# Particle creation due to cosmological contraction of extra dimensions

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(Received 24 October 1988)

Particle production in a cosmological spacetime with extra dimensions is discussed. A five-dimensional cosmological model with a three-dimensional space expanding isotropically like in a radiative Friedmann-Robertson-Walker model and an internal space contracting to a constant small size is considered. The parameters of the model are adjusted so that time variations in internal space are compatible with present limits on time variations of the fundamental constants. By requiring that the energy density of the particles produced be less than the critical density at the radiation era we set restrictions on two more parameters: namely, the initial time of application of the semiclassical approach and the relative sizes between the internal space and the horizon of the ordinary Universe at this time. Whereas the production of massless particles allows a large range of variation to these parameters, the production of massive particles sets severe constraints on them, since, if they are overproduced, their energy density might very soon dominate the Universe and make cosmological dimensional reduction by extradimensional contraction unlikely.

## I. INTRODUCTION

In any realistic theory with extra dimensions the extra space, or internal space, is assumed to be at present a compact manifold of very small size compared with that of the visible space. This size is directly related to the fundamental constants and, consequently, must be stable; it is supposed to be of the order of the Planck length ($l_P=1.6\times10^{-33}$ cm) although present-day accelerators can probe matter at $10^{-16}$ cm only. But this may not have been always so; for one, if we go back in time, according to the standard cosmological model there must have been a time when the visible Universe was of comparable size to that of the internal space. Besides, and within the context of Kaluza-Klein theories and cosmology, one would like to explain the contraction of the internal space as a consequence of cosmological evolution.

Models in which cosmological dimensional reduction by contraction of the internal space may be achieved have been considered in recent years.[1] They are solutions of Einstein's equations in more than four dimensions which differ in their higher-dimensional energy-momentum tensor sources. If one sticks to the original Kaluza-Klein idea that the gauge fields have geometric origin, then there is no source, but the theories one obtains are nonrealistic.[2] Non-null energy-momentum tensors may be due to external fields as, for instance, those involved in eleven-dimensional supergravity[3,4] or those of the Chapline-Manton action[5,6] which is thought to be the field-theory limit of a ten-dimensional superstring theory. Quantum corrections computing the one-loop effective potential lead to a Casimir energy which must be included in the energy-momentum tensor.[7,8] None of the effective models so far studied is completely satisfactory; although some evolve to solutions with a static internal space, they are usually unstable.[1]

The Casimir effect which is one of the quantum consequences of extra dimensions has remarkable features. For instance, Appelquist and Chodos[9] showed that a Casimir force will tend to contract the internal space, thus providing a mechanism for compactification. Another quantum consequence of extra dimensions is particle production.

In this paper we consider production due to the rapid contraction of the internal space. Particle production in the context of Kaluza-Klein theories has been considered before. Thus Koikawa and Yoshimura[10,11] considered a model in which the extra dimensions undergo oscillations around a static solution of Einstein's equations in more than four dimensions. Such oscillations will produce particles which might be seen in the visible Universe as ultrahigh-energy cosmic rays. Also Maeda[12,13] computed the back reaction due to production of scalar particles in anisotropic higher-dimensional cosmologies with contracting extra dimensions, following the computations in four dimensions by Zel'dovich and Starobinsky[14] and by Hu and Parker.[15] He argued that if the contraction starts near the Planck time the created particles will produce a very rapid isotropization of the model, thus making cosmological dimensional reduction ineffective. However, as he pointed out,[12] he ignored the Casimir effect of external matter fields or of gravitons, which will induce contraction of the internal space, or that compactification might have been induced by an antisymmetric tensor field like in eleven-dimensional supergravity with the Freund-Rubin ansatz.[16]

Therefore, since in any realistic model the dynamics of the spacetime will be driven initially by other factors than just the particles produced we shall ignore here the back reaction of the particles on the geometrical background in this semiclassical period. Besides, the back-reaction computation is more difficult and liable to more approximations than the computation of the number and energy density of particles produced, which can be done in some detail.

As a model to carry out our computations we shall choose one which has some of the features relevant for particle production that we might expect in a realistic model: namely, a model with cosmological dimensional reduction, consisting of a rapid internal contraction toward a static size, and an isotropic expansion of the ordinary Universe. Simple models with these features can be found even as solutions of five-dimensional Einstein equations in vacuum[17–19] when the Friedmann-Roberston-Walker (FRW) ordinary universe is open. When the spatial sections are flat the vacuum models are of a five-dimensional Kasner type with the extra dimension contracting towards a crack-of-doom singularity.[20] Although we shall briefly comment on models with nonflat spatial sections we shall mainly deal with a model with a flat spatial section corresponding to a five-dimensional anisotropic fluid. The model is simple enough so that many computations can be done analytically and the dependence of the results on the parameters is clear.

The parameters of the model are severely restricted by the limits on the time rate of change of the fundamental constants, particularly of the fine-structure constant whose value cannot have changed by more than one percent since nucleosynthesis time.[1,21,22] This sets limits on the time variability of the internal dimensions after the nucleosynthesis. Further restrictions on the parameters come from the energy density of the particles produced. When such energy density is imposed to be less than, say, the critical energy density of the radiative universe, the initial time of application of the semiclassical approach and the size of the internal space at this time are tightly bounded.

The plan and a summary of the main results of the paper are the following. In Sec. II we review the standard theory of a quantum scalar field coupled to a four-dimensional isotropic cosmological model with an extra dimension. We set up the equations to determine the Bogoliubov coefficients relating the "in" and "out" modes which determine the number of particles produced.[23–25] One of the technical difficulties in computing particle creation in cosmology is that of the ill-defined nature of the particle concept[24,25] in a curved spacetime near the singularity. This is solved in the usual way[14,26] by introducing a time parameter $t_0$, initial time, before which the metric is matched to the Minkowski metric. This is justified by the fact that, for a time close to the Planck time, the semiclassical approach, i.e., fields quantized on a classical gravitational background, must break down. Now if we want to compute particle creation since $t_0$, we want to assume that there are no "in" particles present[23] and since we have an unambiguous definition of the particle concept in Minkowski spacetime it is natural to assume such "in" particles as metric. We shall, therefore, assume that the contraction of the extra dimension and the expansion of the ordinary Universe starts at $t_0$. This time must be larger than the Planck time ($t_p = 5.39\times10^{-44}$ sec) and smaller than, say, the grand unification time ($\sim 10^{-38}$ sec). For the "out" modes, on the other hand, there is no ambiguity since we have an adiabatic expansion which does not produce particles and consequently an adiabatic vacuum can be defined.

In Sec. III we compute numerically the particle production of scalar particles for a "realistic" model in the sense stated above. The parameters we can adjust in the model are three: $t_0$, $f$, and $A$. The first is the initial time, $f$ defines the ratio between the size of the internal space and the horizon of the ordinary Universe at $t_0$, and $A$ is related to the rate of contraction of the internal space: it is restricted by the limits on the time variability of the fundamental constants. We consider several models for different values of the parameters.

For the massless particles, which correspond to modes with no momentum component in the fifth dimension, the production is more important near $t_0$, when the contraction of the internal space is faster. Once the internal size settles down no massless particles are produced because the dynamics comes only from the visible isotropic Universe which is radiative and therefore has zero curvature scalar so that, independently of the coupling parameter, the particles become conformally coupled. As is well known, massless conformally coupled particles are not produced in an isotropic expansion.[27] The models which give more production of massless particles have a large parameter $A$ and undergo a Kasner-type period with three dimensions expanding isotropically and one contracting. This corresponds to a period with no five-dimensional source when the dynamics is driven by the curvature. This is an interesting fact because with the Kasner metric many results can be found analytically, as we see in the next section.

