Scattering of quantum particles by gravitational plane waves

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(Received 27 August 1990)

We consider the coupling of quantum massless and massive scalar particles with exact gravitational plane waves. The cross section for scattering of the quantum particles by the waves is shown to coincide with the classical cross section for scattering of geodesics. The expectation value of the scalar field stress tensor between scattering states diverges at the points where classical test particles focus after colliding with the wave. This indicates that back-reaction effects cannot be ignored for plane waves propagating in the presence of quantum particles and that classical singularities are likely to develop.

I. INTRODUCTION

Plane-fronted gravitational waves with parallel rays (*PP* waves) have received some attention recently as it has been realized that they are exact classical solutions to string theory.[1-3] *pp* waves are spacetimes admitting a covariantly constant null vector field l, i.e., $l_{\mu;\nu}=0$, and were classified by Ehlers and Kundt.[4,5] They admit a group of isometries G_1 on a null orbit generated by the Killing vector l, all their curvature scalars vanish and therefore are type N in the Petrov classification. Gravitational plane waves are a subclass of *pp* waves that admit a group G_5 of isometries with an Abelian subgroup G_3 acting on null hypersurfaces and were first studied by Baldwin and Jeffrey.[6] They are assumed to describe the gravitational field at great distances from finite radiating bodies[6,5,7] and they can be purely gravitational, purely electromagnetic or both, depending on the source.

As was first pointed out by Penrose[8] gravitational plane waves have interesting geometrical properties such as the absence of a spacelike Cauchy surface as a consequence of the focusing effect they exert on null rays. This was taken as an indication that regular (i.e., nonsingular) plane waves could develop singularities when two of them collided. And, in fact, exact solutions representing two colliding plane waves have been found with singularities at the focusing points.[9-11] Of course such focusing properties are due to the energetic content of the waves and are not expected to be physically significant when the gravitational field far from the source can be treated in the weak-field approximation. In fact, the time of focusing is typically inversely proportional to the energy density per unit surface of the wave. Such effects, however, may be physically relevant when strong time-dependent gravitational fields are involved such as after the collision of black holes,[12,13] the decay of a cosmological inhomogeneous singularity,[14,15] or by traveling waves on strongly gravitating cosmic strings.[16,17] Gravitational plane waves can be expected to provide local models for such processes, at least qualitatively.

When coupled to quantum fields they also have interesting properties. As was first shown by Gibbons[18] and Deser[19] gravitational plane waves propagate without hindrance by quantum effects. Namely, they induce neither particle creation nor vacuum polarization under interaction with a quantum scalar field. In this sense they behave as electromagnetic or Yang-Mills plane waves in flat spacetime.[19,20] Depending on the source, gravitational plane waves may be considered, in a quantum-mechanical language, as a coherent superposition of gravitons, photons, or both.

In this paper we consider the scattering of massive and massless scalar particle states by gravitational plane waves. Since there is no particle creation this may be considered as the quantum analogue of the focusing of geodesics. Physical information on this process is given by the transition amplitude between scattering states (S matrix) and by the expectation value of the stress tensor of the scalar field between scattering states. After evaluation of the S matrix we compute the scattering cross section and it turns out to coincide with the classical cross section for geodesics. Furthermore we find that the expectation value of the stress tensor diverges at the points of geodesic focusing for all gravitational plane waves.

The divergence of the stress tensor may be taken as an indication that back reaction must be important when considering gravitational plane waves in the presence of quantum particle states and that classical singularities may develop at the focusing points. In more realistic situations, however, the wave may have a finite size and the stress tensor may not diverge at the focusing points;[21] still one should expect that the stress tensor be large there and, consequently, back reaction important.

Incidentally on a spacetime where there is no particle production one should be able to use test quantum particles, as opposed to test classical particles,[22] to define the singularities of spacetime. This is similar to the way Horowitz and Steif[3] define singularities by propagating test quantum strings on spacetime. In general the classical and the "semiclassical" definitions of singularities may differ. This is not the case on a gravitational plane-wave background; the reason is that the geometrical optics approximation is exact in this case, and thus the rays of the waves and the geodesics behave similarly. This is also the reason why the classical and quantum cross sections coincide.

A few comments on the relation of our work with that of other authors are in order. 't Hooft[23] has considered the scattering of a quantum scalar particle by the Aichelburg-Sexl (AS) impulsive (or shock) *pp* wave[24] to study the collision of particles at very high energies. The AS metric is supposed to represent the gravitational field of a massless particle[24,25] or equivalently, that of a particle propagating with kinetic energy larger than the Planck mass. The idea is that at very high energies one cannot ignore the gravitational field of the particles. In this case one of the particles is at rest and its field is quantized in the gravitational field of the propagating particle with which it collides. Although the AS metric is not a plane wave, the scattering has some similarities with that of plane waves. In particular, the classical and quantum cross sections also coincide in this case.

Veneziano,[26] using results of high-energy superstring scattering computes the focusing of two colliding finite-size graviton beams, which can be evaluated in the limit of small angles and when string size effects are neglected. He finds that the gravitons of one of the beams are all focused toward a point located on the beam axis at a given distance from the plane of collision. This point coincides with the focusing point of null rays by an impulsive gravitational plane wave with constant energy density along its (infinite) plane.

Recently Klimcik[21] has considered the scattering of quantum particles by impulsive *pp* waves and the expectation value of the energy density of the scalar field between scattering states. Some of his results have been obtained in the particular case of impulsive plane waves. When we restrict ourselves to impulsive plane waves our results are in agreement.

The plan of the paper is as follows. In Sec. II we review the geometrical properties of gravitational plane waves and introduce two sets of coordinates (harmonic and group coordinates) which are useful in this context.[18] The geodesic equations are solved and discussed in Sec. II A. In Sec. II B the classical cross section is computed for a general "sandwich" plane wave. In a sandwich wave the spacetime curvature is only different from zero in a certain compact region in the null coordinate $u=t-z$, where z is the axis of propagation of the wave. In Sec. II C we discuss the focusing of geodesics by sandwich waves, with some attention to the particular case of impulsive plane waves.

In Sec. III we quantize a scalar field of arbitrary mass coupled to the plane-wave background. The Bogoliubov coefficients between the "in" and "out" modes are computed in Sec. III A for a general sandwich plane wave. In this case we have flat "in" and "out" regions and the particle concept is physically unambiguous. The scattering matrix and the cross sections are computed in Sec. III B and, finally, the expectation value of the stress tensor of the scalar field in vacuum and between scattering states is considered in Sec. III C.

II. GEOMETRICAL PROPERTIES

In this section we shall consider the geometrical properties of gravitational plane waves. In particular we shall analyze the behavior of test particles on such backgrounds.

