Cosmic strings and Einstein-Rosen soliton waves

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(Received 5 June 1987)

We consider all generalized soliton solutions of the Einstein-Rosen form in the cylindrical context. They are Petrov type-I solutions which describe solitonlike waves interacting with a line source placed on the symmetry axis. Some of the solutions develop a curvature singularity on the axis which is typical of massive line sources, whereas others just have the conical singularity revealing the presence of a static cosmic string. The analysis is based on the asymptotic behavior of the Riemann and metric tensors, the deficit angle, and a *C*-velocity associated to Thorne's *C*-energy. The *C*-energy is found to be radiated along the null directions.

I. INTRODUCTION

In recent years considerable attention has been devoted to the possible existence of cosmic strings. It is predicted by current grand unified theories (GUT's) with symmetry breaking that a network of cosmic strings may have been formed as a consequence of phase transitions in the early Universe.[1]

As the Universe expands the cosmic strings evolve into essentially two types:[2] closed loops and open infinite strings. Closed loops oscillate, emit their energy into gravitational energy, and eventually disappear; they are supposed to be relevant in the important, and as yet unsolved, problem of the formation of galaxies[3] and clusters of galaxies.[4] Open strings, on the other hand, expand and straighten out and a few of them survive to the present day as straight-line cosmic strings; they also may have cosmological consequences: they produce light deflection and act as gravitational lenses;[2,5] they produce steplike discontinuities in the microwave background,[5,6] and may be responsible for the large-scale structures of the order of 100 Mpc in the Universe.[7]

Straight strings have stimulated the interest in exact solutions of Einstein's equations with cylindrical symmetry. The reason is that their gravitational effects are described by solutions with a cylindrical source formed by an anisotropic fluid. The equation of state of the fluid is that the longitudinal pressure equals the negative of its energy density. The radius of the cylinder is related to the Compton wavelength of the Higgs scalar responsible for the symmetry breaking[1,2] of the corresponding GUT and is therefore very small ($\lambda \sim 10^{-30}$ cm). The assumption of a line source is a good approximation for a macroscopic description.

The gravitational effects produced by an isolated static straight string are described outside the source by flat space minus a wedge and this is true for line sources[8-10] as well as for extended sources.[9,11] For a line source the metric has a conical singularity in the symmetry axis: the metric has no curvature singularities and the axis is quasiregular.[12] This is simply described by a deficit angle that is directly related to the energy density per unit length of the string.[8]

Of course an isolated cosmic string is an idealization. A string will normally interact with particles and radiation surrounding it.[1,13] In particular one might like to consider the interaction with gravitational radiation. The interaction of a string with gravitational radiation may be considered by yet another idealization: namely, that of considering radiation with cylindrical symmetry too. This allows one to deal with exact solutions. In fact, this approach has been followed by Xanthopoulos[14,15] who gives an exact solution describing a rotating string surrounded by nonradiating cylindrical gravitational waves.

In this paper we follow this approach and give a family exact solutions of Einstein's equations representing a cosmic string surrounded by gravitational radiation in the form of Einstein-Rosen solitonlike waves. The metrics have no curvature singularities and approach flat space far from the axis. The string itself is static and does not radiate, it is described by a quasiregular axis and a deficit angle. Besides the above solutions the family also includes solutions with curvature singularities on the symmetry axis which represent a massive line source of ordinary matter surrounded by gravitational radiation.

All the solutions are of Petrov type I and the soliton waves become pure gravitational radiation at null infinity. This makes our solutions qualitatively different from those given by Xanthopoulos, which are Petrov type D and nonradiative.

The plan of the paper is the following. In Sec. II we consider cylindrically symmetric metrics of the Einstein-Rosen form. For these metrics we review Thorne's *C*-energy, introduce an associated *C*-velocity, and consider the deficit angle. These quantities together with the Riemann tensor will then be useful for the study and physical interpretation of our solutions.

In Sec. III we give the explicit solutions. They are generalized soliton solutions which we divide in two classes: class A and class B. Class A solutions are simple generalizations of solutions considered previously[16] and class B solutions are new; although they have partially been considered in a cosmological context[17,18] they were not completely integrated. In this respect our solutions may be also relevant in the cosmological context. We also define four asymptotic regions of the spacetime where the study of the solutions is particularly useful.

The main features of solutions belonging to class A (B) may be inferred from the study of two types of simple models, called A1 (B1) and A2 (B2). To the first type, A1 and B1, we devote Sec. IV, where the asymptotic properties of such solutions are considered. Section V is concerned with the second type, A2 and B2; for these solutions, a perturbative analysis is also performed.

Finally in Sec. VI the physical interpretation of all the solutions is given and the main conclusions summarized. Class A2 and B2 solutions turn out to be the most relevant to the subject of cosmic strings since they admit an interpretation in terms of finite perturbations propagating on the background field of a static cosmic string or a massive line source. Although some of the class A1 and B1 solutions may be seen to contain a cosmic string, the surrounding gravitational field has asymptotic properties which depart from asymptotic flatness.