For the massive particles, which correspond to modes with momentum in the fifth dimension, the number of particles produced is roughly of the same order as in the massless case. However, these modes are highly blueshifted and their masses at present would be of the order of the Planck mass; therefore, if massive modes are excited their contribution to the total energy density would be the dominant one, unless they decay quickly into massless modes. However, this is unlikely: if they decay by interaction with themselves the interaction time is too large, and if they decay by interaction with other particles these should also have components in the fifth dimension and their contribution to the energy density would be important too. The number of massive modes excited depends on the initial size of the internal space: if it is large many modes are excited. Whereas for the models considered the energy density of the massless particles produced is less than the critical energy density at the radiation era, if even a single massive mode is excited, it would very soon dominate the total energy density.

In Sec. IV we study analytically the production of particles during the Kasner period. This is interesting, as many of the known realistic models with extra dimensions undergo a Kasner period.[1] In our model a Kasner period is only possible if $t_0 > 10^3 t_P$. The problem is seen to be analogous to the quantum-mechanical scattering through a potential in one dimension.[26] We use this fact to analyze the "sudden approximation" used as a consequence of the abrupt matching of the cosmological and Minkowski spacetimes at $t_0$. Such analysis plays a crucial role when considering the contribution to the energy density of the massive particles.

For the massless particles the number of particles produced per mode can be computed analytically and only the integration corresponding to the total energy density, which we compare with the critical energy density at the radiation era, must be computed numerically. As in the previous section we find that the energy density due to massless particles is always several orders of magnitude smaller than the critical energy density.

For the massive particles, on the other hand, analytic computations can be done only when the three-momenta of the particles is zero. When the internal and visible sizes are comparable at $t_0$ the contribution to the total energy density of the massive modes may be less than the critical density. If the internal size is too large the massive modes would dominate the dynamics of the model, and the back reaction at late times, i.e., at the classical period, cannot be ignored. Then cosmological dimensional reduction would be unlikely.

Thus, according to our results on particle production in extra dimensions, somewhat realistic models with cosmological dimensional reduction can be constructed compatible with a very small time variation of the coupling constants and with present-day cosmological energy density. The critical energy density at the radiation era sets limits on the initial time and the internal size at such initial time. If we are far off these limits too many massive particles are produced, back reaction should be taken into account at least in the classical period, and, as in Refs. 12 and 13, this might produce a tendency to isotropize the model, making any cosmological dimensional reduction mechanism inefficient.

## II. QUANTUM FIELD COUPLED TO A FRW MODEL WITH AN EXTRA DIMENSION

In this section we set up the framework for computing particle creation for a scalar field coupled to a cosmological model with an extra dimension. The material in this section is standard.[14,15,23,24] Although in this paper we shall mainly deal with the spatially flat ($\kappa=0$) FRW cosmology, it is convenient at this stage to consider the open ($\kappa=-1$) and closed ($\kappa=+1$) models too, in order that similarities among the three cases can be brought into display later on.

We shall consider a scalar field coupled to a metric of the form

$$ds^2=-dt^2+C(t)h_{ij}(\mathbf{x})dx^i dx^j+g_{55}(t)(dx^5)^2 \ , \qquad h_{ij}dx^i dx^j=\frac{dr^2}{1-\kappa r^2}+r^2 d\Omega^2=d\chi^2+f^2(\chi)d\Omega^2 \ , \tag{1}$$

where

$$f(\chi)=r=\begin{cases}\sin\chi, & 0\le\chi\le 2\pi, \quad \kappa=1, \\ \chi, & 0\le\chi<\infty, \quad \kappa=0, \\ \sinh\chi, & 0\le\chi<\infty, \quad \kappa=-1 \ , \end{cases}$$

and

$$d\Omega^2=d\theta^2+\sin^2\theta\, d\phi^2 \ ,$$

and the fifth dimension is assumed to be compact. Therefore, the values of the fifth coordinate are restricted to a certain range $0\le x^5\le 2\pi R_5$, so that the physical size of the extra space is given by $L_{\rm ph}=2\pi R_5[g_{55}(t)]^{1/2}$.

Changing the cosmological time variable $t$ into a conformal time $\eta$ defined by

$$\eta=\int\frac{dt}{C^{1/2}(t)} \ , \tag{2}$$

we can rewrite Eq. (1) as

$$ds^2=C(\eta)[-d\eta^2+d\chi^2+f^2(\chi)d\Omega^2]+g_{55}(\eta)(dx^5)^2 \ . \tag{3}$$

Now, the scalar field operator $\Phi(x)$ may be expanded in terms of creation and annihilation operators, satisfying the usual commutation relations:

$$\Phi(x)=\sum_{k_5=0}^{\infty}\int d\tilde{\mu}(x)[a_{\mathbf{k},k_5}u_{\mathbf{k},k_5}(x) + a^{\dagger}_{\mathbf{k},k_5}u^{*}_{\mathbf{k},k_5}(x)] \ , \tag{4}$$

where $d\tilde{\mu}(x)$ is an appropriate measure depending on the FRW model considered[24] and the complete set of exact modes $u_{\mathbf{k},k_5}(x)$ satisfies the five-dimensional Klein-Gordon equation

$$(\Box-m^2-\xi R)u_{\mathbf{k},k_5}(x)=0 \ , \tag{5}$$

where $\mathbf{k}$ and $k_5$ are quantum numbers labeling a particular mode, $R$ is the five-dimensional curvature scalar, $\xi$ is an arbitrary coupling, and $m$ is the mass of the scalar particle in the five-dimensional world. Equation (5) can be separated by writing

$$u_{\mathbf{k},k_5}(x)=\frac{1}{C^{1/2}(\eta)g_{55}^{1/4}(\eta)}\frac{e^{ik_5x^5}}{(2\pi R_5)^{1/2}}Y_{\mathbf{k}}(\mathbf{x})\chi_{\mathbf{k},k_5}(\eta) \ . \tag{6}$$

Since the fifth dimension is compact we have

$$k_5=\frac{n}{R_5} \ , \tag{7}$$

with $n$ an integer. The functions $Y_{\mathbf{k}}(\mathbf{x})$ are the harmonics of the three-dimensional Laplacian $\Delta^{(3)}$ associated with $h_{ij}(\mathbf{x})$. They satisfy

$$\Delta^{(3)}Y_{\mathbf{k}}(\mathbf{x})=-(k^2-\kappa)Y_{\mathbf{k}}(\mathbf{x}) \ ,$$

and are normalized according to

$$\int d^3x\sqrt{h}\,Y_{\mathbf{k}}(\mathbf{x})Y^{*}_{\mathbf{k}'}(\mathbf{x})=\delta_{\mathbf{k}\mathbf{k}'} \ ,$$

where $h=\det(h_{ij})$ and $\delta_{\mathbf{k}\mathbf{k}'}$ is the $\delta$ function associated with the measure $\tilde{\mu}$. For the flat case, $\kappa=0$, such harmonics are, of course, the familiar plane-wave solutions

$$Y_{\mathbf{k}}(\mathbf{x})=\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}}$$

and the spectrum of eigenvalues is continuous, with $0\le k<\infty$, where $k=|\mathbf{k}|$ is the coordinate wavelength of the corresponding mode. For the $\kappa=\pm1$ cases, explicit expressions for $Y_{\mathbf{k}}(\mathbf{x})$ can be found in Ref. 24. For our present purposes, it is sufficient to recall that for the open

($\kappa=-1$) model, the spectrum is also continuous, with $0 \le k < \infty$, whereas for the closed model it is discrete, with $k$ running over the set of all positive integers.

Substitution of Eq. (6) into Eq. (5) leads to the following equation for $\chi_l(\eta)$, where from now on $l$ stands for $(\mathbf{k}, k_5)$:

$$\chi_l'' + (\Omega_l^2 + Q)\chi_l = 0 \ , \tag{8}$$

where

$$\Omega_l^2(\eta) \equiv k^2 - \kappa + C(\eta)\left[\frac{k_5^2}{g_{55}(\eta)} + m^2\right] ,$$

$$Q(\eta) \equiv \xi C(\eta) R(\eta) - \frac{H''}{H} ,$$

$$H(\eta) \equiv C^{1/2}(\eta) g_{55}^{1/4}(\eta) ,$$

and a prime denotes differentiation with respect to $\eta$.