We start with the most general gravitational *pp* wave. In the standard form,[4,5] its spacetime metric is given by

$$ds^2=-du\,dV+F(u,X^a)du^2+\sum_a dX^a dX^a\ , \tag{1}$$

where (u,V) are null coordinates, X^a are spacelike transverse coordinates, and $F(u,X^a)$ is an arbitrary function. For generality and for its possible relevance to string theory[2] we assume $a=1,\dots,n$, with n an arbitrary number of transverse dimensions. The set of coordinates $\{u,V,X^a\}$ which range over all real values are called harmonic coordinates.[18] From (1) it is clear that $l=2\partial_V$ is a covariantly constant null Killing field and that the metric can be written in the form

$$g_{\mu\nu}=\eta_{\mu\nu}+Fl_\mu l_\nu\ , \tag{2}$$

where $l_\mu=-\delta_{\mu u}$ ($\mu=1,\dots,N$ with $N=n+2$), then $g^{\mu\nu}=\eta^{\mu\nu}-Fl^\mu l^\nu$ and the Riemann and Ricci tensor take the form

$$R_{\mu\nu\rho\sigma}=2l_{[\mu}\partial_{\nu]}\partial_{[\rho}Fl_{\sigma]}\ , \tag{3}$$

$$R_{\mu\nu}=\tfrac{1}{2}(\eta^{\rho\sigma}\partial_\rho\partial_\sigma F)l_\mu l_\nu\ . \tag{4}$$

From this it is clear that all curvature scalar invariants vanish and the metric is Petrov type N; this plays an important role in showing that these metrics are solutions to string theory.[2] In particular the Ricci tensor is proportional to the stress tensor, and from (4) we see that the source is in general a null fluid or, if F is an harmonic function, vacuum.

Gravitational plane waves are the particular case when F is quadratic in $X^a, F(u,X^a)=H_{ab}(u)X^aX^b$, i.e.,

$$ds^2=-du\,dV+H_{ab}(u)X^aX^b du^2+\sum_a dX^a dX^a\ , \tag{5}$$

where $H_{ab}(u)$ is, without loss of generality, a symmetric matrix. From (4) the only nonvanishing component of the Ricci tensor is $R_{uu}=-H^a_a$, which corresponds to the energy density ($R_{00}=R_{uu}$)

$$T_{00}=\frac{-1}{8\pi G}H^a_a\ . \tag{6}$$

The weak-energy condition[22] thus implies

$$H^a_a\le 0\ . \tag{7}$$

Two particular cases of interest are *pure gravitational waves* which correspond to $H^a_a=0$, i.e., a Ricci flat metric, in which the Weyl and Riemann tensors coincide, and *pure null electromagnetic waves*[4] which correspond to $H_{ab}(u)=H(u)\delta_{ab}(H<0)$, in which the Weyl tensor vanishes.

In a quantum-mechanical language pure gravitational waves may be interpreted as a coherent superposition of gravitons and pure null electromagnetic waves as a coherent superposition of photons; in the general case both components are present. Note that the energy density is constant on the plane of the wave which is defined as $u=$const, and that the only curvature singularities we may have are at the points where $H_{ab}(u)$ diverges, in which case one also finds incomplete geodesics.[2]

Harmonic coordinates are a convenient set of coordinates because they cover the whole plane-wave spacetime with a single chart and also because direct information on the curvature is contained in a unique metric component. However they do not display some of the symmetries of the spacetime and thus they are not the best coordinates to use for studying the kinematics of test particles or the wave equation on such spacetime. It is convenient to introduce the so-called *group coordinates* $\{u,v,x^a\}$ where $a=1,\ldots,n$ and the coordinates range over all real values. In these coordinates the metric (5) takes the form

$$ds^2=-du\,dv+g_{ab}(u)dx^a dx^b \ . \tag{8}$$

The relationship between the two sets of coordinates is given by

$$V=v+\tfrac{1}{2}\dot{g}_{ab}(u)x^a x^b,\quad X^a=P^a_b(u)x^b \ , \tag{9}$$

where

$$g_{ab}(u)=P^c_a(u)P^c_b(u) \ , \tag{10}$$

and the matrix $P^a_b(u)$ is determined by solving the differential equation

$$\ddot{P}^a_b(u)=H_{ac}(u)P^c_b(u) \ . \tag{11}$$

Moreover, the system of equations (11) must be solved with initial conditions satisfying the constraint

$$\dot{P}^c_a(u)P^c_b(u)-\dot{P}^c_b(u)P^c_a(u)=0 \ , \tag{12}$$

which is stable against evolution; i.e., once imposed on the initial conditions it holds for all values of u, as can be seen using (11) and the fact that $H_{ab}(u)$ is symmetric. Conversely, given a plane wave in group coordinates (8) one can find its form in harmonic coordinates (5) by solving Eqs. (10) and (12) to find the matrix $P^a_b(u)$ and then using Eq. (11) in order to obtain $H_{ab}(u)$.

The nonzero Christoffel symbols and nonzero components of the Riemann and Ricci tensors in these coordinates are

$$\begin{aligned}&\Gamma^a_{bu}=\tfrac{1}{2}g^{ac}\dot{g}_{cb},\quad \Gamma^v_{ab}=\dot{g}_{ab} \ ,\\ &R^{\ a}_{ubu}=-\partial_u\Gamma^a_{bu}-\Gamma^a_{cu}\Gamma^c_{bu} \ ,\\ &R_{uu}=-\tfrac{1}{2}g^{ab}\ddot{g}_{ab}-\tfrac{1}{4}\dot{g}^{ab}\dot{g}_{ab} \ .\end{aligned} \tag{13}$$

The group coordinates do not cover the whole spacetime with a single chart, because they become singular for some value of the null coordinate u. The reason is that, defining $\gamma(u)=|\det g_{ab}(u)|^{1/2n}$, the weak energy condition implies[18]

$$\frac{\ddot{\gamma}}{\gamma}\leq 0 \ , \tag{14}$$

which means that $\gamma(u)$ is a convex function [the equality in (14) can only hold for flat space] and since γ is positive for some value of u, then it must vanish for at least some other value $u=u_f,\gamma(u_f)=0$. Let us prove (14). Denoting by g the matrix whose components are $g_{ab}(u)$ and using (13) it is easily seen that

$$\frac{\ddot{\gamma}}{\gamma}=\frac{1}{2n}\left[-2R_{uu}+\frac{1}{2}\mathrm{tr}[(\dot{g}^{-1})\dot{g}]+\frac{1}{2n}\mathrm{tr}^2[(g^{-1}\dot{g})]\right] \ . \tag{15}$$

Defining $T=g^{-1}\dot{g}$, Eq. (15) can be rewritten

$$\frac{\ddot{\gamma}}{\gamma}=\frac{1}{2n}\left[-2R_{uu}-\frac{1}{2}\mathrm{tr}(T^2)+\frac{1}{2n}\mathrm{tr}^2(T)\right] \ . \tag{16}$$

Since g is by hypothesis nondegenerate and positive definite it follows that the quadratic forms g and $\dot{g}$ can be simultaneously diagonalized and therefore T is diagonalizable. (In fact, T is the endormorphism associated with the simultaneous diagonalization of g and $\dot{g}$.) Then the traces in Eq. (16) can be expressed as sums of eigenvalues and it is a simple matter to show that the contribution of the terms involving traces is strictly negative unless all eigenvalues of T are equal, in which case these terms just cancel out. On the other hand, the weak-energy condition implies $R_{uu}\geq 0$, which completes the proof of (14). If all eigenvalues of T are equal it is easy to see that $R_{uu}=0$ implies $H_{ab}(u)=0$, so the equality in (14) can only hold for flat space. [Incidentally, there seems to be a misprint in Ref. 18 since it asserts that $(\ddot{\gamma}^2)/\gamma^2\leq 0$, for which counterexamples can be found.]

In summary, there is a coordinate singularity at u_f where $\det g_{ab}(u_f)=0$ and at least two coordinate charts are needed in group coordinates. As we shall see shortly this coordinate singularity is related to the focusing properties of the plane waves.