II. CYLINDRICAL SYMMETRY, C-ENERGY, C-VELOCITY, AND DEFICIT ANGLE

A metric with whole-cylinder symmetry, i.e., cylindrical symmetry with hypersurface orthogonal Killing fields,[19] can be written in the Einstein-Rosen form[20] as

$$ds^2=f(t,\rho)(d\rho^2-dt^2)+\rho(e^{\phi(t,\rho)}d\varphi^2+e^{-\phi(t,\rho)}dz^2)\ , \tag{2.1}$$

and Einstein's field equations in vacuum can be written as

$$\phi_{,\rho\rho}+\rho^{-1}\phi_{,\rho}-\phi_{,tt}=0\ , \tag{2.2a}$$

$$(\ln f)_{,t}=\rho\phi_{,t}\phi_{,\rho}$$
$$(\ln f)_{,\rho}=-(2\rho)^{-1}+(\rho/2)[(\phi_{,\rho})^2+(\phi_{,t})^2]\ , \tag{2.2b}$$

$\phi(t,\rho)$ represents the unique degree of freedom (one polarization) of the Einstein-Rosen radiation. A solution of the linear equation (2.2a) determines the $f(t,\rho)$ coefficient by means of (2.2b).

A. C-energy

For a cylindrical system Thorne[21] was able to introduce a total energy called the C-energy, and a contravariant C-energy flux vector P^μ obeying a conservation law $P^\mu_{;\mu}=0$. The energy density is localizable and locally measurable. An observer with four-velocity u^μ measures a C-energy density $P^\mu u_\mu$ and if X^μ is a spacelike vector such that $X_\mu u^\mu=0$, the C-energy flux is $P^\mu X_\mu$.

The four-vector P^μ is derived from a C-energy potential $C(\rho,t)$,

$$2C(\rho,t)=\ln f+\ln\rho-\phi\ , \tag{2.3}$$

which is proportional to the total C-energy contained inside a cylinder of radius ρ per unit coordinate length. It is given by

$$(P^t,P^\rho)\equiv(8\pi\rho f)^{-1}(C_{,\rho},-C_{,t}) \tag{2.4}$$

and its components represent the local energy density measured by a local observer with four-velocity $u^\mu=(1,0,0,0)$ and the local flux measured by that observer along the ρ direction. We take $G=c=1$.

B. C-velocity

Some of the metrics considered in this paper can be interpreted as gravitational localized soliton waves propagating on the background produced by a cosmic string or a massive line source. One would like to define a velocity associated to these waves since they presumably carry gravitational energy. In Ref. 16 a coordinate velocity of the maximum of the perturbations was considered, it was found that such velocity was greater than one and one could not interpret the perturbations as the propagation of a physical effect but rather as an interference phenomenon. Here we shed light on such interpretation by following a different approach based on the C-energy flux.

Since for metric (2.1) the C-energy flux vector P^μ is a timelike vector, we may define a four-velocity $u^\mu=P^\mu/(-P^\alpha P_\alpha)$. Then an observer with four-velocity u^μ will measure no flux of the C-energy. We shall call the C-velocity to the coordinate three-velocity of such an observer: $v^i=u^i/u^0$ $(i=1,2,3)$. It may be written, in our coordinate system, in terms of $\psi=\ln\rho-\phi$ as

$$v^\rho=-2\psi_{,\rho}\psi_{,t}[(\psi_{,\rho})^2+(\psi^2_{,t})]^{-1}\ . \tag{2.5}$$

We note that a similar three-velocity may be defined in Minkowski space with the electromagnetic field using the Poynting flux "vector" $T^{0\mu}$ which also verifies $T^{0\mu}{}_{,\mu}=0$ (T^μ is the electromagnetic energy-momentum tensor).

The three-velocity v^i may be written in this case as[22] $v^i=\tanh(\alpha)n^i$, where

$$\tanh(2\alpha)n^i=T^{0i}/T^{00}=2(\mathbf{E}\times\mathbf{B})^i(\mathbf{E}^2+\mathbf{B}^2)^{-1}$$

(the factor 2 in the tanh is due to the tensorial character of $T^{\mu\nu}$). An observer propagating at such velocity measures no flux of electromagnetic energy. For a null electromagnetic field ($\mathbf{E}^2=\mathbf{B}^2,\mathbf{E}\cdot\mathbf{B}=0$) such speed is the speed of light. In analogy we shall see later that for the gravitational case, in the regions where gravitational radiation dominates, the C-velocity approaches unity.

C. Deficit angle

A straight-line cosmic string is being characterized by a conical singularity along the axis. The condition for regularity in the coordinate system (2.1) is that[19] $\lim_{\rho\to 0}X_{,\mu}X^{\mu}_{,}/4X\to 1$, where $X\equiv|\partial_\varphi|^2$. When a metric has no curvature singularity along the axis and it fails to be regular, i.e., the above limit differs from unity, the axis contains a cosmic string. Equivalently, a cosmic string may also be identified by the deficit angle near the axis which measures the deviation from local flatness around the axis.[2] The deficit angle is defined by

$$\Delta\varphi(\rho)\equiv 2\pi-\int_0^\rho(g_{\varphi\varphi})^{1/2}d\varphi\Big/\int_0^\rho(g_{\rho\rho})^{1/2}d\rho\ . \tag{2.6}$$

In the limit $\rho\to 0$, $\Delta\varphi(0)$ is directly related to the energy density per unit length of the string (string tension) and it is often used in the literature because it has a direct observational manifestation in terms of a light-deflection angle.[2,5] Moreover using the field equations (2.2) and (2.3) it is easily shown that

$$\Delta\varphi(\rho=0)=2\pi[1-\exp(-C)] \; ; \tag{2.7}$$

that is, for Einstein-Rosen metrics the deficit angle on the axis is related to the C-energy. It turns out that for the metrics which approach Minkowski space far from the axis, the relation (2.7) between the deficit angle and the C-energy is still valid for large ρ.