In addition, the modes $u_l$ must be normalized in the usual Klein-Gordon scalar product, which lead to the Wronskian condition on $\chi_l(\eta)$:

$$\chi_l'^{*}\chi_l - \chi_l'\chi_l^{*} = i \ . \tag{9}$$

We shall concentrate on massless particles $m=0$, in the five-dimensional sense. Since the anisotropy introduced by the fifth dimension breaks conformal invariance, such particles will be created as a consequence of cosmological evolution.

In what follows and unless it is otherwise stated, we shall refer to the $\kappa=0$ case only. In order to compute the number of particles created we must find the Bogoliubov transformation relating the set of "in" and "out" modes. Since these cosmological models have a singularity at $t=0$, it is difficult to find a natural candidate for the in modes. For cosmological models having flat spatial sections, as is the case in our $\kappa=0$ model, such difficulty is usually circumvented[26] by assuming that before some time $t_0$ larger than the Planck time the metric is matched to flat spacetime, thus providing the unambiguous standard Minkowski in vacuum state. An abrupt matching at $t=t_0$, i.e., the metric is continuous but its first derivatives are not, implies a so-called "sudden approximation" which gives rise to ultraviolet divergences that must be removed using an ultraviolet cutoff parameter related to the inverse of the transition time from the Minkowski to the curved spacetime, here typically of order $t_0^{-1}$.

On the other hand, it is easy to find a natural candidate for the out vacuum state, since for large $t$ the universe expansion is an adiabatic one and we can take the usual adiabatic out vacuum (no particles are created at later times). In order to solve Eq. (7) we note that it admits formal WKB solutions of the form[14]

$$\chi_l = \frac{\alpha_l}{\sqrt{2\Omega_l}} \exp\left[-i\int^{\eta}\Omega_l d\eta\right] + \frac{\beta_l}{\sqrt{2\Omega_l}} \exp\left[+i\int^{\eta}\Omega_l d\eta\right] , \tag{10}$$

where we have replaced $\chi_l$ by two complex functions $\alpha_l$ and $\beta_l$; therefore, we can still impose a further restriction:

$$\chi_l' = -i\Omega_l \left[\frac{\alpha_l}{\sqrt{2\Omega_l}} \exp\left[-i\int^{\eta}\Omega_l d\eta\right] - \frac{\beta_l}{\sqrt{2\Omega_l}} \exp\left[+i\int^{\eta}\Omega_l d\eta\right]\right] . \tag{11}$$

Then introducing Eqs. (10) and (11) into Eqs. (8) and condition (9) we get

$$\alpha_l' = \frac{1}{2}\left[\frac{\Omega_l'}{\Omega_l} - i\frac{Q}{\Omega_l}\right] \exp\left[+2i\int\Omega_l d\eta\right]\beta_l - i\frac{Q}{2\Omega_l}\alpha_l ,$$

$$\beta_l' = \frac{1}{2}\left[\frac{\Omega_l'}{\Omega_l} + i\frac{Q}{\Omega_l}\right] \exp\left[-2i\int\Omega_l d\eta\right]\alpha_l + i\frac{Q}{2\Omega_l}\beta_l , \tag{12}$$

and

$$|\alpha_l|^2 - |\beta_l|^2 = 1 \ . \tag{13}$$

For times prior to $t=t_0$ the spacetime is assumed to be flat so that

$$\Omega_l^2 = k^2 + \frac{C}{\sqrt{g_{55}}} k_5^2 = \text{const}$$

and the modes $\exp(\pm i\int^{\eta}\Omega_l d\eta)$ occurring in Eq. (10) are the positive- and negative-frequency modes defining the Minkowski in vacuum. After $t_0$ such modes are the zeroth-order adiabatic positive- and negative-frequency modes. In the large-$t$ limit the expansion rate vanishes and the adiabatic approximation becomes exact. Therefore these modes define the adiabatic out vacuum. Note that in the $k_5=0$ case such out vacuum may also be regarded from the four-dimensional point of view as a conformal vacuum if we assume that, as $\eta \to \infty$, the physical size of the extra dimension $L_{\text{ph}}$ approaches a constant value. Indeed, we have

$$u_{\mathbf{k},0}^{(+)}(x) \underset{\eta\to\infty}{\sim} \frac{1}{L_{\text{ph}}^{1/2}} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2k}} \frac{e^{-ik\eta}}{C^{1/2}(\eta)} ,$$

where $u_{\mathbf{k},0}^{(+)}$ is the positive-frequency part of Eq. (6). Then the right-hand side of this expression is nothing but a conformal positive-frequency mode for a four-dimensional conformally flat metric with conformal factor $C(\eta)$.

In order to find the Bogoliubov transformation between the in and out modes we pick up a positive-frequency in mode, i.e., we set initial conditions

$$\alpha_l(\eta_0) = 1, \quad \beta_l(\eta_0) = 0 \ , \tag{14}$$

in Eq. (10) and let this mode evolve in time. As $t$ becomes large enough ($t \to \infty$) the functions $\alpha_l$ and $\beta_l$ must settle down to constant values which tell us what linear combination of positive- and negative-frequency out modes makes up the positive-frequency in mode. The number of particles created in mode $l$ is then given by

$$N_l = \lim_{\eta\to\infty} |\beta_l|^2 . \quad (15)$$

Following Zel'dovich and Starobinsky,[14] Eqs. (12) and (13) with initial conditions (14) can be cast in the form

$$\frac{ds_l}{d\eta} = \frac{\Omega_l'}{2\Omega_l} v_l + \frac{Q}{2\Omega_l} r_l ,$$
$$\frac{dv_l}{d\eta} = \frac{\Omega_l'}{\Omega_l}(1+2s_l) - \left[\frac{Q}{\Omega_l} + 2\Omega_l\right] r_l , \quad (16)$$
$$\frac{dr_l}{d\eta} = \frac{Q}{\Omega_l}(1+2s_l) + \left[\frac{Q}{\Omega_l} + 2\Omega_l\right] v_l ,$$

with initial conditions

$$s_l = r_l = v_l = 0 , \quad (17)$$

at $\eta=\eta_0$, where $s_l \equiv |\beta_l|^2$,

$$v_l \equiv 2\,\mathrm{Re}\left[\alpha_l \beta_l^* \exp\left[-2i\int^{\eta} \Omega_l d\eta\right]\right] ,$$

and

$$r_l \equiv 2i\,\mathrm{Im}\left[\alpha_l \beta_l^* \exp\left[-2i\int^{\eta} \Omega_l d\eta\right]\right] .$$

Thus, to determine the amount of particles produced in mode $l$ we must solve the first-order differential system (16) with the initial conditions (17) and determine $s_l$ when $\eta\to\infty$.

Let us conclude this section with a few remarks on the $\kappa=\pm 1$ models. The initial conditions (14) have been derived for the $\kappa=0$ case as a consequence of its matching to Minkowski space at $\eta=\eta_0$. However, in a different spirit, they may also be seen as a choice of a zeroth-order adiabatic vacuum state at $\eta=\eta_0$. This later method is often used[12,13] for metrics which do not have flat spatial sections, as is the case in our $\kappa=\pm 1$ models. Therefore, the system (16) can also be used for computing the amount of particles produced in the $\kappa=\pm 1$ models. The initial conditions (17) are the same, although their interpretation is different.