A. Geodesic equations

It is a simple task to compute and solve the geodesic equations for test particles in group coordinates. These may be deduced from the Lagrangian

$$L_g=-\frac{du}{d\lambda}\frac{dv}{d\lambda}+g_{ab}(u)\frac{dx^a}{d\lambda}\frac{dx^b}{d\lambda} \ , \tag{17}$$

where we have introduced the affine parameter $\lambda=\tau/m$, with τ the proper time of the particle and m its rest mass. This also includes massless particles, taking the limit in which both m and τ tend to zero. The momentum of the particle is then $p^\mu=dx^\mu/d\lambda$. Since the Lagrangian does not depend on the coordinates v and x^a (i.e., ∂_v and ∂_a are Killing vectors) we have that $p_v=g_{vu}p^u$ and $p_a=g_{ab}p^b$ are constants of motion. The conservation of momentum in the v direction implies that u is proportional to the affine parameter, $u=2p_-\lambda$, where we have introduced $p_-\equiv -p_v$ for convenience in the notation. p_- is positive definite for timelike or null geodesics and it vanishes only for massless particles traveling in the same direction that the wave, but these are not very interesting because they never collide with the wave. The conservation of the transverse momentum can be rewritten as $p_a = 2p_- g_{ab}\dot{x}^b$, where the overdot means derivative with respect to u as usual. All that remains, thus, is an equation for $v(u)$ that can be obtained from the constraint $p_\mu p^\mu = -m^2$, which implies $4p_-^2\dot{v} = 2p_- p_a \dot{x}^a + m^2$. The general solution is thus

$$x^a = x_0^a + \frac{p_b}{2p_-}\int_{u_0}^{u} g^{ab}(u')du' ,$$
$$v = v_0 + \frac{m^2}{4p_-^2}(u-u_0) + \frac{p_a p_b}{4p_-^2}\int_{u_0}^{u} g^{ab}(u')du' , \tag{18}$$

where x_0^a, v_0 are the initial positions at $u = u_0$ and we have assumed that the coordinates do not become singular in the interval (u_0, u). If that were the case then one should use two overlapping charts near the singularity of one of the charts. We see that the explicit solution of the geodesic equations is reduced to single integrations. A particularly simple case is that of *perpendicular incidence*, i.e., $p_a = 0$, then

$$x^a = x_0^a, \quad v = v_0 + \frac{m^2}{4p_-^2}(u-u_0) ; \tag{19}$$

i.e., the particles always remain in the same transversal coordinates and describe a straight line in the (u,v) plane.

The focusing of geodesics will be discussed in detail in Sec. II C using harmonic coordinates, which are better suited for this purpose. However, some intuitive arguments can be advanced using the results obtained in group coordinates. Let us define a family $\mathcal{F}(v_0, p_-, p_a)$ of parallel geodesics as the set of all geodesics having the same parameter v_0 and p_-, p_a but different values for x_0^a (i.e., same momentum but different impact parameters). For the particular case of perpendicular incidence we are interested in families with $p_a = 0$. Consider two geodesics from $\mathcal{F}$ whose transverse coordinates differ by Δx_0^a, and recall that at $u = u_f$ we have $\det g_{ab}(u_f) = 0$. Therefore $g_{ab}(u_f)$ has at least one null vector and if Δx_0^a is chosen to be proportional to this null vector then $g_{ab}(u_f)\Delta x_0^a \Delta x_0^b = 0$, which means that the two particles converge at the same point at the same time; i.e., their transverse distance becomes zero and their z coordinate is the same.

An interesting question that arises now is whether the perpendicularly incident parallel beam of geodesics $\mathcal{F}$ focuses into a point or into a more extended structure. This depends obviously on the number of null vectors of $g_{ab}(u_f)$. We may define an equivalence relation in the space of transverse coordinates at $u = u_f$ as follows: two coordinates x_1^a, x_2^a correspond to the same point if their transverse distance is zero. Then if there are n transverse coordinates and the number of null vectors is k, the family $\mathcal{F}$ will focus into a space of dimension equal to the rank of $g_{ab}(u_f)$, $r = n-k$. For instance, if $n = 2$ and $g_{ab}(u_f)$ has two null vectors the geodesics will focus into a point, but if there is only one null vector they will focus into a line. The shape of the focusing surface, however, is better elucidated when we use harmonic coordinates. This is specially true in the case of oblique incidence $(p_a \neq 0)$ since $g_{ab}(u_f)$ is not invertible and the integrals in (18) are ill defined.

Therefore, let us now consider the geodesic equations in harmonic coordinates. These equations are interesting because the harmonic coordinates can be interpreted as ordinary Minkowski coordinates in the "in" and "out" regions, far from the wave. They can be derived from the Lagrangian

$$L_h = \frac{du}{d\lambda}\frac{dV}{d\lambda} + H_{ab}(u)X^a X^b \left[\frac{du}{d\lambda}\right]^2 + \sum_a \frac{dX^a}{d\lambda}\frac{dX^a}{d\lambda} , \tag{20}$$

where again $\lambda = \tau/m$ is the affine parameter. The Euler-Lagrange equations imply that

$$p_- \equiv -p_V = \frac{1}{2}\frac{du}{d\lambda} = \text{const} , \tag{21}$$

so u is proportional to the affine parameter, and

$$\ddot{X}^a = H_{ab}(u)X^b , \tag{22}$$

$$\dot{V} = H_{ab}X^a X^b + \sum_c \dot{X}^c \dot{X}^c + \frac{m^2}{4p_-^2} , \tag{23}$$

where (22) indicates that the geodesics suffer a transverse force in these coordinates and (23) is, as previously, the constraint equation. Overdots indicate derivatives with respect to u.

It is interesting to consider the geodesic deviation suffered by two nearby geodesics of the family $\mathcal{F}$ (defined as before). Let us call ΔX^a their transverse coordinate separation, then using (3) and (5), or more easily from (22), the geodesic deviation equation is simply

$$\Delta \ddot{X}^a = H_{ab}\Delta X^b , \tag{24}$$

which gives the local tidal forces between the geodesics. Note that when we have a pure gravitational wave $(H_a^a = 0)$ these geodesics suffer the same pattern of tidal forces that geodesics in a weak gravitational field in the transverse-traceless gauge.[27] On the other hand when we have a pure null electromagnetic wave $[H_{ab} = H(u)\delta_{ab},\ H \leq 0]$ the geodesics suffer symmetric attractive tidal forces. It becomes apparent here that only when $H_{ab}(u)$ diverges arbitrarily rapidly near some value of u the tidal forces become infinite and the geodesics [see Eq. (22)] meet a singularity of the spacetime.

B. Sandwich waves. Classical cross section

In order to define a cross section the spacetime must have asymptotically flat "in" and "out" regions. For simplicity we shall restrict ourselves to sandwich waves. In harmonic coordinates a sandwich wave is simply defined as a plane wave for which $H_{ab}(u) = 0$ for $u < u_1$ and $u > u_2$, where u_1 and u_2 are finite real numbers that without loss of generality can be assumed to have negative values $u_1 < u_2 < 0$. We shall further assume that the wave is "sufficiently weak,"[8] which means that focusing occurs at some $u > 0$. Since there is no focusing inside the wave we avoid the need of having to use different group coordinate overlapping charts.