D. Curvature tensor

In the following sections we shall often need the Riemann tensor in different asymptotic regions. Using the null tetrad adapted to the Einstein-Rosen metric,

$$n=(2f)^{-1/2}(\partial_\rho+\partial_t) \; ,$$
$$l=(2f)^{-1/2}(\partial_\rho-\partial_t) \; ,$$
$$m=(2g_{zz})^{-1/2}\partial_z - i(2g_{\varphi\varphi})^{-1/2}\partial_\varphi \; ,$$

and its complex conjugate m^*, the non-null components of the Riemann tensor are

$$\psi_0=R_{\mu\nu\alpha\beta}n^\mu m^\nu n^\alpha m^\beta \; ,$$
$$\psi_4=R_{\mu\nu\alpha\beta}l^\mu m^{*\nu} l^\alpha m^{*\beta} \; ,$$
$$\psi_2=\tfrac{1}{2}R_{\mu\nu\alpha\beta}n^\mu l^\nu(n^\alpha l^\beta - m^\alpha m^{*\beta}) \; . \tag{2.8}$$

III. GENERALIZED CYLINDRICAL SOLITON SOLUTIONS

The solutions we consider are called generalized soliton solutions because they can be obtained from the usual soliton solutions[23–26] of Eq. (2.2a) by taking advantage of the linearity of such an equation. Generalized soliton solutions for the "potential" ϕ have been given in Refs. 17–19 in the cosmological context. The solutions in the cylindrical case can be borrowed from the cosmological ones by simply applying the following transformation: $t\to\rho$, $z\to t$, $x\to i\varphi$, $y\to iz$. These solutions may be classified in two classes: class A and class B. We shall denote solutions of class A as (ϕ_A, f_A) referring to the metric coefficients of (2.1). Following Ref. 18 we define

$$\sigma_i^\pm \equiv L_i \pm (L_i{}^2-1)^{1/2} \; ,$$
$$L_i \equiv (t_i{}^2+w_i{}^2)\rho^{-2} + [1+(t_i{}^2+w_i{}^2)^2\rho^{-4} - 2(t_i{}^2-w_i{}^2)\rho^{-2}]^{1/2} \; , \tag{3.1}$$

where

$$t_i \equiv t_i^0 - t \; ,$$

and t_i^0, w_i are arbitrary real parameters. The solution of class A is the generalization of the soliton solutions with complex pole trajectories obtained by taking the integers which give the degeneracies of the poles as real numbers. We have

$$\phi_A = d\ln\rho + \sum_{i=1}^{s} h_i \ln\sigma_i{}^{1/2} \; , \tag{3.2a}$$

where d and h_i are real parameters, s is integer, and σ_i indicates one pole trajectory. The function f_A can be found by either directly integrating (2.2b) or, more easily, by taking the appropriate limits in the corresponding soliton solution.[18] The result is

$$f_A=\alpha^2\rho^{(d^2-1-g^2)/2}\prod_{i=1}^{s}\sigma_i^{h_i(2h_i+d-g)/2}(1-\sigma_i)^{-h_i{}^2/2}H_i^{-h_i^2/4} \times \prod_{\substack{i,j=1\\ i>j}}^{s}\left[\left[(\sigma_i+\sigma_j)\rho^2-\frac{8t_it_j\sigma_i\sigma_j}{(1+\sigma_j)(1+\sigma_i)}\right]^2-\frac{64w_i{}^2w_j{}^2\sigma_i{}^2\sigma_j{}^2}{(1-\sigma_i)^2(1-\sigma_j)^2}\right]^{h_ih_j/2} , \tag{3.2b}$$

where

$$g\equiv\sum_{i=1}^{s}h_i, \quad H_i\equiv(1-\sigma_i)^2+16w_i{}^2\sigma_i{}^2\rho^{-2}(1-\sigma_i)^{-2} \; ,$$

and α is an arbitrary parameter arising from the integration.

In the cosmological context this has been given in Ref. 27. It reduces to the solutions given previously in Refs. 24 and 16 for one pole ($i=1$) and in Ref. 28 for two poles with $h_1=h_2=2$.

Solutions of class B (ϕ_B, f_B) are obtained[18] from the imaginary part of the complex pole trajectories making use again of the linearity of Eq. (2.2a). This method is equivalent to the complexification method of Feinstein and Charach.[17] The potential ϕ is given by

$$\phi_B=d\ln\rho+\sum_{i=1}^{s}h_i\gamma_i \; ,$$
$$\gamma_i\equiv\arccos[2t_i\rho^{-1}\sigma_i{}^{1/2}(1+\sigma_i)^{-1}] \; . \tag{3.3a}$$

The corresponding function f_B has not been given previously. It may be obtained by direct integration of (2.2b) or more easily by making appropriate use of the soliton solution with complex poles. The computation here is more complex than in the previous case, however. The final result is

$$f_B=\alpha^2\rho^{(d^2+2g-1)/2}\prod_{\substack{i,j=1\\ i>j}}^{s}(A_{ij}^-/A_{ij}^+)^{h_ih_j/2}\exp\left[d\sum_{i=1}^{s}h_i\gamma_i\right]$$
$$\times\prod_{i=1}^{s}\sigma_i^{h_i(h_i-1)/2}[(t_i^2-w_i^2-\rho^2)^2+4w_i^2t_i^2]^{h_i(h_i-1)/4}(1-\sigma_i)^{-h_i^2/2}H_i^{h_i(2-h_i)/4}\rho^{-h_i^2}\ , \qquad (3.3b)$$

where

$$A_{ij}^{\pm}=(\sigma_i+\sigma_j)(1-\sigma_i^2)(1-\sigma_j^2)\rho^2-8\sigma_i\sigma_j[t_it_j(1-\sigma_i)(1-\sigma_j)\pm w_iw_j(1+\sigma_i)(1+\sigma_j)]\ .$$

We shall now introduce four asymptotic spacetime regions which will prove useful in the analysis of the solutions to be done in the following sections. Besides the metric components we shall consider in each region different quantities which we shall now specify.

