi^{-2}\phi_{,\mu}\phi_{,\nu}+\tfrac{1}{4}\hat{\gamma}^{ab}{}_{,\mu}\gamma_{ab,\nu}\ , \tag{7}$$

$$(\phi\gamma_{ab,\mu}\hat{\gamma}^{ac})^{;\mu}=0\ , \tag{8a}$$

$$\phi_{,\mu}{}^{;\mu}=0\ , \tag{8b}$$

where $\bar{R}_{\mu\nu}$ denotes the Ricci tensor for the metric $g_{\mu\nu}$ and all derivatives are taken in terms of such four-dimensional metrics. Equation (7) is a consequence of $R_{\mu\nu}=0$, Eq. (8a) follows from $R_{ab}=0$ and (8b) is just the trace of (8a), the equations $R_{\mu a}=0$ are identically verified in view of the assumption $\gamma_{\mu a}=0$.

We now restrict the field equations to $N=6$ and introduce the ξ and ψ fields:

$$\gamma_{ab}\equiv\phi\xi^{-1}\begin{bmatrix}1 & \psi\\ \psi & \psi^2+\epsilon\xi^2\end{bmatrix}\ . \tag{9}$$

Equation (7) now reads

$$\bar{R}_{\mu\nu}=-\phi^{-1}\phi_{\mu;\nu}+\tfrac{1}{2}\phi^{-2}\phi_{,\mu}\phi_{,\nu}-\tfrac{1}{2}\xi^{-2}(\xi_{,\mu}\xi_{,\nu}+\epsilon\psi_{,\mu}\psi_{,\nu})\ , \tag{10}$$

which together with (8) are the field equations coupling gravity with a Brans-Dicke massless scalar field ϕ or dilaton with the coupling constant[10] $\omega=-\frac{1}{2}$ and an energy-momentum tensor $T_{\mu\nu}$ which we can extract from the ξ and ψ parts of (10); assuming $8\pi G=c=1$, it is

$$T_{\mu\nu}=\xi^{-2}[\xi_{,\mu}\xi_{,\nu}+\epsilon\psi_{,\mu}\psi_{,\nu}-\tfrac{1}{2}g_{\mu\nu}(\xi_{,\alpha}\xi^{,\alpha}+\epsilon\psi_{,\alpha}\psi^{,\alpha})]\ . \tag{11}$$

The fluid interpretation of (11) will be discussed in the next section.

We shall now discuss related scalar theories. First we should remark that the Brans-Dicke coupling constant ω has been determined as a consequence of the six-dimensional general covariance; moreover, the coupling of the fluid tensor $T_{\mu\nu}$ with the Brans-Dicke field is also determined by this general covariance and is a consequence of the Bianchi identities for the Ricci tensor $\bar{R}_{\mu\nu}$:

$$(\bar{R}^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}\bar{R})_{;\nu}=0 \ . \tag{12}$$

Looking at (11) as an effective four-dimensional theory, it may be related to other scalar theories by means of conformal transformations.[12,20] In fact, define a new four-dimensional metric tensor $\hat{g}_{\mu\nu}$:

$$\hat{g}_{\mu\nu}=\phi^{1-\Omega}g_{\mu\nu} \ , \tag{13}$$

where Ω is an arbitrary parameter. In terms of this new metric the Ricci tensor $\hat{R}_{\mu\nu}$ may be written

$$\hat{R}_{\mu\nu}=-\Omega\phi^{-1}\phi_{,\mu;\nu}+\frac{1+(\Omega-1)(\Omega+3)}{2}\phi^{-2}\phi_{,\mu}\phi_{,\nu} -\tfrac{1}{2}\xi^{-2}(\xi_{,\mu}\xi_{,\nu}+\epsilon\psi_{,\mu}\psi_{,\nu}) \ , \tag{14}$$

an equation similar to (8) but with all derivatives taken in terms of the metric $\hat{g}_{\mu\nu}$.

Now for $\Omega=0$ we may define a new field

$$\Phi=\ln\phi \tag{15}$$

and (14) becomes

$$\hat{R}_{\mu\nu}=-\Phi_{,\mu}\Phi_{,\nu}-\tfrac{1}{2}\xi^{-2}(\xi_{,\mu}\xi_{,\mu}+\epsilon\psi_{,\mu}\psi_{,\nu}) \ , \tag{16a}$$

with

$$\Phi_{,\mu}{}^{;\mu}=0 \ . \tag{16b}$$

When $\Phi_{,\mu}$ is timelike the massless scalar field Φ can be interpreted as the potential field for a Tabensky-Taub fluid[13] (stiff fluid) with energy and pressure densities

$$\rho=p=-\tfrac{1}{2}\Phi_{,\mu}\Phi^{,\mu} \ , \tag{17a}$$

and an irrotational four-velocity

$$U_\mu=\Phi_{,\mu}/(-\Phi_{,\alpha}\Phi^{,\alpha})^{-1/2} \ . \tag{17b}$$

Therefore, Eqs. (16) together with (8a) describe the coupling of a stiff fluid with the fluid described by (11).