The matrix $P^a_b(u)$ defined in Sec. II will play an important role in the following discussion. When dealing with sandwich waves we shall impose that the metric in group coordinates has the Minkowski form in the "in" region; i.e., the group and harmonic coordinates coincide in this region. This is achieved by solving Eq. (11) with initial conditions $P^a_b(u_0)=\delta_{ab}$, $\dot{P}^a_b(u_0)=0$ [which also satisfy Eq. (12) trivially] for some $u_0<u_1$. Note, however, that both sets of coordinates will be different inside the wave and also in the "out" region.

The classical cross section for scattering of geodesics can be calculated for a general sandwich wave. This will be done in harmonic coordinates, since it is in these coordinates that the metric takes the Minkowski form both in the "in" and "out" regions. Denoting by p^μ the momentum of the incident particle before scattering and by q^μ the momentum of the same particle after scattering we have [see Eq. (21)]

$$q^a=\left.\frac{dX^a}{d\lambda}\right|_{\text{out}}=2p_-\dot{X}^a|_{\text{out}} , \tag{25}$$

where, as indicated, the derivatives must be calculated in the "out" region. Now, $\dot{X}^a$ is given according to (9) by $\dot{X}^a(u)=\dot{P}^a_b(u)x^b(u)+P^a_b(u)\dot{x}^b(u)$, and x^a and its derivatives can be obtained from (18). For simplicity let us consider the case of perpendicular incidence first, i.e., $p_a=0$. From (25) and (18) we have

$$q^a=2p_-\dot{P}^a_b(\text{out})X^b_0 , \tag{26}$$

where we have used the fact that $x^a_0=X^a_0$ in the "in" region (where the initial conditions are taken).

By Eq. (21) we have that $q_-=p_-$ (this equation is analogous to the conservation of energy in flat space), so regardless of the direction of scattering q_- will be the same and equal to the initial value p_- and Eq. (26) can be viewed as a linear relation which gives the transverse momentum q^a in terms of the position X^b_0 where the particle "hits" the wave. Denoting by J the Jacobian corresponding to such linear transformation we can write

$$\begin{aligned} d\sigma|_{\text{classical}} &\equiv d^nX^a_0=|J|d^nq^a \\ &=\frac{1}{2^n|\det[\dot{P}^a_b(u)]|}\frac{d^nq^a}{(q_-)^n} , \end{aligned} \tag{27}$$

where $\dot{P}^a_b(u)$ has to be calculated in the "out" region (i.e., $u>0$). Here we have defined the classical cross section in the usual way as given by the surface element in n transverse dimensions. From Eq. (11) we have that $\ddot{P}^a_b=0$ in the "out" region. This has the general solution

$$P^a_b=B^a_b+uA^a_b , \tag{28}$$

where A and B are constant matrices, which by (12) must satisfy the relation $A^TB=B^TA$.

Therefore $\det(\dot{P}^a_b)$ is constant as required for the consistency of (27). Introducing (28) into (27) we get

$$d\sigma|_{\text{classical}}=\frac{1}{2^n|\det A|}\frac{d^nq^a}{(q_0)^n} . \tag{29}$$

Although in the preceding discussion we have assumed perpendicular incidence for simplicity, Eq. (29) is also valid for the case of oblique incidence. The reason is that the only modification required is to replace Eq. (26) by [see Eq. (18)]

$$\begin{aligned} q^a= 2p_-\dot{P}^a_b&\left[X^b_0+\frac{p_c}{2p_-}\int_{u_0}^u g^{bc}(u')du'\right] \\ &+2P^a_b\left[\frac{p_c}{2p_-}g^{bc}(u)\right] , \end{aligned}$$

but this does not affect the Jacobian of the linear transformation relating q^a to X^b_0, which is all that is needed.

Of course Eq. (29) is only valid as long as the matrix A is nonsingular. If A were a singular matrix then the differential cross section defined in (27) is physically infinite because the linear relation (26) is degenerate and one should have to modify the definition of cross section in order to obtain a finite answer. Since we are mainly interested in comparing the classical and quantum results and, moreover, the singular case should be obtained from the nonsingular one by a suitable limiting procedure, we shall assume for simplicity that $\det A\neq 0$ throughout.

C. Focusing of geodesics

As mentioned in the Introduction, plane waves exert a focusing effect on null rays and also on massive particles. Before proceeding to consider the case of a general sandwich wave, it is convenient to review the simpler case of an impulsive plane wave.[21,13]

The metric for an impulsive plane wave is given by (5) with $H_{ab}(u)=A_{ab}\delta(u)$, and it can be understood as the matching of two pieces of flat space through a hyperplane $u=0$ which has an energy density per unit transverse surface, see (6), given by $\rho=-A^a_a/8\pi G$. Without loss of generality one can choose A_{ab} to be diagonal, by an appropriate rotation of the transverse plane. So let

$$A_{ab}\equiv-\frac{1}{\lambda_a}\delta_{ab} . \tag{30}$$

In spite of the Dirac δ function appearing in $H_{ab}(u)$ it is easy to see, using group coordinates, that the metric is continuous. For this we solve (11) taking as initial conditions $P^a_b(u_0)=\delta^a_b$ and $\dot{P}^a_b(u_0)=0$ for some $u_0<0$. Then $P^a_b(u)=\delta^a_b+u\Theta(u)A^a_b$, where $\Theta(u)$ is the Heaviside step function, and

$$g_{ab}=\delta_{ab}\frac{[\lambda_a-u\Theta(u)]^2}{\lambda_a^2} . \tag{31}$$

In these coordinates the interpretation of the metric as the matching of two flat regions at $u=0$ is clear. In one of the regions $u<0$ we have chosen that the metric has the Minkowskian form, but in the other $u>0$ the transverse coefficients are proportional to $(1-u/\lambda_a)^2$, i.e., flat space in non-Minkowskian coordinates. The matching is continuous but it has discontinuous first derivatives, a feature of the impulsive waves.

Now from (7) and (30) we see that at least one of the λ_a must be positive. Let λ_1 be the minimum positive value in the set $\{\lambda_a\}$. Then for $u=\lambda_1$ we have from (31) that $\det g_{ab}(\lambda_1)=0$. This means, according to Sec. II A that the focusing point is at $u_f=\lambda_1$ for perpendicularly incident geodesics. Note that the time of focusing is roughly proportional to the inverse of the energy density per unit surface of the wave. For a purely gravitational impulsive wave $\rho=0$, but one may be tempted to define the energy of the wave as the inverse of the focusing time.

The geodesic equations (22) and (23) can be easily integrated for impulsive plane waves. One finds

$$X^a(u)=b^a+\frac{p^a}{2p_-}u-\frac{b^a}{\lambda_a}u\Theta(u)\ , \tag{32}$$

$$V(u)=c_0+\frac{p^ap_a+m^2}{4p_-^2}u-\sum_a\frac{b_a^2}{\lambda_a}\Theta(u)+\sum_a\left[\frac{b_a^2}{\lambda_a^2}-\frac{b_ap_a}{\lambda_ap_-}\right]u\Theta(u)\ , \tag{33}$$

where the impact parameters $b^a\equiv X^a(u=0)=x_0^a-p^au_0/2p_-$ and the constant $c_0\equiv V(u=0)=v_0-(p^ap_a+m^2)u_0/4p_-^2$ have been introduced. In (33) we see the typical shift[13] in $V(u)$ at $u=0$, $\Delta V=-\sum_a(b_a^2/\lambda_a)$ due to the impulsive nature of the wave. The transverse coordinates on the other hand simply change direction as the wave is crossed.