Near the symmetry axis ($\rho\to 0$). The behavior of the gravitational field near the symmetry axis is essential to determine the source of this field. The main features to look for in the region are the deficit angle and the curvature tensor. For a cosmic string no curvature singularities are allowed and the string is identified by means of the deficit angle. For the metrics having curvature singularities we may identify, following Israel,[29] a massive line source.

Null infinity ($\rho\simeq|t|\to\infty$). In this region we shall look for gravitational radiation. The radiation will be characterized in two independent ways. One, simply algebraic, is based on the Petrov classification of the dominant terms of the Riemann tensor (according to the peeling-off property). The other is based on the C-energy; we shall evaluate the rate of C-energy radiation[15] $C_{,v}(u\equiv t+\rho,\ v\equiv t-\rho)$ and also the C-velocity. From these indicators we shall conclude that our solutions are radiative.

Spacelike infinity ($|t|\ll\rho\to\infty$). We do not need to evaluate the curvature tensor in this region because from the metric components one can easily see that the pole trajectories contribution to the metric becomes small. This is the region where an observer far from the string will be located. The deficit angle $\Delta\varphi(\rho\gg)$ and the C energy, will give information about the gravitational energy which lies between the observer and the string or the massive line source.

Timelike infinity ($\rho\ll|t|\to\infty$). This region gives information about the final state (and initial state since there is $t\to -t$ symmetry) of the system. Some solutions develop singularities and will be disregarded as candidates for a reasonable physical interpretation. For the cosmic-string solutions the deficit angle will prove useful to characterize the final state.

IV. SOLUTIONS OF CLASS A1 AND CLASS B1

In this and the next section we analyze the solutions of class A and class B. As usual for soliton solutions[24] we need only to consider solutions with one pole trajectory, which we denote A1 and B1, and solutions with two pole trajectories, A2 and B2, to be studied in the next section. The large-scale structure and the main features of all the solutions are characterized by these simpler solutions.[24]

A. Class A1

Solutions of class A1 (ϕ_{A1},f_{A1}) are obtained from (2.2) by taking $h_1=h$, $s=1$, and $w_1\equiv w$. We shall consider this solution in the asymptotic regions defined in Sec. III.

1. Near the axis

The metric coefficients behave like

$$\phi_{A1}=(d+h)\ln\rho-(h/2)\ln[4(w^2+t_1^2)][1+O(\rho^2)]\ ,$$
$$f_{A1}=\alpha^2\rho^{[(d+h)^2-1]/2}[4(w^2+t_1^2)]^{-h(d+h)/2}[1+O(\rho^2)]\ . \qquad (4.1)$$

The curvature tensor (2.8) becomes singular on the axis, unless

$$(d+h)^2=1\ , \qquad (4.2)$$

in which case the axis is quasiregular. The deficit angle in such case is

$$\Delta\varphi(0)=2\pi(1-\alpha^{-1})\ . \qquad (4.3)$$

2. Null infinity

The metric coefficients approach the static solution, which corresponds to the first term in (3.2a), in the form

$$\phi_{A1}=d\ln\rho+O(\rho^{-1/2})\ ,$$
$$f_{A1}=\alpha^2(32w^2)^{-h^2/4}\rho^{(d^2-1)/2}[1+O(\rho^{-1/2})]\ .$$

However the Riemann tensor (2.8) differs from that of the static solution:

$$\psi_0=-(4w^{3/2}f_{A1})^{-1}h(1+h^2/2)\rho^{-1/2}[1+O(\rho^{-1/2})]\ ,$$
$$\psi_2=(f_{A1})^{-1}O(\rho^{-3/2}),\quad \psi_4=(f_{A1})^{-1}O(\rho^{-1})\ , \qquad (4.4)$$

for $t>0$. Thus the leading Riemann components in this region give an algebraic classification of Petrov type N (Ref. 19). This may be taken as an indication that the metric is radiative here.[28] Such an indication is reinforced by the fact that the rate of C-energy radiation is finite in this region ($t>0$):

$$C_{,v}=-(8w)^{-1}h^2[1+O(\rho^{-1/2})] . \tag{4.5}$$

The C-velocity approaches unity.

3. *Spacelike infinity*

The metric coefficients approach the static solution differing from it at $O(\rho^{-1})$. The deficit angle when (4.2) is verified (the case relevant for a cosmic string) is $\Delta\varphi(\rho\gg)=2\pi$—an indication that the spacetime does not become Minkowskian in this region; in fact, it corresponds to the static spacetime of a massive line source. When $d=1$ the metric approaches the Minkowski metric and it gives

$$\Delta\varphi(\rho\gg)=2\pi[1-(4w)^{h^2/4}\alpha^{-1}] , \tag{4.6}$$

but now the axis is not regular (if $h\neq 0$) and such a solution is not relevant for cosmic strings.