## III. NUMERICAL SOLUTION FOR A "REALISTIC" MODEL

In this section we shall present the numerical solution of Eq. (16) for a "realistic" model ("realistic" in the sense specified in the Introduction). Let us consider the space-time metric

$$ds^2 = -dt^2 + Bt(dx^2+dy^2+dz^2) + \frac{1}{4}\frac{(A+2t)^2}{At+t^2}(dx^5)^2 . \quad (18)$$

It corresponds to a radiative flat FRW model with an extra dimension which contracts to a constant value. The arbitrary parameter $B$ is physically irrelevant because it can be rescaled or eliminated through a redefinition of the $\mathbf{x}$ coordinates; we shall give it dimensions of length, so that $\mathbf{x}$ has no dimensions. On the other hand, the parameter $A$ ($A\geq 0$) is a meaningful one. It tells us how large the physical size of the extra dimension is at $t=t_0$ as compared to its asymptotic size for large times. If $A=0$, then $g_{55}=$const. We are mainly interested in the case $A\neq 0$, because we want to study the effect of cosmological contraction on particle production. For instance, when $A \gg t_0$ the physical size of the extra space at $t=t_0$ is $(A/4t_0)^{1/2}$ times larger that its size for large cosmological times. The metric (18) does not correspond to a solution of the five-dimensional Einstein equations in vacuum, but to an anisotropic source (responsible for the anisotropy) of the type $T_{\mu\nu}=\mathrm{diag}(\rho,p,p,p,p_5)$. At early times $t \ll A$ the metric has a Kasner behavior and therefore $T_{\mu\nu}\simeq 0$, whereas at late times $t \gg A$ we have $p=\frac{1}{3}\rho$, $p_5=0$ with $\rho=\rho_c=3/(32\pi G_5 t^2)$, where $G_5$ is the five-dimensional gravitational constant; i.e., it is a radiative isotropic universe with an extra dimension of constant size.

The parameter $A$ must be chosen in order that the metric (18) accommodates the constraints on the time variation of fundamental constants. These are the fine-structure constant ($\alpha_{\mathrm{em}}$), the strong ($\alpha_s$) and weak ($\alpha_w$) couplings, and the gravitational constant ($G$). They are related to the size of the internal space as[1]

$$\alpha_{\mathrm{em}},\alpha_s,\alpha_w \propto g_{55}^{-1}(t), \quad G \propto g_{55}^{-D/2}(t) ,$$

where $D$ is the dimension of the internal space, $D=1$ in our case. The time variations are severely constrained by experimental observations. A typical bound based on nucleosynthesis and the observed $^4$He abundance is[21,22]

$$\frac{g_{55}^{1/2}(t_n)}{g_{55}^{1/2}(t_1)} = 1\pm 0.01 ,$$

where $t_n$ is the nucleosynthesis time ($t_n\simeq 1$ sec) and $t_1$ is the age of the Universe ($t_1\simeq 2\times 10^{17}$ sec). In order to make Eq. (18) compatible with such a constraint, the parameter $A$ must be restricted by

$$A < 0.4 t_n . \quad (19)$$

Now, the present size $L_{\mathrm{ph}}(t_1)$ of the internal space can be estimated from the value of the electric charge to be[20] $L_{\mathrm{ph}}(t_1)\simeq 2.38\times 10^{-31}$ cm, but $L_{\mathrm{ph}}(t_1)\simeq 2\pi R_5$ because $g_{55}(t_1)\simeq 1$. Therefore,

$$R_5 \simeq 3.78\times 10^{-32}\ \mathrm{cm} , \quad (20)$$

which in turn determines the spectrum of $k_5$ through Eq. (7). In what follows we shall keep $R_5$ as a parameter in the relevant expressions.

We can now proceed to the numerical evaluation of $N_l$. With the metric (18) we can write in Eq. (8) the functions

$$\Omega_l^2 = k^2 + \left[4t_0 k_5 \frac{\eta}{\psi}\right]^2 ,$$
$$Q = (2\xi - \tfrac{1}{2})\left[\frac{\psi''}{\psi} + \frac{2}{\eta}\frac{\psi'}{\psi}\right] + \frac{1}{4}\left[\frac{\psi'}{\psi}\right]^2 , \quad (21)$$

with

$$\psi \equiv \frac{(A/t_0)+2\eta^2}{\eta\sqrt{(A/t_0)+\eta^2}} ,$$

where, for convenience, we have set $B=4t_0$ so that $\eta=(t/t_0)^{1/2}$. We can now compute $N_l$ for different values of $k$ and $k_5$, numerically integrating the system of Eq. (16) with the boundary conditions (17) at $t=t_0$ and using Eq. (15). We shall consider the $k_5=0$ and the $k_5 \neq 0$ sectors separately. The latter represents massive particles in the four-dimensional universe.

We shall consider the following cases corresponding to different values of the parameters $A$ and $\alpha$, where

$$\alpha \equiv t_0/t_P , \tag{22}$$

all of which are roughly compatible (we shall be more specific later on) with the condition that the physical size of the extra dimension at $t_0$, $2\pi R_5[g_{55}(t_0, A)]^{1/2}$, be of the same order of magnitude as the horizon, $t_0$:

case $a_1$: $A=0$, $\alpha \sim 2\pi R_5/l_P$ ,

case $a_2$: $A=10t_0$, $\alpha \sim 4\pi R_5/l_P$ ,

case $b$: $A=(\alpha l_P/\pi R_5)^2 t_0$, $\alpha > 10\pi R_5/l_P$ ,

where $\pi R_5/l_P \simeq 73.5$ if we take the value of $R_5$ given by Eq. (20). Case $b$ falls into the class with $A \gg t_0$ and corresponds to models that have a Kasner period.

### A. Massless particles

First of all we concentrate on the $k_5=0$ sector, which corresponds to massless particles in the four-dimensional picture.

(i) Case $a_1$: In this case Eq. (8) reduces to

$$\chi''_{\mathbf{k},0}+k^2\chi_{\mathbf{k},0}=0$$

and therefore we have $N_{\mathbf{k},0}=0$ for all $\mathbf{k}$. This is because it is equivalent to the problem of massless particle creation in a conformally flat four-dimensional cosmology. We do not have to worry about the coupling $\xi$ because for $A=0$ and a radiative universe we have $R=0$.

(ii) Case $a_2$: On physical grounds we expect that a significant contribution to the energy density of created particles will come from modes such that

$$\omega(k) \sim t_0^{-1} , \tag{23}$$

where $\omega(k)=k/\sqrt{C(t_0)}$ is the physical frequency of the mode $k$ at $t_0$. With our choice of parameters, Eq. (23) implies $k \sim 2$. On the other hand, the sudden approximation is not reliable when the frequencies involved are greater than the inverse of the transition time ($\sim t_0^{-1}$); we shall discuss this is some detail in Sec. IV. Therefore we shall not be concerned too much about the region $k>2$ because we expect that in it the particle number will be exponentially suppressed. We shall introduce a cutoff parameter $k_{\max}$ to account for this.

In Fig. 1 we have represented the numerical result for the evolution of $|\beta_{1,0}(\eta)|^2$. In this case the coupling is relevant and we have used the conformal one: $\xi=\frac{3}{16}$. We see from the figure that $|\beta_{1,0}(\eta)|^2$ grows very fast from $\eta=\eta_0=1$ to $\eta=2$ and then, after a mild oscillation, quickly stabilizes to a constant value $N_{1,0}$ of order $10^{-4}$. Similarly, we compute $N_{k,0}$ for different values of $k$ ranging from 0 to 8.

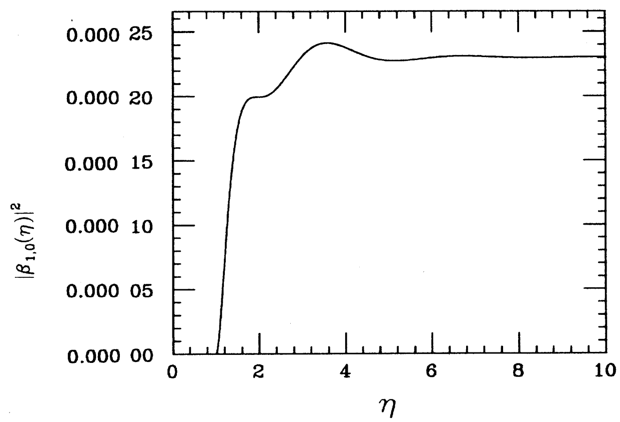


FIG. 1. Time evolution, in conformal time $\eta$, of the function $|\beta_{1,0}(\eta)|^2$ obtained by numerical integration of Eq. (16). It stabilizes at $2.3\times10^{-4}$ giving the number of massless particles produced in mode $k=1$ according to Eq. (15). The coupling parameter is $\xi=\frac{3}{16}$ and $A=10t_0$.