In the degenerate case all $\lambda_a>0$ are equal $\lambda=\lambda_a=u_f$. For the case of *perpendicular incidence*, $p_a=0$ we can introduce polar coordinates $r=\sqrt{X^aX_a}$, $b=\sqrt{b^ab_a}$, and Eqs. (32) and (33) become

$$r=b\left[1-\frac{u}{\lambda}\Theta(u)\right]\ , \tag{34}$$

$$V(u)=c_0+\frac{m^2}{2p_-}u+\frac{b^2}{\lambda}\Theta(u)\left[\frac{u}{\lambda}-1\right]\ . \tag{35}$$

In this case all geodesics of the family $\mathcal{F}(v_0,p_-,p_a=0)$ focus at the same point $r(u_f)=0$, $V(u_f)=c_0+m^2u_f/2p_-$ independently of the impact parameter b.[13] This result was already anticipated in the last subsection using group coordinates. In an analogous way we may consider the case of *oblique incidence* ($p_a\neq0$). Without going into details we shall only mention that subfamilies of $\mathcal{F}$ with the same value of the projection b^ap_a meet at one point in the degenerate case and that for different values of b^ap_a these focusing points span a straight line parallel to the Z axis, at transverse coordinates given by $X^a(u_f)=p_a\lambda/2p_-$.

In the nondegenerate case, or when some of the λ_a are different, not all geodesics will meet at the same point. For instance, let us assume that $n=2$ and $\lambda_1\neq\lambda_2$. As we have argued in the previous section we expect a one-dimensional focusing structure. In fact, for perpendicular incidence all geodesics in a subfamily with fixed b_2 meet at the same point:

$$X^1(u_f)=0,\quad X^2(u_f)=b_2(1-\lambda_1/\lambda_2)\ , \tag{36}$$

$$Z(u_f)=\frac{V-u}{2}=\frac{1}{2}\left[c_0-\lambda_1+\frac{m^2}{2p_-}\lambda_1-\frac{b_2^2}{\lambda_2}\left[1-\frac{\lambda_1}{\lambda_2}\right]\right]\ . \tag{37}$$

Subfamilies labeled by different values of b_2 meet at different points and the focusing points span the parabola (eliminating b_2 in the above equations) $X^1=0$,

$$Z=[c_0+(m^2/2p_--1)\lambda_1-X_2^2/(\lambda_2-\lambda_1)]/2$$

in the (X^2,Z) plane.

The shape of this parabola, however, is not very relevant if we are interested in comparing with the scattering of quantum particles. Indeed, geodesics of a new family $\mathcal{F}(v_0+\Delta v_0,p_-,p_a=0)$ will focus into a similar parabola shifted an amount Δv_0 in the Z direction, and the set of all parabolas that one could build using different values for v_0 would span the whole (X^2,Z) plane. We can define a larger family of geodesics $\hat{\mathcal{F}}(p_-,p_a)$ as the union of families $\mathcal{F}(v_0,p_-,p_a)$ for all possible values of v_0. In the "in" region $\hat{\mathcal{F}}$ represents a swarm of particles filling the whole space and all of them having the same momentum. This would be the classical analogue of an "in" quantum scattering state with well-defined momentum $|p_-,p_a\rangle$. What is important in comparing with the quantum case, therefore, is not the parabola itself but its projection onto the transverse space X^a; i.e., the focusing is characterized by the equation $X^1(u_f)=0$ in the preceding example.

Let us go on to consider the focusing of geodesics for the case of a general sandwich wave. Although Eqs. (22) and (23) cannot be integrated in general, a formal solution can be obtained by simply substituting Eqs. (18) into the change of variables (i). Restricting attention to the transverse coordinates (because shifts in the z direction are generated by similar shifts in v_0 as explained in the preceding paragraph) we have

$$X^a(u)=P_b^a(u)\left[x_0^b+\frac{p_c}{2p_-}\int_{u_0}^u g^{bc}(u')du'\right]\ . \tag{38}$$

As mentioned before the integral in Eq. (38) diverges as the upper limit approaches u_f because $g_{ab}(u_f)$ is not invertible, but as we shall see the factor $P_b^a(u)$ appearing at the beginning of the right-hand side (RHS) compensates for the divergence. The integral can be split into two terms:

$$\int_{u_0}^0 g^{bc}(u')du'+\int_0^u g^{bc}(u')du'\ . \tag{39}$$

The first term is just a finite constant because $g^{bc}(u)$ is well behaved for $u<0$. The second term can be explicitly integrated by noting that the matrix $-[A^TP(u)]^{-1}$ is a primitive for the matrix $[g]^{-1}$ in the "out" region, so that

$$\int_0^u g^{bc}(u')du'=-(P^{-1})_d^b(A^{-1})_d^c+(B^{-1})_d^b(A^{-1})_d^c\ . \tag{40}$$

Introducing this into (38) we have

$$X^a(u)=P^a_b(u)\left[x^b_0+\frac{p_c}{2p_-}K^{bc}\right]-\frac{p_c}{2p_-}(A^{-1})^c_a\ , \tag{41}$$

where u has to lie in the "out" region and K^{bc} is just a constant matrix which is the sum of the first term in (39) plus the second term on the RHS of (40). Note that the potentially divergent matrix P^{-1} appearing in (40) is not present in the final equation (41).

In the fully degenerate case $g_{ab}(u_f)$ has n null vectors, i.e., $g_{ab}(u_f)=0$ and $P^a_b(u_f)=0$, so the geodesics in $\mathcal{F}$ focus into a single point in transverse space

$$X^a(u_f)=-\frac{p_c}{2p_-}(A^{-1})^c_a\ , \tag{42}$$

which is independent of v_0 and the impact parameters x^a_0.

In the general case $g_{ab}(u_f)$ will have rank r; i.e., it will have $k=n-r$ independent null vectors, which are also the null eigenvectors of $P^a_b(u_f)$. We define the matrix $M^a_b(u)\equiv(i/4p_-)P^a_c(u)(A^{-1})^c_b$, which has also rank r because A is nonsingular (the numerical factor in the definition has been introduced for later convenience). Equation (41) can be rewritten as

$$X^a(u)+\frac{p_c}{2p_-}(A^{-1})^c_a = -4ip_-M^a_c(u)A^c_b\left[x^b_0+\frac{p_c}{2p_-}K^{bc}\right]. \tag{43}$$

It is easy to see, using Eqs. (12) and (28), that M is symmetric and therefore it can be diagonalized by means of a rotation matrix R (in transverse space) whose columns are the eigenvectors of M (i.e., R^TMR is diagonal). Let R^a_m for $m=1$ to k be the null eigenvectors. Then from Eq. (43) the pattern of focusing points is characterized by the set of equations

$$Y_m(u_f)=-\frac{p_c}{2p_-}(A^{-1})^c_aR^a_m\ , \tag{44}$$

where $Y_m=X_aR^a_m$ are the first k rotated coordinates. We shall find the quantum analogue of Eq. (44) in Sec. III C.