4. *Timelike infinity*

Here,

$$\begin{aligned}\phi_{A1}&=(d+h)\ln\rho-(h/2)\ln\{4t_1{}^2[1+O(t^{-2})]\} ,\\ f_{A1}&=\alpha^2\rho^{[(d+h)^2-1]/2}(4t_1{}^2)^{-h(h+d)/2}\\ &\quad\times[1+O(t^{-2})] ,\end{aligned} \tag{4.7}$$

and the Riemann tensor gives curvature singularities unless one of the two following conditions is verified:

$$\text{(a)}\quad h/|h|=d/|d| \quad\text{and}\quad |d|\geq|h| , \tag{4.8a}$$

$$\text{(b)}\quad (d+h)^2=1 \quad\text{and}\quad h(h+d)\leq 2 . \tag{4.8b}$$

The deficit angle for case (b) is

$$\Delta\varphi(\rho)=2\pi(1-\alpha^{-1}) , \tag{4.9}$$

which agrees with $\Delta\varphi(0)$ of (4.3). The rate of C-energy radiation and the C-velocity vanish in this region.

The physical interpretation of these solutions will be discussed in Sec. VI. We now consider solutions of class B1, because in many respects they are similar to class A1.

B. Class B1

Class B1 solutions (ϕ_{B1}, f_{B1}) are obtained from (3.3) by taking $h_1=h$, $s=1$, and $w_1\equiv w$. The asymptotic behavior is as follows.

1. *Near the axis*

Here

$$\phi_{B1}=d\ln\rho+h\gamma_1 ,$$

$$f_{B1}=\alpha^2 2^{-h(h-1)}\rho^{(d^2-1)/2}e^{dh\gamma_1}[1+O(\rho^2)] ,$$

and the curvature tensor becomes singular, unless

$$d^2=1 . \tag{4.10}$$

In such a case the axis is quasiregular with a deficit angle

$$\Delta\varphi(0)=2\pi(1-2^{h(h-1)/2}\alpha^{-1}) , \tag{4.11}$$

which may be compared with (4.3) for the A1 solutions.

2. *Null infinity*

It approaches the static solution in a way similar to class A1, and the Riemann components (2.8) become

$$\begin{aligned}\psi_0&=-(4w^{3/2}f_{B1})^{-1}h(1-h^2/2)(\rho^{-1/2})\\ &\quad\times[1+O(\rho^{-1/2})] ,\\ \psi_2&=(f_{B1})^{-1}O(\rho^{-3/2}),\quad \psi_4=(f_{B1})^{-1}O(\rho^{-1}) ,\end{aligned} \tag{4.12}$$

which implies a Petrov type-N behavior for the leading Riemann components as for class A1. Similarly the rate of C-energy radiation is finite,

$$C_{,v}=-(8w)^{-1}h^2[1+O(\rho^{-1/2})] , \tag{4.13}$$

and the C-velocity approaches unity.

3. *Spacelike infinity*

Now the metric coefficients become

$$\begin{aligned}\phi_{B1}&=d\ln\rho+h\gamma_1 ,\\ f_{B1}&=\alpha^2 2^{-h(h-1)}\rho^{(d^2+h^2-1)/2}e^{dh\gamma_1}[1+O(\rho^{-2})] ,\end{aligned} \tag{4.14}$$

where

$$\gamma_1=\pi/2-t_1\rho^{-1}[1+O(\rho^{-2})]$$

and the Riemann components behave like $\psi\sim(f_{B1})^{-1}O(\rho^{-2})$.

As in the case of class A1 solutions, it is not possible to impose that the axis be quasiregular together with the requirement that the metric becomes flat in this region. For a quasiregular axis, i.e., (4.10), we find a deficit angle $\Delta\varphi(\rho\gg)=2\pi$ (if $h\neq 0$). On the other hand, if we take $d^2+h^2=1$, which from (4.14) implies an asymptotically constant f_{B1}, we have

$$\Delta\varphi(\rho\gg)=2\pi(1-2^{h(h-1)/2}e^{h(1-d)\pi/2}\alpha^{-1}\rho^{(d-1)/2}) , \tag{4.15}$$

comparable to (4.6) when $d=1$ (thus $h=0$, i.e., the static solution). A peculiar feature of this solution is that the C-velocity (2.5) does not vanish:

$$v^\rho=2h(d-1)[h^2+(d-1)^2]^{-1}+O(\rho^{-1}) . \tag{4.16}$$

This is not very significant in terms of energy radiation, however, because the C-energy goes like $\rho^{-(d^2+h^2+3)/2}$ (it goes like $\rho^{-(d^2+1)/2}$ at null infinity) and the rate of C-energy radiation vanishes.

4. *Timelike infinity*

In this region the metric approaches the static solution differing from it at $O(t^{-1})$. The deficit angle corresponding to a quasiregular axis, i.e., with (4.10), is

$$\Delta\varphi(\rho)=2\pi(1-2^{h(h-1)/2}\alpha^{-1}) . \quad (4.17)$$

The C-velocity and the rate of C-energy radiation vanish as expected. We shall now go on to consider solutions with two pole trajectories.

V. SOLUTIONS OF CLASS A2 AND CLASS B2

The solutions considered in this section generally represent gravitational finite perturbations propagating on the background field of a static string or a massive line source. Besides the asymptotic analysis similar to that of the last section, a perturbative analysis will be done.

A. Class A2

They are denoted by (ϕ_{A2}, f_{A2}) and defined by (3.2) with $h_1=h_2\equiv h$ $(s=2)$ chosing $\sigma_1^{(+)}$, $\sigma_2^{(-)}$ according to (3.1); this guarantees that the poles give localized contributions only.[24]

Perturbative analysis. We may now perform a perturbative analysis similar to that of Ref. 16. For this we take $t_1^0=t_2^0$, define $\delta w\equiv w_2-w_1$, and call $w\equiv w_1$, $\sigma\equiv\sigma_2^{(-)}$. Since the nonstatic term in (3.2) is now $\ln\sigma_1\sigma_2$ we may expand this to first order in δw. The function $\sigma_1\sigma_2$ differs from unity only on a small localized region. The spacetime trajectory $\rho(t)$ of the maximum of this perturbation with respect to ρ was evaluated analytically in Ref. 16 and it was found to "propagate" at a speed greater than light. Here we analytically compute the C-velocity and shed light on the meaning of that perturbation.