Similarly, for $\Omega\neq0$ we may define a new scalar field

$$\Phi=\phi^{\Omega}$$

and (14) becomes

$$\hat{R}_{\mu\nu}=-\Phi^{-1}\Phi_{,\mu;\nu}-\omega\Phi^{-2}\Phi_{,\mu}\Phi_{,\nu} -\tfrac{1}{2}\xi^{-2}(\xi_{,\mu}\xi_{,\nu}+\epsilon\psi_{,\mu}\psi_{,\nu}) \ , \tag{18a}$$

with

$$\Phi_{,\mu}{}^{;\mu}=0 \ , \tag{18b}$$

where

$$\omega\equiv\Omega^{-2}(1-\Omega-3\Omega^2/2) \ .$$

This describes the Brans-Dicke theory[9,10] for a massless scalar field with arbitrary parameter ω coupled to a fluid described by (11).

These field equations represent the natural generalization of the Jordan and Thiry theory[8] when the Yang-Mills fields A_μ^a are absent; of course the introduction of the fields A_μ^a is straightforward using (2)–(5) but we have included the scalar sector only (γ_{ab}) because it leads to a simple fluid interpretation.

III. PARTICLE AND STRING FLUID INTERPRETATION FOR $T_{\mu\nu}$

We turn now to the fluid interpretation for the energy-momentum tensor $T_{\mu\nu}$ defined in (11). To this aim we introduce two new vector fields A_μ and B_μ,

$$A_\mu\equiv\cos(\sqrt{\epsilon}\theta)\xi_{,\mu}+\sqrt{\epsilon}\sin(\sqrt{\epsilon}\theta)\psi_{,\mu} \ ,$$
$$B_\mu\equiv-\frac{1}{\sqrt{\epsilon}}\sin(\sqrt{\epsilon}\theta)\xi_{,\mu}+\cos(\sqrt{\epsilon}\theta)\psi_{,\mu} \ , \tag{19a}$$

and impose the orthonormality of the vectors $A^\mu B_\mu=0$ to fix θ:

$$\frac{\tan(2\sqrt{\epsilon}\theta)}{2\sqrt{\epsilon}}=\frac{\psi_{,\alpha}\psi^{,\alpha}}{\xi_{,\alpha}\xi^{,\alpha}-\epsilon\psi_{,\alpha}\psi^{,\alpha}} \ . \tag{19b}$$

If we assume that A_μ is timelike then B_μ is spacelike and we may define unitary timelike and spacelike vectors

$$U_\mu\equiv A_\mu/(-A_\alpha A^\alpha)^{-1/2}, \quad \chi_\mu\equiv B_\mu/(B_\alpha B^\alpha)^{-1/2} \ , \tag{19c}$$

as a fluid four-velocity and anisotropy direction which verify

$$U_\mu U^\mu=-\chi_\mu\chi^\mu=-1, \quad \chi_\mu U^\mu=0 \ . \tag{20}$$

Now making use of the relations from (19),

$$A_\mu A^\mu=\tfrac{1}{2}(\xi_{,\alpha}\xi^{,\alpha}+\epsilon\psi_{,\alpha}\psi^{,\alpha})+\tfrac{1}{2}D \ ,$$
$$B_\mu B^\mu=\frac{\epsilon}{2}(\xi_{,\alpha}\xi^{,\alpha}+\epsilon\psi_{,\alpha}\psi^{,\alpha})-\tfrac{1}{2}D \ , \tag{21a}$$

$$\sin(2\sqrt{\epsilon}\theta)=2\sqrt{\epsilon}\xi_{,\alpha}\psi^{,\alpha}/D \ ,$$
$$\cos(2\sqrt{\epsilon}\theta)=(\xi_{,\alpha}\xi^{,\alpha}-\epsilon\psi_{,\alpha}\psi^{,\alpha})/D \ , \tag{21b}$$

where

$$D\equiv[(\xi_{,\alpha}\xi^{,\alpha}-\epsilon\psi_{,\alpha}\psi^{,\alpha})^2+4\epsilon(\psi_{,\alpha}\xi^{,\alpha})^2]^{1/2} \ , \tag{22}$$