III. QUANTUM FIELD THEORY IN A PLANE-WAVE SPACETIME

In this section we review the quantization of a field coupled to a gravitational plane wave.[18] As it was the case for the geodesic equations, the use of group coordinates simplifies the problem. As usual, we shall consider a scalar field Φ with mass m. When coupled to the metric (8) this field satisfies the Klein-Gordon equation

$$(\Box_g-m^2)\Phi(u,v,x^a)=0\ , \tag{45}$$

where the g in the d'Alembertian operator refers to the metric (8). Note that the coupling parameter ξ between Φ and the curvature does not appear in this equation because the Ricci scalar is zero, $R=0$. It will be present, however, in the expression for the stress tensor of the field. The hyperbolic partial differential equation (45) can be separated in all coordinates and it is easy to see that any solution can be written (in a region where the coordinates are not singular) in terms of the following normalized modes and their complex conjugates:

$$u_{k_-k_a}=\frac{\gamma(u)^{-n/2}}{\sqrt{2k_-}(2\pi)^{(n+1)/2}}\exp\left[ik_ax^a-ik_-v-\frac{i}{4k_-}\int_0^u(g^{ab}k_ak_b+m^2)du\right]\ , \tag{46}$$

where k_-,k_a are the separation constants, $\gamma(u)\equiv[\det g_{ab}(u)]^{1/2n}$ as in the previous section, and a normalization constant has been introduced according to the scalar product of Eq. (48) that follows. In fact, the scalar product between two solutions ϕ_1 and ϕ_2 of the Klein-Gordon equation is defined as usual[28] by $\langle\phi_1,\phi_2\rangle=N\int_\Sigma\mathbf{J}\cdot\tilde{\omega}$, where $\mathbf{J}$ is the Klein-Gordon conserved current, $J_\mu=i(\phi_1\partial_\mu\phi_2^*-\phi_2^*\partial_\mu\phi_1)$, N the spacetime dimension ($N=n+2$), and $\tilde{\omega}$ the volume form restricted to the Cauchy hypersurface Σ. Now for the modes we are interested in, modes which represent particles incident to the wave front, the hypersurface $u=u_0=$const is a good substitute Cauchy surface,[18] and we have

$$\langle\phi_1,\phi_2\rangle=-i\int_{u_0}\gamma^n(u)(\phi_1\partial_v\phi_2^*-\phi_2^*\partial_v\phi_1)dv\,\Pi_a dx^a\ . \tag{47}$$

The modes (46) have been normalized so that

$$\langle u_{k_-k_a},u_{k'_-k'_a}\rangle=\delta^{(n)}(k_a-k'_a)\delta(k_--k'_-)\ . \tag{48}$$

We can now proceed to quantize the field Φ by imposing the usual commutation relations between Φ and its conjugate momentum π_Φ.[28]

The field operator Φ can then be expanded by means of the modes (46) in terms of creation $a^\dagger_{k_-k_a}$ and annihilation $a_{k_-k_a}$ operators as

$$\Phi(u,v,x^a)=\int d^nk_a dk_-(u_{k_-k_a}a_{k_-k_a}+u^*_{k_-k_a}a^\dagger_{k_-k_a})\ , \tag{49}$$

where the operators $a^\dagger_{k_-k_a}$ and $a_{k_-k_a}$ satisfy the commutation relations

$$[a_{k_-k_a},a^\dagger_{k_-k_a}]=\delta^{(n)}(k_a-k'_a)\delta(k_--k'_-)\ . \tag{50}$$

A. Bogoliubov transformations for sandwich waves

As we have defined earlier, a sandwich wave is a plane wave in which spacetime is flat for $u<u_1$ and $u>u_2$ for finite values of u_1 and u_2. In order to be able to use a single group coordinate chart we assume as before that the chart is "sufficiently weak" and that $u_1<u_2<0$. The practical convenience of using sandwich waves is that now we have "in" and "out" modes defining unambiguous "in" and "out" physical vacua. Our purpose in this section is to find the Bogoliubov transformations relating the "in" and "out" modes.

As usual the normal modes defining the "in" vacuum are the positive frequency modes that are proportional to $\exp(ik_a x^a + ik_z z - i\omega t)$, where $\omega = (\mathbf{k}^2 + m^2)^{1/2}$. If we define

$$k_- = (\omega - k_z)/2, \quad k_+ = (\omega + k_z)/2 , \tag{51}$$

the "in" modes can be written for $u \le u_1$ as

$$u^{\text{in}}_{k_- k_a} = \frac{1}{\sqrt{2k_-}(2\pi)^{(n+1)/2}} \exp\left[ik_a X^a - ik_- V - \frac{i}{4k_-}(m^2 + k_a k^a)u + i\varphi \right] , \tag{52}$$

where the constant phase

$$\varphi \equiv \frac{1}{4k_-}\left[(m^2 + k_a k^a)u_1 + \int_{u_1}^{0} [g^{ab}(u)k_a k_b + m^2] du \right] . \tag{53}$$

has been introduced so that these "in" modes agree with (46) in the "in" region ($u \le u_1$). We have also used the fact that in this region we can take the transverse metric $g_{ab} = \delta_{ab}$ so that the group and canonical coordinates coincide ($x^a = X^a, v = V$). Recall, however, that once this election has been made the group and harmonic coordinates will differ in the "out" region ($u \ge u_2$). Since (52) and (46) coincide in the "in" region it is clear that the "in" modes $u^{\text{in}}_{k_- k_a}$ are given by (46) in the whole coordinate chart where the group coordinates are regular.

We need now the expression of the "out" modes in the "out" region. These are simply expressed in terms of harmonic coordinates, since in such coordinates the metric takes the Minkowski form there, as

$$u^{\text{out}}_{l_- l_a} = \frac{1}{\sqrt{2l_-}(2\pi)^{(n+1)/2}} \exp\left[il_a X^a - il_- V - \frac{i}{4l_-}(l_a l^a + m^2)u \right] \quad (u > u_2) . \tag{54}$$

The Bogoliubov coefficients relating the two kinds of modes are defined by[28]

$$\alpha_{ij} = \langle u_i^{\text{out}}, u_j^{\text{in}} \rangle, \quad \beta_{ij} = -\langle u_i^{\text{out}}, u_j^{\text{in}*} \rangle , \tag{55}$$

where i and j stand here for the quantum numbers k_a, k_- and l_a, l_-. We shall use the hypersurface $u=0$ to compute the scalar products. Since $u=0$ lies in the "out" region and no derivatives with respect to u are involved in the scalar product, we only need $u_i^{\text{out}}(u=0)$, which is given by (54), and $u_i^{\text{in}}(u=0)$, which is given by (46). This means that we need to know $u_i^{\text{in}}(u=0)$ in terms of harmonic coordinates. For this we note that, from (9),

$$x^a = [P^{-1}(u=0)]^a_b X^b , \qquad v = V - \tfrac{1}{2}\dot{g}_{ab}(u=0)x^a x^b . \tag{56}$$

It is very easy to see that $\beta_{ij}=0$. For this it suffices to note that the integrand in the corresponding scalar product, see (47), has all the dependence on V in the form $\exp[-i(k_- + l_-)V]$ which, after integration with respect to V, will become proportional to $\delta(k_- + l_-)$. But $k_- + l_- \ge 0$ because $2k_- = \omega - k_z \ge 0$ and similarly $l_- \ge 0$. As a result,

$$\beta_{l_- l_a, k_- k_a} = 0 , \tag{57}$$

which means that the positive-frequency "in" modes do not develop negative-frequency "out" mode parts and consequently the "in" vacuum and "out" vacuum can be identified; i.e., there is no particle creation as was first shown by Gibbons.[18]