Following Ref. 16 we introduce new R,T coordinates

$$\rho=w\cosh(2T)\sinh(2R), \quad 0\le R\le\infty ,$$
$$t=w\sinh(2T)\cosh(2R), \quad -\infty\le T\le\infty , \quad (5.1)$$

in terms of which $\sigma=\tanh^2(R)$. The C-velocity (2.5) is now easily found to be

$$v^\rho=-2\chi(1+\chi^2)^{-1} \quad (\chi\equiv\xi+\eta) ,$$
$$\xi\equiv\tfrac{1}{2}\sinh 4R\coth 2T(\cosh^2 2R-3\sinh^2 2T)F , \quad (5.2)$$
$$\eta\equiv\frac{4}{h}(d-1)(\cosh 2R+\sinh 2T)^2(\sinh 2R\sinh 4T)^{-1}$$
$$\times[(w/\delta w)(\cosh^2 2R+\sinh^2 2T)-2\cosh 2R]F ,$$

where

$$F\equiv[\sinh^2 2R(3\cosh^2 2T-\cosh^2 2R-1)+2\cosh^2 2T]^{-1} .$$

Recalling the meaning of the C-velocity of Sec. II we see that, according to the sign of v^ρ, there are two competing fluxes of C-energy: one of them ingoing (towards the axis) the other outgoing (from the axis). If we take now $d=1$, which is the case relevant for a cosmic string, then $\eta=0$ and the two fluxes cancel on the world line $\cosh^2(2R)-3\sinh^2(2T)=0$ which is just the maximum of the perturbation $\sigma_1\sigma_2$ (see Ref. 16). Thus the localized perturbation may be understood as a superposition effect produced by the interference of the incoming and outgoing fluxes of C-energy. We now go back to the analysis of the solution in the asymptotic regions.

1. Near the axis

In this region

$$\phi_{A2}=d\ln\rho+\frac{h}{2}\ln[(w_1{}^2+t_1{}^2)(w_2{}^2+t_2{}^2)^{-1}]$$
$$+O(\rho^2) ,$$
$$f_{A2}=\alpha^2\rho^{(d^2-1)/2}[(w_1{}^2+t_1{}^2)(w_2{}^2+t_2{}^2)^{-1}]^{hd/2} \quad (5.3)$$
$$\times[1+O(\rho^2)] ,$$

and the Riemann components show that there is a curvature singularity unless

$$d^2=1 , \quad (5.4)$$

in which case the axis is quasiregular and the deficit angle is

$$\Delta\varphi(0)=2\pi(1-\alpha^{-1}) . \quad (5.5)$$

2. Null infinity

The metric components differ from the static solution by terms of $O(\rho^{-1/2})$, but the Riemann components (2.8) become

$$\psi_0=(4f_{A2})^{-1}h(w_1w_2)^{-3/2}$$
$$\times\left[w_2{}^{3/2}-w_1{}^{3/2})+\frac{h^2}{2}(w_2{}^{1/2}-w_1{}^{1/2})^3\right]$$
$$\times\rho^{-1/2}[1+O(\rho^{-1/2})] ,$$
$$\psi_2=(f_{A2})^{-1}O(\rho^{-3/2}), \quad \psi_4=(f_{A2})^{-1}O(\rho^{-1}) . \quad (5.6)$$

Thus, as typical of these generalized soliton solutions, the dominant components indicate a Petrov type-N behavior for the metric. The rate of C-energy radiation

$$C_{,v}=-h^2(8w_1w_2)^{-1}(w_2{}^{1/2}-w_1{}^{1/2})^2$$
$$\times[1+O(\rho^{-1/2})] \quad (5.7)$$

reaches a finite value and the C-velocity approaches unity.

3. Spacelike infinity

In this region the metric coefficients approach the static values, differing at $O(\rho^{-1})$, and so do the Riemann components. The deficit angle for $d=1$ is now

$$\Delta\varphi(\rho>>)=2\pi\{1-(4w_1w_2)^{h^2/4}[(w_1+w_2)^2+(t_1^0-t_2^0)^2]^{-h^2/4}\alpha^{-1}\}+O(\rho^{-1}) , \quad (5.8)$$

and the spacetime becomes asymptotically Minkowskian in this region. The rate of C-energy radiation and the C-velocity both vanish as expected.

4. *Timelike infinity*

The metric components and the Riemann components approach the static solution components with differences of $O(t^{-1})$. The deficit angle for $d=1$ is

$$\Delta\varphi(\rho)=2\pi(1-\alpha^{-1}) , \tag{5.9}$$

which may be compared to the deficit angle near the axis (5.5). The rate of C-energy radiation and the C-velocity both vanish in this region. Leaving the physical discussion of these solutions for the next section we shall now consider the class B2 solutions.

B. Class B2

Such solutions, (ϕ_{B2}, f_{B2}), are obtained from (3.3) when $h_1=h_2\equiv h$ $(s=2)$ and taking $\gamma_1^{(+)}$ and $\gamma_2^{(-)}$ according to (3.1); as in the previous case this electron guarantees that the poles give localized contributions only.