we may substitute into (11) to find

$$T_{\mu\nu}=(\rho+\Pi)U_\mu U_\nu+(\epsilon\sigma-\Pi)\chi_\mu\chi_\nu+\Pi g_{\mu\nu} \ , \tag{23a}$$

where

$$\Pi=\tfrac{1}{2}(\xi_{,\alpha}\xi^{,\alpha}+\epsilon\psi_{,\alpha}\psi^{,\alpha}) \ , \tag{23b}$$

$$\rho=\epsilon\sigma=(2\xi^2)^{-1}D \ . \tag{23c}$$

For $\epsilon=1$, this is the energy-momentum tensor for an anisotropic fluid[14] with four-velocity U^μ and anisotropic direction χ^μ. It has a rest energy density ρ, σ pressure along the anisotropic direction, and Π pressure on the direction perpendicular to χ^μ. This anisotropic fluid may be formed by two irrotational perfect fluids with fluid potentials ξ^{-1} and ψ, respectively.[21]

Note that the anisotropic fluid verifies a stiff equation of state along the anisotropic direction ($\rho=\sigma$).

For $\epsilon=-1$ the energy-momentum tensor has a quite different interpretation. In fact, we define the bivector $\Sigma^{\mu\nu}$,

$$\Sigma^{\mu\nu}\equiv 2U^{[\mu}\chi^{\nu]} , \tag{24a}$$

as the surface-forming bivector of the average string world sheet[15] for a fluid element of strings; much as the field U^μ represents the average velocity for a fluid element of pointlike particles. Then we have

$$\Sigma^{\mu\alpha}\Sigma_\alpha{}^\nu=U^\mu U^\nu-\chi^\mu\chi^\nu \tag{24b}$$

and

$$\Sigma^{\mu\alpha}\Sigma_{\alpha\mu}=2 . \tag{25}$$

This last relation is the usual gauge for strings which corresponds to the usual normalization (20) for U_μ when dealing with pointlike particles.

Thus the energy-momentum tensor (23) for $\epsilon=-1$ can be written as

$$T^{\mu\nu}=(\rho+\Pi)\Sigma^{\mu\alpha}\Sigma_\alpha{}^\nu+\Pi g_{\mu\nu} , \tag{26}$$

which is the energy-momentum tensor for a perfect fluid of ordered strings[15] with pressure Π and energy (string tension) density ρ. When $\Pi=0$ it reduces to a cloud of ordered strings which is the generalization for strings of dust for point particles. A cloud of ordered strings is formed by a set of parallel strings which are ramdomly distributed on a plane perpendicular to them. This cloud is similar to a dust of directed radiation.[22] The inclusion of Π in (26) introduces an interaction among the strings which form the cloud. The fluid of ordered strings should not be mistaken with a cloud of completely ramdonized strings which is equivalent to a usual perfect fluid of point particles with $p=-\frac{1}{3}\rho$ equation of state.[23] Fluids of strings and generalizations in multidimensional spaces are under active consideration by the authors.

If we take the particular solution $\phi=1$ of (8b), that is, we assume no Brans-Dicke field equations (10) are just Einstein field equations for either an anisotropic fluid of particles or a perfect fluid of strings and for both fluids the energy-momentum tensor obeys, as a consequence of the Bianchi identities (12), the conservation law

$$T^{\mu\nu}{}_{;\nu}=0 , \tag{27}$$

then the fluid fields ξ and ψ are determined by Eq. (8a).

When ϕ is different from a constant, then Eq. (27) is not verified by the fluids described. The conservation equation derived from (12) does include the Brans-Dicke field, thus representing the coupling of the Brans-Dicke field with these fluids. Of course the same applies if we consider the conformal transformed equations (16) or (18) for the coupling of stiff fluid or a Brans-Dicke field with an arbitrary coupling parameter.

We may generalize this description to any even number of extra dimensions. By decomposing the extra metric in 2×2 boxes we obtain a description of a multifluid when the extra metric has signature $++++\cdots$, a superposition of perfect fluids of strings when we consider $+-+-\cdots$, or a superposition of the two systems if we consider $+++-\cdots$. The individual fluids do not verify the conservation law (27) but their superposition does, representing the interaction among them. The anisotropic fluid interpretation for the superposition of an arbitrary number of scalar fields with two Killing vectors can be found in Ref. 24.

IV. INTEGRATION OF THE FIELD EQUATIONS

An interesting feature of the unified model presented here is that it is exactly solvable when the metric has certain symmetries. In fact, we may find exact solutions when the four-dimensional metric has two commuting Killing vectors.