A physical quantity of interest to us is the total number density, in ordinary space, of particles created in mode $k_5=0$. This is given by[24]

$$n_{k_5=0}=\frac{1}{2\pi^2c^3C^{3/2}(t)}\int_0^{k_{\max}} k^2N_{k,0}dk , \tag{24}$$

where $C(t)=4t_0t$, $k_{\max}$ is the appropriate cutoff, and we have reintroduced the speed of light $c$. In order to estimate the integral occurring in Eq. (24) it is useful to represent the spectrum $k^2N_{k,0}$ in the relevant range for $k$. This is depicted in Fig. 2(a); from it we see that the integral in Eq. (24) is of order $10^{-5}$ when we take $k_{\max}\sim 2$. Therefore we have

$$n_{k_5=0} \sim 10^{-38}(t_0t)^{-3/2}\left[\frac{\text{sec}}{\text{cm}}\right]^3$$
$$\sim \alpha^{-3/2}10^{27}\left[\frac{t}{\text{sec}}\right]^{-3/2} \text{particles/cm}^3 .$$

Since the spectrum of these particles is not thermal, one may gain a better insight by comparing its contribution to the energy density with the critical density for a FRW radiation-dominated universe.

The energy density of the created particles in ordinary space is given by[24]

$$\rho_{k_5=0}=\frac{\hbar}{2\pi^2c^3C^2(t)}\int_0^{k_{\max}} k^3N_{k,0}dk . \tag{25}$$

This integral can be estimated from the spectrum $k^3 N_{k,0}$ depicted in Fig. 2(b) and it turns out to be also of order $10^{-4}$. Since the critical energy density of a FRW radiation universe in ordinary space is given by[28]

$$\rho_c = \frac{3c^2}{32\pi G t^2}\ , \tag{26}$$

we have

$$\frac{\rho_{k_5=0}}{\rho_c} = \frac{1}{3\pi\alpha^2}\int_0^{k_{\max}} k^3 N_{k,0}\,dk \sim \frac{10^{-5}}{\alpha^2}\ , \tag{27}$$

where $\alpha \sim 10^2$ and therefore $\rho_{k_5=0}/\rho_c \sim 10^{-9}$.

So far, we have presented the results for $\xi = \frac{3}{16}$. To see what is the effect of changing the coupling parameter we have computed $N_{k,0}$ for $k=1$, $A=0.4t_0$, and different values of $\xi$:

$$\xi = \tfrac{3}{16}:\ N_{1,0} = 1.7\times 10^{-6}\ ,$$

$$\xi = 0:\ N_{1,0} = 2.9\times 10^{-5}\ ,$$

$$\xi = \tfrac{1}{4}:\ N_{1,0} = 1.3\times 10^{-9}\ .$$

Therefore we see that for minimal coupling the number of particles is enhanced by 1 order of magnitude while for $\xi = \frac{1}{4}$ it is suppressed by 3 orders of magnitude. This can easily be understood from the functions in Eq. (21).

(iii) Case $b$: In this case we have $A > 100t_0$, and therefore the size of the extra space at $t = t_0$ is considerably larger than its asymptotic size for large $t$. Such metrics undergo a Kasner period for times $t \ll A$, because then we have $g_{55} \sim A/4t$.

First of all we can start by computing $|\beta_{k,0}(\eta)|^2$ for $k=1$ as we did in case $a_2$. The results for $A = 10^2 t_0$ and $A = 10^4 t_0$ are depicted in Fig. 3, curves (a) and (b), respectively. We can see that in passing from $A = 10^2 t_0$ to $A = 10^4 t_0$, the number of produced particles increases very little. Moreover, there is no point in further increasing $A$. For $A > 10^4 t_0$, the function $|\beta_{1,0}|^2$ looks exactly like curve (b) in Fig. 3. In fact, the limit $A \to \infty$ can be performed in Eq. (21), and the result of the numerical integration in this case is also the same.

This has a simple explanation. We note from Fig. 3 that the particle production occurs for times $\eta < 10$, i.e., $t = \eta^2 t_0 \ll A$ and therefore the Kasner approximation $A \to \infty$ is a good one. This is a very interesting result, because the problem of particle creation in the Kasner model can be analytically solved for the massless $k_5 = 0$ modes, as will be shown in Sec. IV. Although $A \to \infty$ is not in agreement with the bound (19), it should be regarded as a good approximation to the large $A$ case. From the results obtained for the $k=1$ mode we can estimate the integral occurring in Eq. (27) to be of the order of $10^{-3}$ (this estimate is confirmed in Sec. IV when we present the analytical results); therefore, we have

$$\frac{\rho_{k_5=0}}{\rho_c} \sim \frac{10^{-4}}{\alpha^2}\ , \tag{28}$$

where $\alpha > 10^3$. The effect of the coupling $\xi$ is not relevant in this case, because for the Kasner metric we have $R = 0$.

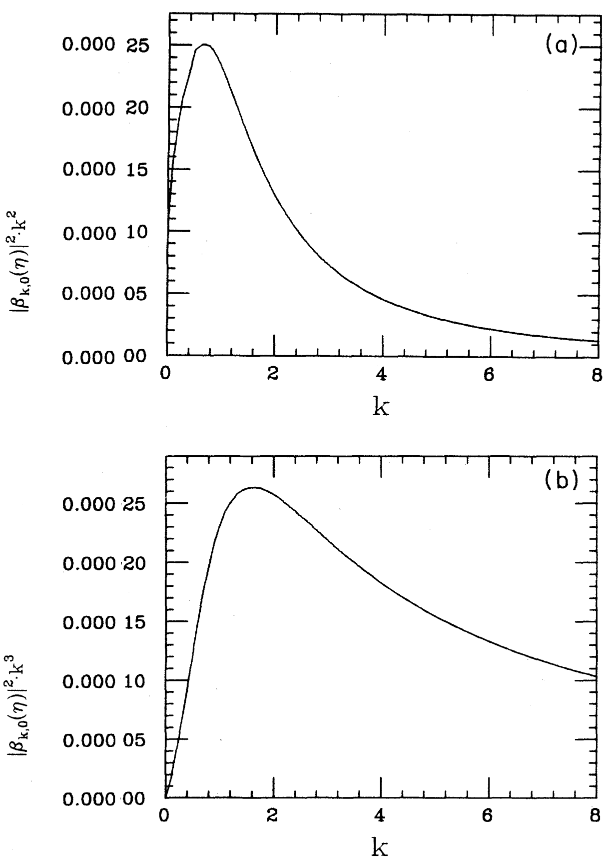


FIG. 2. (a) This shows the integrand in Eq. (24) to compute the number density of massless particles produced by the same model as in Fig. 1. The results for large momentum are overestimated since we have used the "sudden approximation" which implies that a cutoff momentum should be taken at $k_{\max} \sim 2$. (b) This shows the integrand in Eq. (25) to compute the energy density of massless particles produced. The same model and comments of (a) apply.

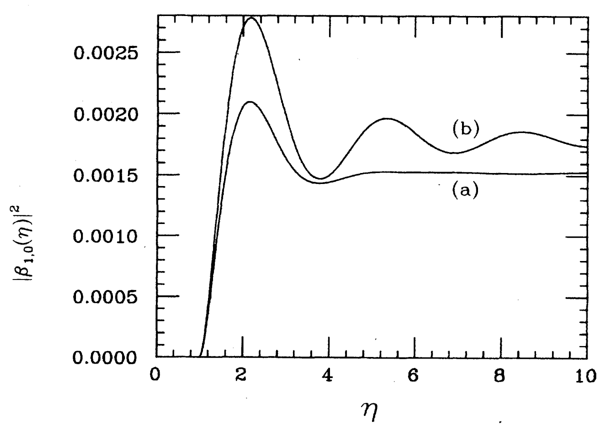


FIG. 3. Time evolution, in conformal time, of the function $|\beta_{1,0}(\eta)|^2$ obtained by numerically integrating Eq. (16). As in Fig. 1 the asymptotic values give the number of particles produced in mode $k=1$: they are $1.5\times 10^{-3}$ for curve (a), corresponding to a model with $A = 10^2 t_0$, and $1.7\times 10^{-3}$ for curve (b), corresponding to a model with $A = 10^4 t_0$. Curve (b) remains unchanged when we increase $A$. In both cases the coupling parameter is $\xi = \frac{3}{16}$.