Perturbative analysis. We may now perform an analysis similar to that of class A2 solutions. The relevant term for the analysis is now $\gamma_1^{(+)}+\gamma_2^{(-)}$. This term differs from zero in a small-(ρ,t) region, and proceeding as for class A1 we may expand it in terms of δw. Introducing new coordinates (R,T) as in (5.1) we have that $(\gamma\equiv\gamma_2)$

$$\gamma^{(-)}=\arccos[\tanh(2T)] . \tag{5.10}$$

The maximum of the perturbation $\gamma_1^{(+)}+\gamma_2^{(-)}$ with respect to ρ is found to be located on the world line $\sinh^2(2T)=3\cosh^2(2R)$. In terms of the coordinates (ρ,t) such trajectory is the same as in class A1 solutions but for a shift, $2w/\sqrt{3}$, along the t axis.

The C-velocity is now given by (5.2) with

$$\begin{aligned}\xi&=\tfrac{1}{2}\sinh 4T\tanh 2R\,(\sinh^2 2T-3\cosh^2 2R)G ,\\ \eta&=2w^2(1-d)(\rho h\cosh 2R\,\delta w)^{-1}\\ &\quad\times(\cosh^2 2T+\sinh^2 2R)^3 G ,\end{aligned} \tag{5.11}$$

where

$$\begin{aligned}G\equiv{}&[\cosh^2 2T(\cosh^2 2R-\sinh^2 2T)\\ &+2\sinh^2 2R\,\sinh^2 2T]^{-1} .\end{aligned}$$

Again for $d=1$, the C-velocity vanishes along the trajectory of the maximum of the perturbation. The same conclusions as for class A1 solutions apply in this case. We can now go back to the asymptotic analysis of the exact solutions.

1. *Near the axis*

In this region we have

$$\begin{aligned}\phi_{B2}&=d\ln\rho+h(\gamma_1^{(+)}+\gamma_2^{(-)})\\ f_{B2}&=\alpha^2 4^{(1-h)h}\rho^{(d^2-1)}\exp[hd(\gamma_1^{(+)}+\gamma_2^{(-)})]\\ &\quad\times[1+O(\rho^2)] ,\end{aligned} \tag{5.12}$$

where

$$\begin{aligned}\gamma_k^{(-)}&=-\gamma_k^{(+)}\\ &=\arcsin[w_k(w_k^2+t_k^2)^{-1/2}]+O(\rho^2) .\end{aligned}$$

From the Riemann tensor one sees that the metric becomes singular unless

$$d^2=1 , \tag{5.13}$$

in which case the axis is quasiregular and has a deficit angle

$$\Delta\varphi(0)=2\pi(1-2^{h(h-1)}\alpha^{-1})+O(\rho^2) . \tag{5.14}$$

2. *Null infinity*

We may note that in this region the metric components are determined by

$$\gamma_k^{(-)}=\pi-(w_k)^{1/2}\rho^{-1/2}[1+O(\rho^{-1/2})] . \tag{5.15}$$

The metric coefficients approach the static values, but the curvature tensor (2.8) becomes

$$\begin{aligned}\psi_0&=(4f_{B2})^{-1}h(w_1w_2)^{-3/2}\\ &\quad\times\left[\left|w_2^{3/2}-w_1^{3/2})-\frac{h^2}{2}(w_2^{1/2}-w_1^{1/2})^3\right|\right.\\ &\quad\times\rho^{-1/2}[1+O(\rho^{-1/2})] ,\\ \psi_2&=(f_{B2})^{-1}O(\rho^{-3/2}),\quad \psi_4=(f_{B2})^{-1}O(\rho^{-1}) ,\end{aligned} \tag{5.16}$$

thus the leading terms behave as those of a Petrov type-N metric. The rate of C-energy radiation approaches a constant value,

$$\begin{aligned}C_{,v}&=-h^2(8w_1w_2)^{-1}(w_2^{1/2}-w_1^{1/2})^2\\ &\quad\times[1+O(\rho^{-1/2})] ,\end{aligned} \tag{5.17}$$

and the C-velocity approaches unity.

3. *Spacelike infinity*

The metric coefficients and the Riemann components approach the static values, differing at $O(\rho^{-1})$. The deficit angle for $d=1$ is

$$\Delta\varphi(\rho>>)=2\pi\{1-2^{h(h-1)}(4w_1w_2)^{h^2/4}[(w_1+w_2)^2+(t_1^0-t_2^0)^2]^{-h^2/4}\alpha^{-1}\}+O(\rho^{-1}) . \tag{5.18}$$

One may compare this with the deficit angle for class A2, (5.8), in both cases the deficit angle is larger than near the axis. The metric is asymptotically flat in this region. The rate of C-energy radiation and the C-velocity both vanish.

4. *Timelike infinity*

The metric coefficients and the Riemann components approach the static solution, differing at $O(t^{-1})$. The deficit angle for $d=1$ is

$$\Delta\varphi(\rho)=2\pi(1-2^{h(h-1)}\alpha^{-1}) , \qquad (5.19)$$

as near the axis (5.14). The rate of C-energy radiation and the C-velocity both vanish.