The reason is that N-dimensional Einstein equations with a massless scalar field Φ,

$$R_{AB}=-\Phi_{,A}\Phi_{,B},\quad \Phi_{,A}{}^{;A}=0 , \tag{28}$$

can be solved by means of the Belinsky and Zakharov[16] (BZ) generating technique (BZ transformation) when the metric γ_{AB} depends on two coordinates only, say (t,z). This means that given a seed solution γ^0_{AB}, which may be flat space in appropriate coordinates, new solutions can be generated following a precise algorithm. The new solutions are usually called soliton solutions. Explicit solutions generalizing the original BZ four-dimensional solutions are known in five[17] and N dimensions.[18]

Now we can find solutions of the system (7) and (8) in the following way. Given a particular seed solution $g^0_{\mu\nu}$, ϕ^0, γ^0_{ab}, which depends at most on two coordinates (t,z), use (4) to obtain the N-dimensional metric $\gamma^0_{AB}(t,z)$ and apply the BZ transformation to obtain a new solition solution $\gamma_{AB}(t,z)$. By using Eqs. (4) again soliton solutions for $g_{\mu\nu}(t,z)$, $\phi(t,z)$, and $\gamma_{ab}(t,z)$ are obtained. Of course this system also applies when A^a_μ are non-null.

Explicit solutions involving the anisotropic fluid and the string fluid in a cosmological context are being considered by the authors.[25,26]

Note that if we assume the Brans-Dicke scalar field is constant, $\phi=1$, then Eq. (8) becomes the usual BZ equations to which the BZ transformation applies. In matrix form they are

$$(t\gamma_{,t}\gamma^{-1})_{,t}-(t\gamma_{,z}\gamma^{-1})_{,z}=0 , \tag{29}$$

where we have used the usual canonical coordinates[16] (t,z) with $t=\sqrt{\det g_{\mu\nu}}$ and

$$(\gamma_{ab,\mu}\hat{\gamma}^{bc})^{;\mu}=\frac{1}{\sqrt{-g}}(\sqrt{-g}\,g^{\mu\nu}\gamma_{ab,\nu}\gamma^{bc})_{,\mu} .$$

Equation (29) in cylindrical space-times with $\gamma(t,z)$ a $(N-4)\times(N-4)$ matrix is the BZ equation, and may be solved by the BZ transformation independently of the metric tensor $g_{\mu\nu}$. Thus one may find soliton solutions for the scalar fields γ_{ab} independently of the metric tensor $g_{\mu\nu}$. As shown in the preceding section, when $N=6$ and $\epsilon=1$ the systems (7) and (29) may be used to represent an anisotropic fluid in a gravitational field; soliton solutions of (29) were found in Ref. 21 to represent solitons of matter.

In general, when the Brans-Dicke field is considered, the system (8) may be written as

$$(t\phi\gamma_{,t}\gamma^{-1})_{,t}-(t\phi\gamma_{,z}\gamma^{-1})_{,z}=0\ ,$$
$$\phi_{,tt}+\frac{1}{t}\phi_{,t}-\phi_{,zz}=0\ , \tag{30}$$

which are not equations in the BZ form. This is essentially because of the cylindrical wave equation verified by $\phi=\sqrt{\det\gamma_{ab}}$ as opposed to the plane-wave equation verified by $\sqrt{\det g_{\mu\nu}}$.

The BZ transformation is not directly applicable to the system. However, soliton solutions for this system may be found as shown in the beginning of this section as part of the larger N-dimensional system, which implies a coupling of the fields $g_{\mu\nu}$, γ_{ab}, and ϕ. That is, we solve (7) and (30) simultaneously. We can see this more explicitly by noting that with the use of the canonical coordinate (α,β) part of the Einstein equations in vacuum (2), or with the massless scalar field (28), may be written in the BZ form as

$$(\alpha\gamma_{,\alpha}\gamma^{-1})_{,\alpha}-(\alpha\gamma_{,\beta}\gamma^{-1})_{,\beta}=0\ , \tag{31}$$

where γ is the $N\times N$ matrix $\gamma_{AB}(\alpha,\beta)$ and $\alpha=\sqrt{\det\gamma_{AB}}$.

The remaining Einstein equations are not relevant for the BZ transformation and can be solved once $\gamma_{AB}(\alpha,\beta)$ are known. The field equations (30) can be regarded as a subsystem of the field equations (31); soliton solutions for the scalar sector γ_{ab} cannot be solved independently of the metric tensor $g_{\mu\nu}$ when there is a Brans-Dicke field or dilaton.

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