This, of course, does not mean that the coefficients α_{ij} are "trivial," since all the focusing properties of the spacetime on the modes are described by these coefficients. Fortunately these coefficients can be explicitly evaluated. Using (47), (55), and (56), after the trivial V integration we have

$$\alpha_{l_- l_a, k_- k_a} = \delta(k_- - l_-)\frac{\gamma^{-n/2}}{(2\pi)^n}\int_{u_0=0} \exp\left[-il_a P^a_b x^b - ik_a x^a - \frac{i}{2}k_- \dot{g}_{ab} x^a x^b \right] \Pi_a dX^a , \tag{58}$$

where we have written the integrand in terms of the group coordinates which are better suited for this integration. In fact, the Jacobian of the transformation between $\{x^a\}$ and $\{X^a\}$ is, see (56), just $\det(P^a_b) = \sqrt{\det g_{ab}} = \gamma^n$ which depends only on u, and therefore is a constant on the surface $u=0$. We can write

$$\alpha_{l_- l_a, k_- k_a} = \delta(k_- - l_-)\frac{\gamma^{n/2}}{(2\pi)^n}\exp\left[\frac{-i}{2}k_- \dot{g}_{ab} x_0^a x_0^b \right] \int d^n x^a \exp\left[\frac{i}{2}k_- \dot{g}_{ab}(x^a - x_0^a)(x^b - x_0^b) \right] , \tag{59}$$

where we have introduced

$$x_0^a \equiv \frac{-1}{k_-}[(\dot{g})^{-1}]^{ab}(P^c_b l_c - k_b) , \tag{60}$$

so that we are finally left with a Gaussian integral. The result is

$$\alpha_{l_-l_a,k_-k_a}=\frac{\delta(k_- - l_-)}{(-2\pi i k_-)^{n/2}}\frac{[\det g_{ab}(u=0)]^{1/4}}{[\det \dot{g}_{ab}(u=0)]^{1/2}}\exp\left[\frac{-i}{2}k_-\dot{g}_{ab}x_0^a x_0^b\right], \tag{61}$$

which can be simplified using the results in Sec. II B to yield

$$\alpha_{l_-l_a,k_-k_a}=\frac{\delta(k_- - l_-)}{(-2\pi i k_-)^{n/2}}\frac{1}{\sqrt{2^n|\det A|}}\times\exp\left[\frac{-i}{2}k_-\dot{g}_{ab}x_0^a x_0^b\right]. \tag{62}$$

Note that the term $\delta(k_- - l_-)$ is expected because ∂_v is a Killing vector, which implies conservation of momentum along the v direction. It should be emphasized that the expression (62) is valid for any sufficiently weak sandwich wave and for any scalar field with arbitrary mass m and coupling parameter ξ.

B. S matrix and cross section

In this section we compute the scattering cross section for quantum scalar particles colliding with a sandwich plane wave. We need to evaluate the scattering matrix (S matrix) between "in" and "out" particles, i.e., the transition amplitude from an "in" one-particle state with quantum numbers (k_-,k_a) to an "out" one-particle state with quantum numbers (l_-,l_a):

$$\langle \text{out},l_-l_a|k_-k_a,\text{in}\rangle=\langle \text{out},0|a_{l_-l_a}^{\text{out}}a_{k_-k_a}^{\dagger\text{in}}|0,\text{in}\rangle . \tag{63}$$

Since there is no particle creation the two vacua can be identified $|0\rangle\equiv|0,\text{in}\rangle\equiv|0,\text{out}\rangle$, and since $\beta_{ij}=0$ there is a simple relation between the "in" and "out" annihilation (and creation) operators,[28] namely, $a_l^{\text{out}}=\int dk\,\alpha_{lk}^* a_k^{\text{in}}$. Using the commutation relations (50) the transition amplitude becomes

$$\langle \text{out},l_-l_a|k_-k_a,\text{in}\rangle=\langle 0|0\rangle\alpha^*_{l_-l_a,k_-k_a}, \tag{64}$$

which gives explicitly the elements of the scattering matrix.

In order to calculate the scattering cross section we start with the transition probability

$$W(i\to\Delta f)=|\langle \text{out},l_-l_a|k_-k_a,\text{in}\rangle|^2 dl_-d^nl_a . \tag{65}$$

For sandwich plane waves,

$$|\langle \text{out},l_-l_a|k_-k_a,\text{in}\rangle|=\frac{\delta(k_- - l_-)}{(2\pi k_-)^{n/2}}\frac{1}{\sqrt{2^n|\det A|}}, \tag{66}$$

Of course the $\delta^2(k_- - l_-)$ term that will appear in (65) is ill defined because the wave travels an infinite amount of time, so we calculate the transition probability per unit coordinate v:

$$\frac{dW(i\to\Delta f)}{dv}=\frac{1}{(2\pi)^{n+1}}\frac{1}{2^n|\det A|}\frac{d^nl_a}{(l_-)^n}. \tag{67}$$

To obtain the cross section this has to be divided by the incident flux of probability, i.e, the probability per unit transverse surface and unit v coordinate: $1/(2\pi)^{n+1}$. Therefore the quantum differential cross section is given by

$$d\sigma(i\to\Delta f)=\frac{1}{2^n|\det A|}\frac{d^nl_a}{(l_-)^n}. \tag{68}$$

which is exactly the same as the classical differential cross section.[29] For an impulsive wave, for instance, in the degenerate case this cross section is simply proportional to $(G\rho)^{-n}$, where ρ is the energy density per unit surface of the wave.

The coincidence of the classical and the quantum results can be traced to the fact that for a plane-wave spacetime the optical approximation is exact and thus the rays of the waves follow geodesic paths. In fact one may write the solutions (46) as $\phi=f(u)\exp\{iS(u)\}$ and note that they satisfy $\nabla S\nabla S=m^2$ exactly. But this is the equation of ray propagation (i.e., eikonal equation of geometrical optics) or alternatively it can be seen as the Hamilton-Jacobi equation for the geodesics with action S. The equivalence of the classical and quantum cross sections, though, is not an exclusive property of plane waves. For instance, it is easy to check that for the Aichelburg-Sexl (AS) metric[24] (which is an impulsive *pp* wave but not a plane wave) the quantum cross section derived by 't Hooft[23] also coincides with the classical cross section for geodesics. This seems to be, thus, a feature of high-energy scattering, meaning that the relative velocity between the wave and the test particle is the speed of light.

C. Focusing of modes and the energy-momentum tensor

The quantum analogue of the focusing of geodesics discussed in Sec. II is the focusing of the normal modes $u_{k_-k_a}^{\text{in}}$. In order to investigate this issue we have to find the form of such modes in the "out" region. We already know that from Eq. (46). However, Eq. (46) is given in terms of group coordinates, which break down precisely at the points where the focusing phenomenon is expected to occur. Therefore we should find a corresponding expression in harmonic coordinates which should also hold at $u=u_f$. This can be accomplished using the Bogoliubov transformations and the relation

$$u_{k_-k_a}^{\text{in}}=\int dl_-dl_a\alpha_{l_-l_a,k_-k_a}u_{l_-l_a}^{\text{out}} . \tag{69}$$