VI. PHYSICAL INTERPRETATION AND CONCLUSIONS

First, we may note that the generalized soliton solutions of class A, (3.2), and class B, (3.3), reduce to a static metric when the parameters h_i are taken null ($h_i=0$). It is worth reviewing some of the properties of the static metric. We shall consider $d\geq 0$ only because for negative values of d the metric has the same interpretation, as for positive ones, if we exchange the roles of the coordinates φ and z. This metric, also known as the Kasner metric, was first studied by Levi-Civita[30] who found that in the Newtonian limit it describes the gravitational field due to an infinite cylinder of relativistic mass per unit length, M given by[31]

$$d\simeq(1+2M)(1-2M)^{-1} . \qquad (6.1)$$

For $d<1$, Newtonian test particles far from the cylinder suffer a repulsive force[32] and therefore this cannot be a cylinder with ordinary matter but rather with negative mass. For $d\neq 1$ the metric has a naked curvature singularity at the axis: $\rho=0$. Following Israel[29] the structure of the inferred energy-momentum tensor on the axis is compatible with the above interpretation.

For $d=1$ the metric has no curvature singularities but a conical singularity with a quasiregular axis. The singularity is characterized by a deficit angle

$$\Delta\varphi(0)=2\pi(1-\alpha^{-1}) , \qquad (6.2)$$

where α is the arbitrary parameter in the f coefficient of (2.1). For the value $\alpha=1$ the metric is just that of a region of flat space. For $\alpha\neq 1$ the metric is that of flat space minus a wedge and this is the spacetime created by an isolated static cosmic string[8,9] with a string tension μ related to the deficit angle by

$$\Delta\varphi(0)=8\pi G\mu . \qquad (6.3)$$

In current GUT's $G\mu\sim 10^{-6}$ and therefore α is nearly unity.[2]

The deficit angle may also be calculated far from the axis $\Delta\varphi(\rho\gg)$. For $d=1$ this agrees with (6.2); however, for $d\neq 1$,

$$\Delta\varphi(\rho\gg)=2\pi . \qquad (6.4)$$

This may be taken as an indication that the spactime is not asymptotically Minkowskian in such a region; it is, in fact, the field of a massive line source.

Now, we turn our attention to the solutions with two poles: class A2 and class B2. As emphasized in the perturbative analysis of Sec. V these solutions represent small gravitational perturbations propagating on the Levi-Civita background. The gravitational perturbations may be interpreted as gravitational radiation as the analysis at null infinity indicates. Such an analysis was based on the algebraic classification of the Riemann tensor (which is asymptotically Petrov type N in agreement with the peeling-off property[32]), the evaluation of the rate of C-energy radiation which is finite, (5.7) and (5.17), and the C-velocity which approaches unity.

The metrics have the same singularities that the Levi-Civita background; therefore,[29] the source to be located at $\rho=0$ is that of a massive line source (for curvature singularity, $d\neq 1$) or a cosmic string (for conical singularity, $d=1$). Far from the axis the spacetime for these metrics is also that of the corresponding Levi-Civita metrics, it approaches Minkowski of $d=1$ only.

For $d=1$ the solutions are relevant for cosmic strings. We have already said that the string is being characterized near the symmetry axis by its deficit angle $\Delta\varphi(0)$ which is given in (5.5) and (5.14). This angle is independent of t and the string is thus static. We may ask now what an observer sitting far from the string would see. Initially, $t\to-\infty$, before the incoming radiation reaches the string he would simply measure a flat space with a deficit angle $\Delta\varphi(\rho)$ given by (5.9) and (5.19) which agrees with that of the isolated string; this is simply reflecting the fact that between the observer and the string no energy is present. At some finite time $|t|\ll\rho$ the observer would measure a deficit angle, given as $\Delta\varphi(\rho\gg)$ in (5.8) and (5.18), which is also that of flat space minus a wedge but larger than before just as if the string had a larger energy density (observationally this angle may be measured from the deflection of light from objects at the opposite side of the string for different impact parameters). As mentioned before the string has not changed its energy density but now, between the observer and the string, we have a lump of gravitational C-energy and this deficit angle is a measure for it. Finally at $t\to\infty$, the energy has been radiated and the observer will measure the same deficit angle as at the beginning; the radiated energy is found at null infinity. These solutions, thus, represent the interaction of a static string with incoming and outgoing gravitational waves localized basically along the light cones. This may be taken as an idealization of gravitational radiation, not necessarily with cylindrical symmetry, surrounding the string. For $d>1$ the wave interpretation is similar but now the waves propagate on the background of massive line sources.

We may turn now to the solutions with one pole: class A1 and class B1. As for the solutions just described these metrics have radiation at null infinity, however this radiation may not be interpreted as localized and propagating on a static background.

For class A1, from (4.1) we see that near the axis the metric on $t=\text{const}$ hypersurfaces behaves like a Levi-Civita metric with parameter $d'\equiv d+h$, instead of d.

However at spacelike infinity it behaves like a Levi-Civita metric with d. In some sense these metrics can be seen as connecting two Levi-Civita metrics with different parameters through the light cones which contain gravitational radiation. In the cosmological context such metrics have been considered as composite universes.[17,27]

Now, from the viewpoint of cosmic strings, only the metrics with $d'=1$, see (4.1), have a string on the axis and its deficit angle is given by (4.3). At spacelike infinity, however, the spacetime approaches that of Levi-Civita with $d=1-h$ $(h\neq 0)$ and it is not asymptotically Minkowskian. This may be seen as a consequence of the presence of C-energy surrounding the string. Note the remarkable fact that if we take $0<h<1$ the gravitational effect produced by the gravitational energy far from the axis is similar to that of a massive rod with negative energy density. Besides, we have asymptotically flat solutions $(d=1)$ which have a massive line source with $d'=1+h$.

Not all these solutions admit a physical interpretation because some of them develop singularities at $|t|\to\infty$, only those satisfying (4.8) are free from singularities.

ACKNOWLEDGMENT

This work was partially supported by research project Comisión Asesora de Investigación Científica y Técnica Spain.

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