As a conclusion to the $k_5=0$ case we may note that Eq. (27) for $\alpha=10^2$ and Eq. (28) for $\alpha=10^3$ both give similar results of order $10^{-9}-10^{-10}$. In the intermediate region $10^2\leq\alpha\leq10^3$, Eq. (28) gives a larger number, while Eq. (27) leads to a smaller estimate. Therefore we may use Eq. (28) as an upper bound to the contribution of the created massless particles to the total energy density, for every reasonable value of $\alpha$.

### B. Massive particles

First of all we would like to be a bit more specific about the condition relating the size of the internal space to the horizon size at $t=t_0$ by introducing a parameter $f$:

$$2\pi R_5\sqrt{g_{55}(t_0)}=ft_0 \ . \tag{29}$$

Recalling Eq. (7) we have from Eq. (29) that

$$q_5^{(n)}\equiv\omega(k_5)t_0=\frac{2\pi n}{f} \ , \tag{30}$$

where $\omega(k_5)=k_5/\sqrt{g_{55}(t_0)}$ is the physical frequency of a wave in mode $k_5$ at $t_0$.

As we have stressed before, the number of particles created in modes such that $q_5>1$ is exponentially suppressed. Therefore, if we assume $f\sim1$, only a few massive modes may become significantly excited.

The total number density of particles created in a particular mode $k_5$ is [see Eq. (24)]

$$n_{k_5}=\frac{I_{k_5}}{2\pi^2c^3C^{3/2}(t)} \ , \tag{31}$$

where

$$I_{k_5}=\int_0^{k_{\max}}k^2N_{k,k_5}dk \ ,$$

and $k_{\max}$ is the suitable cutoff parameter. The integral $I_{k_5}$ can be roughly estimated to be of the same order as $N_{1,k_5}$ (as we did in the case $k_5=0$). Therefore we write

$$I_{k_5}=\lambda N_{1,k_5} \ ,$$

where $\lambda$ is presumably of order one, its precise value depending on the function $k^2N_{k,k_5}$ and the cutoff $k_{\max}$. However, our conclusions will not depend strongly on such details, and they would also hold even if $\lambda$ differed from unity by, say, 1 order of magnitude.

We have numerically computed $N_{1,k_5}$ for the cases $a_1$, $a_2$, and $b$ defined above. The computations have been done for the first massive mode ($n=1$) and for different values of $f$ ranging from 1 to $4\pi$. The corresponding values for $q_5$ are obtained from Eq. (30). In cases $a_1$ and $a_2$ we have definite values for the parameter $A$, while in case $b$ our results are within the same order of magnitude for every $A$, providing that it is in the allowed range $A>100t_0$. The results are the following.

Case $a_1$, $N_{1,k_5}$:

$$4\times10^{-4}(f=1),\ 6\times10^{-3}(f=2\pi),\ 6\times10^{-3}(f=4\pi) \ ,$$

Case $a_2$, $N_{1,k_5}$:

$$1\times10^{-2}(f=1),\ 2\times10^{-2}(f=2\pi),\ 2\times10^{-2}(f=4\pi) \ ,$$

Case $b$, $N_{1,k_5}$:

$$10^{-3}(f=1),\ 10^{-2}(f=2\pi),\ 10^{-2}(f=4\pi) \ .$$

From these results we can see that $I_{k_5}$ is a small quantity, typically of order $10^{-3}-10^{-2}$. Note that we have included the results for $q_5=2\pi$, although these modes are outside the region where the sudden approximation is valid, and the number of particles is overestimated as we shall see in detail in the next section.

As we did for the particles in the $k_5=0$ mode, the contribution of one of these massive modes to the energy density $\rho_{k_5}$ can be compared with the critical energy density $\rho_c$. We have

$$\rho_{k_5}=\frac{\hbar}{2\pi^2c^3C^{3/2}(t)}\times\int_0^{k_{\max}}\left[\frac{k_5^2}{g_{55}(t)}+\frac{k^2}{C(t)}\right]^{1/2}k^2N_{k,k_5}dk$$
$$>\frac{\hbar}{2\pi^2c^3C^{3/2}(t)}\frac{q_5}{t_0}I_{k_5} \ ,$$

so that

$$\frac{\rho_{k_5}}{\rho_c}>\frac{2q_5I_{k_5}}{3\pi\alpha^{5/2}}\left[\frac{t}{t_P}\right]^{1/2} . \tag{32}$$

Therefore, as time evolves, the contribution of the massive modes becomes more and more important, and it would very soon dominate over $\rho_c$. For sensible values of $q_5$ and $\alpha$ this would happen even before the nucleosynthesis time. This is because, whereas, as a consequence of cosmological evolution, particles in the massless sector lose their energy through red-shift, the physical frequency in the fifth dimension is blue-shifted and particles with $k_5\neq0$ acquire energies of the order of the Planck mass.

Equation (32) raises a problem, because even if a single massive mode is excited, then its contribution to the energy density becomes so large that back reaction should be taken into account and the cosmological model that we started with breaks down. One may think that a possible solution to this problem is that the massive modes decay into massless particles very soon after $t_0$, so that they would also become red-shifted. The scalar field that we have considered is a free field, but realistic fields are interacting, and a decay of the massive modes can be imagined in principle. However, such a mechanism seems to us very inefficient. Indeed, conservation of momentum implies that a massive particle in the $n=1$ sector, for instance, can only decay into massless particles through interaction with another $n=1$ massive particle. The mean free path of such particles in ordinary space is given by $\lambda_{\text{free}}=(n_{k_5}\sigma)^{-1}$, where $\sigma$ is the cross section corresponding to the process of interaction. It will be typically of the form $\sigma\simeq g^2E^{-2}$, where $g$ is a coupling constant and $E$ is the energy at which the interaction takes place. Taking $f\sim1$ we have $E\geq2\pi/t_0$, hence

$$\lambda_{\rm free} \geq \frac{64\pi^4}{I_{k_5} g^2} \left[\frac{t}{t_0}\right]^{1/2} t \; . \tag{33}$$

Therefore, the mean free path is larger, and increases with time faster, than the horizon size, thus rendering the process of decay completely ineffective.

Nevertheless there is still a way out to the problem raised by Eq. (32) which will be presented in Sec. IV (after the sudden approximation has been discussed in some detail) because it falls beyond the reach of the numerical analysis presented in this section.

## IV. ANALYTICAL RESULTS

In this section we shall work in the approximation $A \to \infty$; i.e., a Kasner metric. As we have seen in Sec. III, this is a good approximation to our model (18) whenever $A > 100 t_0$.

### A. Massless particles

In this approximation, the problem of particle creation for the massless ($k_5=0$) modes can be solved analytically. In fact, we have $\Omega_{k,0}^2 = k^2$ and $Q=(4\eta^2)^{-1}$ and Eq. (8) becomes

$$\chi_{k,0}'' + \left[k^2 + \frac{1}{4\eta^2}\right] \chi_{k,0} = 0 \; , \tag{34}$$

whose general solution can be expanded in terms of Bessel functions as

$$\chi_{k,0} = \sqrt{\eta}[C_1 J_0(k\eta) + C_2 Y_0(k\eta)] \; .$$

A particular combination of the coefficients $C_1$ and $C_2$ can be used to construct modes normalized according to Eq. (9) which approach positive-frequency adiabatic modes in the large time limit $\eta \to \infty$:

$$\chi_{k,0}^{\rm out}(\eta) = \left[\frac{\pi\eta}{4}\right]^{1/2} H_0^{(2)}(k\eta) \underset{\eta\to\infty}{\sim} \frac{e^{-i(k\eta-\pi/2)}}{\sqrt{2k}} \; ,$$

where $H_n^{(2)}(k\eta) \equiv [J_n(k\eta) - iY_n(k\eta)]$ is the Hankel function of the second kind and order $n$. At time $\eta = \eta_0$ these out modes are matched to a combination of positive- and negative-frequency Minkowski modes:

$$\chi_{k,0}^{\rm out}(\eta) = \begin{cases} \sqrt{\pi\eta/4} H_0^{(2)}(k\eta), & \eta > \eta_0 & (35a) \\ A_1 e^{-ik(\eta-\eta_0)} + A_2 e^{ik(\eta-\eta_0)}, & \eta < \eta_0 \; . & (35b) \end{cases}$$

It is now clear that the coefficients $A_1$ and $A_2$ in Eq. (35b) are proportional to the Bogoliubov coefficients $\alpha_{k,0}$ and $\beta_{k,0}$ which give us the number of created particles.

From Eq. (35b) we have

$$\lambda \equiv \frac{\chi_{k,0}'^{\rm out}(\eta_0)}{\chi_{k,0}^{\rm out}(\eta_0)} = \frac{-ik(A_1 - A_2)}{A_1 + A_2} \; , \tag{36a}$$

while from Eq. (35a) we get

$$\lambda = \frac{1}{2\eta_0} - k\frac{H_1^{(2)}(k\eta_0)}{H_0^{(2)}(k\eta_0)} \; . \tag{36b}$$

Equation (36a) leads to $\alpha_{k,0}/\beta_{k,0} = A_1/A_2 = (1-\lambda/ik)/(1+\lambda/ik)$; then substituting $\lambda$ from Eq. (36b) and using Eq. (13), i.e., $|\alpha_{k,0}|^2 - |\beta_{k,0}|^2 = 1$, we find $|\beta_{k,0}|^2$ and thus

$$N_{k,0} = |\beta_{k,0}|^2 = \frac{\pi q}{8}\left[ [(Y_0 - J_1)^2 + (J_0 + Y_1)^2] - \frac{1}{q}(J_0 J_1 + Y_0 Y_1) + \frac{1}{4q^2}(J_0^2 + Y_0^2) \right] \; , \tag{37}$$

where $q \equiv k\eta_0$ is the argument of the Bessel functions appearing in Eq. (37). Such a variable $q$ is similar to $q_5$ introduced in Sec. III, since it can be written as $q = 2\omega(k) t_0$, where $\omega(k)$ is the physical frequency of a mode $k$ at $t = t_0$.

The number density of created particles and the energy density per unit proper volume are given by Eqs. (24) and (25), which can now be cast in the form

$$n_{k_5=0} = \frac{1}{16\pi^2 c^3 (t t_0)^{3/2}} \int q^2 N_{q,0} dq \tag{38}$$

and

$$\rho_{k_5=0} = \frac{\hbar}{32\pi^2 c^3 t^2 t_0^2} \int q^3 N_{q,0} dq \; . \tag{39}$$

Note that if we decrease $t_0$, then the energy density increases like $t_0^{-2}$. This can also be derived on dimensional grounds. In fact, setting $\hbar = c = 1$, we know that the energy density has dimensions of $t^{-4}$, and this must be made out of the only time parameters at hand, $t$ and $t_0$. Now, the dependence on the cosmological time $t$ is through the well-known relation $\rho_{k_5=0} \sim C^{-2}(t)$ (expansion plus redshift) which means $\rho_{k_5=0} \propto t^{-2}$; therefore we are bound to write $\rho_{k_5=0} \sim t^{-2} t_0^{-2}$.

However, we want to stress that this simple power-law dependence is only valid for $t_0 > 10^3 t_P$ (case $b$ of Sec. III), where the Kasner approximation is a good one, and it cannot be extrapolated back to arbitrary small values of $t_0$. This is because in case $a_2$, the quantity $N_{k,0}$ depends on $(A/t_0)$ in a complicated way. Now, such dependence does not appear in Eq. (37), because we have sent $A$ to infinity.

As we did in case $a_2$ of the previous section, we plot the spectra $q^2 N_{q,0}$ and $q^3 N_{q,0}$ as a function of $q$ in Figs. 4(a) and 4(b), in order to estimate the integrals in Eqs. (38) and (39). However, the integral in Eq. (39) has an ultraviolet logarithmic divergence. In fact, the large-$q$ expansion of Eq. (37) reads

$$N_{q,0} = \frac{1}{(4q)^4} - \frac{9}{1024 q^6} + \cdots \; . \tag{40}$$

In order to clarify the origin of such divergence, the following discussion may prove useful. As stressed by Hu,[26] there is a close analogy between the problem of particle creation in an expanding homogeneous cosmology and the problem of scattering over a potential barrier in quantum mechanics: compare Eq. (8), or Eq. (34), with

$$\frac{d^2\psi}{dx^2}+(E-V)\psi=0 \ . \tag{41}$$

In the particle creation problem we have positive-energy in modes which can be expanded in terms of out modes of positive and negative frequency with some coefficients $\alpha_l$ and $\beta_l$. The relevant coefficient for particle creation is that of the negative-frequency wave. The equivalent scattering problem in quantum mechanics is that of a left-traveling incoming wave, playing the role of a positive-frequency out mode, which is partially transmitted through the barrier and partially reflected back as a right-traveling outgoing wave, playing the role of a negative-frequency out mode. The reflection coefficient is then relevant for particle production.

In the case at hand, we are dealing with the potential

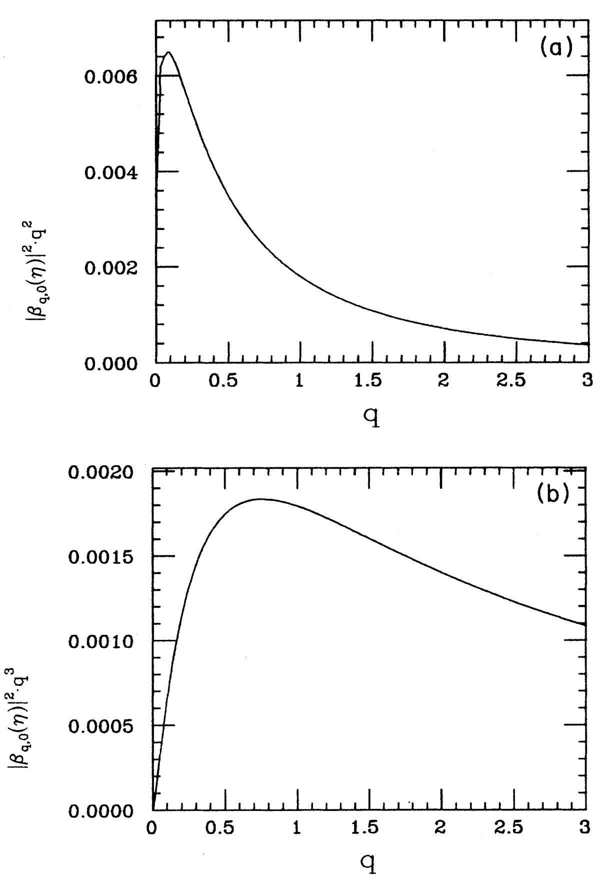


FIG. 4. (a) This shows the integrand in Eq. (38) to compute the number density of massless particles in the model with a Kasner period. As in Fig. 2(a) the results for large $q$ are overestimated due to the "sudden approximation" used and we need to introduce a cutoff parameter $q \sim 2$. (b) This shows the integrand in Eq. (39) to compute the energy density of massless particles. Same model and comments as in (a).

$$V(\eta)=-Q(\eta)=\begin{cases}0, & \eta<\eta_0 \ \text{(Minkowski space)},\\ -(4\eta^2)^{-1}, & \eta>\eta_0 \ .\end{cases} \tag{42}$$

It is known[24] that if an expansion is bounded by adaibatic in and out regions, then the number of quanta produced as a consequence of such expansion falls off faster than any inverse power of $k$ in the ultraviolet limit. Therefore, the power-law behavior that we get in Eq. (40) must be because of the discontinuity in the potential (42) at $\eta=\eta_0$, which is in fact the so-called "sudden approximation." To illustrate how the divergence is due to such gap, let us consider the potential

$$V(\eta)=-\frac{V_0}{2}\left[1+\tanh\frac{\eta}{\Delta\eta}\right] , \tag{43}$$

which is schematically depicted in Fig. 5, where $\Delta\eta$ is a parameter that can be understood as the transition time between $V(\eta)=0$ and $V(\eta)=-V_0$. This example can be analytically solved as a scattering problem in quantum mechanics and the result is given by[24]

$$N_k=\frac{\sinh^2(\pi\Delta\eta\omega_-)}{\sinh(\pi\Delta\eta k)\sinh(\pi\Delta\eta\omega_{\text{out}})} , \tag{44}$$

where

$$\omega_{\text{out}}=(k^2+V_0)^{1/2} ,$$

$$\omega_-=(\omega_{\text{out}}-k)/2 .$$

When $k\Delta\eta>1$ we can approximate $N_k$ by

$$N_k \sim e^{-2\pi\Delta\eta k} ; \tag{45}$$

i.e., we have an exponential suppression in agreement with the preceeding remarks. However, if we previously perform the limit $\Delta\eta \rightarrow 0$, then the potential (43) turns into a step function (also represented in Fig. 5), which corresponds to the "sudden approximation." In this case the scattering problem leads to

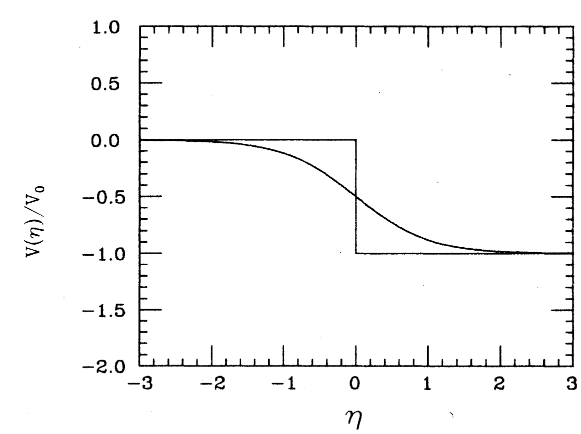


FIG. 5. The potential of Eq. (43) is represented with a transition time parameter $\Delta\eta=1$. It is compared with the step potential, i.e., the limit $\Delta\eta \rightarrow 0$, which corresponds to the "sudden approximation."

$$N_k=\frac{\omega_-^2}{k\omega_{\rm out}}\ , \tag{46}$$

which in the ultraviolet limit gives

$$N_k=\frac{1}{16}\frac{V_0^2}{k^4}+\cdots\ . \tag{47}$$

Note that using in Eq. (47) the value $V_0=-1/4\eta_0^2$, which corresponds to the gap in the potential (42), we arrive precisely at the result given in Eq. (40), in agreement with the claim that the divergences in the energy density are entirely because of the sudden approximation. If our model incorporated a smooth matching, with a transition time $\Delta t$ between flat space and the cosmological model, then one would expect a falloff behavior like the one in Eq. (45). Taking Eq. (2) into account and recalling the meaning of $q$, this could be rewritten as

$$N_{k,0}\simeq e^{-2\pi\omega(k)\Delta t}=e^{-\pi q}\ , \tag{48}$$

where we have used $\Delta t=t_0$. In practice one uses the "sudden approximation" and a cutoff at a frequency such that $\omega(k_{\max})\simeq(\Delta t)^{-1}$, for instance, $q\simeq 2$, as we did in the previous section. This is justified because, according to Eq. (48), increasing $q$ by 1 means decreasing $N_{k,0}$ by more than one order of magnitude.

### B. Massive particles

We can now turn to the massive modes ($k_5\neq 0$). Incidentally, we note that in the $A\to\infty$ limit, the number of particles in a massive mode can also be found analytically in the infrared $k=0$ limit. As usual, the in vacuum is defined by the Minkowski modes and the out vacuum is the adiabatic one. The result is given by

$$N_{0,k_5}=\frac{\pi}{192q_5}\{16q_5^2[(Y_0-J_1)^2+(J_0+Y_1)^2] -8q_5(J_0J_1+Y_0Y_1)+J_0^2+Y_0^2\}\ , \tag{49}$$

where the Bessel functions are evaluated at $\frac{2}{3}q_5$ and $q_5$ is given in Eq. (30). One may note that for the massive particles there is no infrared divergence, contrary to what happens in the massless case, i.e., the limit $k\to 0$ in Eq. (37). Equation (49), however, is not sufficient to estimate the total number density of particles even approximately, since for this we should know $N_{k,k_5}$ for $k\sim 1$ from where the dominant contribution comes. Therefore we shall refer to the numerical results presented in Sec. III.

According to such results, if we are in the region where the "sudden approximation" is reliable, say $q_5\sim 1$, then Eq. (32) predicts that the massive particles would very soon dominate the Universe and back reaction would spoil our model. On the other hand, if $q_5>1$ we expect a falloff behavior of the form (48); i.e.,

$$N_{k,k_5}\simeq e^{-2\pi\omega(k,k_5)\Delta t}=\exp\{-2\pi[q_5^2+(q^2/4)]^{1/2}\Delta t/t_0\}\ .$$

In particular, setting $f\simeq 1$, and recalling that $q_5=2\pi/f$ (for the first massive mode $n=1$) we have

$$N_{k,k_5}\le e^{-4\pi^2/f_1}\simeq 10^{-17/f_1}$$

and

$$I_{k_5}\le q_5^2e^{-2\pi q_5\Delta t/t_0}\simeq\left[\frac{4\pi^2}{f^2}\right]10^{-17/f_1}\ , \tag{50}$$

where $f_1\equiv ft_0/\Delta t$.

Now, for $t>t_n$ when the size of the extra space is constant, the energy of a particle in the first massive mode can be approximated by $\hbar c/R_5$, hence

$$\rho_{k_5}(t>t_n)=\frac{\hbar c}{R_5}n_{k_5}=\frac{\hbar}{R_5}\frac{I_{k_5}}{2\pi^2c^2C^{3/2}(t)}$$

and

$$\frac{\rho_{k_5}(t>t_n)}{\rho_c(t)}\le\frac{1}{3\alpha f^2}\left[\frac{8\pi l_P}{R_5}\right]\left[\frac{t}{t_0}\right]^{1/2}\times 10^{-17/f_1}\ . \tag{51}$$

From Eq. (51) we can see that if we take $f_1\le\frac{1}{2}$, then $\rho_{k_5}\ll\rho_c$ even for times as large as $t=10^{13}$ sec, which is the end of the radiation era, and the contribution of the massive modes is completely irrelevant. However, if we take $f_1\ge 1$ then they could dominate the Universe by nucleosynthesis time.

As a conclusion to this section we can say that if we are careful in the choice of our parameters so that a relation of the form

$$f\le\frac{1}{2}\frac{\Delta t}{t_0} \tag{52}$$

is satisfied, then we can expect that the effect of the particles produced as a consequence of cosmological evolution will be small compared to the driving forces that determine such evolution, and therefore the back reaction can be neglected. On the contrary, if Eq. (52) is violated, which will be the case when the size of the extra dimension is much larger than the horizon at $t_0$, then the dynamics of the model we started with may be substantially altered and cosmological dimensional reduction spoiled.

### ACKNOWLEDGMENTS

It is a pleasure to thank Enrique Alvarez and Diego Harari for valuable suggestions. This work was partially supported by research project Comisión Asesora de Investigación Científica y Técnica.

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