Using (54) and (62) we have, in the "out" region,

$$u^{\text{in}}_{k_-k_a}=\frac{1}{(-4\pi i k_-)^{n/2}\sqrt{|\det A|}}\frac{e^{-i(k_-V+m^2u/4k_-)}}{(2\pi)^{(n+1)/2}\sqrt{2k_-}}e^{(-i/4k_-)k^T[A^{-1}(B^T)^{-1}]k}I\ , \tag{70}$$

where k is the vector with components k^a and I is the Gaussian integral,

$$I(u)=\int dl_a e^{-l^TMl+w^Tl}\ , \tag{71}$$

where l is the vector with components l^a and M and w are given by

$$M=\frac{i}{4k_-}[B+uA]A^{-1}\ ,$$
$$w_a=\frac{i}{2k_-}[k_b(A^{-1})^b_a+2k_-X^a]\ . \tag{72}$$

If we are not at a focusing position, i.e., $u\neq u_f$, then we have $\det P(u)\neq 0$ the matrix M is nonsingular, and the Gaussian integral yields

$$I(u\neq u_f)=\frac{\pi^{n/2}}{\sqrt{\det M}}e^{w^TM^{-1}w/4}\ . \tag{73}$$

Obviously the preceding expression does not hold at the focusing position, $u=u_f$. Let us see what happens in the case of full degeneracy, i.e., when $g_{ab}(u_f)=0$. In this case $M(u_f)=0$ and one obtains, from (71),

$$I(u=u_f)=(2\pi)^n\delta^{(n)}\left[X^a+\frac{1}{2k_-}k_b(A^{-1})^b_a\right]\ , \tag{74}$$

so the modes $u^{\text{in}}_{k_-k_a}$ become completely focused by the δ function into a single point in the transverse space, in analogy to what happens when classical geodesics are considered [see Eq. (42)].

In a more general situation the matrix $M(u_f)$ will have a zero eigenvalue of multiplicity k, and the rank of $g_{ab}(u_f)$ will be $r=n-k$. As explained in Sec. II C M can be diagonalized by means of a rotation matrix R in transverse space whose columns are the eigenvectors of M. We denote by R^a_m for $m=1$ to k the null eigenvectors. Then the integral (71) can be expressed as a product of two factors, one of them similar to (73) and the other similar to (74):

$$I(u=u_f)=\frac{\pi^{(n-k)/2}}{\sqrt{\det M_R}}e^{(w_R^TM_R^{-1}w_R)/2}(2\pi)^k\prod_{m=1}^{k}\delta\left[Y_m+\frac{1}{2k_-}k_c(A^{-1})^c_aR^a_m\right]\ , \tag{75}$$

where M_R and w_R are the restrictions of M and w to the subspace orthogonal to the null eigenvectors and $Y_m=X_aR^a_m$ are, as in Eq. (44) the first k-rotated coordinates. Thus, similarly to what happens for classical geodesics, the "in" modes become focused by the δ functions into the same pattern of dimension $r=n-k$ [see Eq. (44)].

This is also true for the expectation value of the energy-momentum tensor in a one-particle scattering state, as we shall see shortly. But let us first consider the vacuum polarization. The renormalized vacuum expectation value (VEV) of the energy-momentum tensor vanishes in a plane-wave spacetime.[18] This is easily seen using the Pauli-Villars regularization prescription.[29,30] One defines a regularized VEV as

$$\langle 0|T_{\mu\nu}(x)|0\rangle_{\text{reg}}=\int_0^\infty d\lambda\,\rho(\lambda)\langle 0|T_{\mu\nu}(x;\lambda)|0\rangle\ , \tag{76}$$

where $\langle 0|T_{\mu\nu}(x;\lambda)|0\rangle$ is the nonregularized expression for a field of mass $\sqrt{\lambda}$ and the distribution $\rho(\lambda)$ satisfies the relation

$$\int_0^\infty d\lambda\,\rho(\lambda)\lambda^n=0 \tag{77}$$

for $n=0,1,2$. Moreover, the integral (76) must contain a term representing the nonregularized VEV corresponding to the physical field of mass m itself, so we write

$$\rho(\lambda)=\delta(\lambda-m^2)+\rho_1(\lambda)\ ,$$

where $\rho_1(\lambda)$ is different from zero only for $\lambda>\Lambda$ and the limit $\Lambda\to\infty$ is taken at the end of the calculation. In addition it is required that

$$\lim_{\Lambda\to\infty}\int_0^\infty d\lambda\,\rho_1(\lambda)\lambda^{-1}=0\ ,$$

although this condition does not play any role in our case. The nonregularized expression of the VEV has the form

$$\langle 0|T_{\mu\nu}(x;\lambda)|0\rangle=\int dk_-dk_a T^{(\lambda)}_{\mu\nu}[u^{\text{in}}_{k_-k_a}(x;\lambda)]\ , \tag{78}$$

where $u^{\text{in}}_{k_-k_a}(x;\lambda)$ is given by (46) with $m^2=\lambda$ and[28]

$$\begin{aligned}T^{(\lambda)}_{\mu\nu}[u_i]=&(1-2\xi)u_{i,\mu}u^*_{i,\nu}+(2\xi-\tfrac{1}{2})g_{\mu\nu}g^{\rho\sigma}u_{i,\rho}u^*_{i,\sigma}\\&-2\xi u_{i;\mu\nu}u^*_i+\frac{2}{N}\xi g_{\mu\nu}u_i\Box u^*_i+\xi R_{\mu\nu}|u_i|^2\\&-2\left[\frac{1}{4}-\left[1-\frac{1}{N}\right]\xi\right]\lambda g_{\mu\nu}|u_i|^2\ ,\end{aligned} \tag{79}$$

where ξ is the parameter of direct coupling of the field to the curvature and we have used that the Ricci scalar vanishes for a plane wave. Since all the dependence of $u^{\text{in}}_{k_-k_a}$ on λ is in the form $\exp(-i\lambda u/4k_-)$ [see Eq. (46)], it is easy to see by inspection that the dependence of $T^{(\lambda)}_{\mu\nu}[u^{\text{in}}_{k_-k_a}(x;\lambda)]$ on λ is a polynomial of degree 2.

It is essential to the Pauli-Villars method that the (continuous) mode summation in Eq. (78) should be done after the integration over λ in Eq. (76), but this integration vanishes due to Eq. (77) and the fact that $T_{\mu\nu}^{(\lambda)}$ is a polynomial of degree 2 in λ. As a result we have $\langle 0|T_{\mu\nu}(x)|0\rangle_{\text{reg}}=0$.

Finally, let us consider the expectation value of the energy-momentum tensor in a scattering state $|k_{-}k_a\rangle$. This is given by

$$\langle k_{-}k_a|T_{\mu\nu}(x)|k_{-}k_a\rangle=2T_{\mu\nu}^{(m^2)}[u_{k_{-}k_a}^{\text{in}}] . \tag{80}$$

The RHS of Eq. (80) is indeed a complicated expression, but from (79) it is easy to see that it will have focusing properties similar to those of the modes $u_{k_{-}k_a}^{\text{in}}$. In particular it is easy to compute the expectation value of the trace in the conformally coupled case, $\xi=\xi_c=(N-2)/(4N-4)$, since in this case all terms involving derivatives in Eq. (79) disappear and we obtain, from (80),

$$\langle k_{-}k_a|T_{\mu}^{\mu}(x)|k_{-}k_a\rangle_{\xi_c}=2m^2|u_{k_{-}k_a}^{\text{in}}|^2 . \tag{81}$$

From this expression it is obvious that the expectation value of the trace will share the same focusing properties than the modes: using Eq. (73) in (70) we see that for $u\neq u_f$ (81) is finite and well behaved, but for $u=u_f$ we should use (75) instead of (73) and then $\langle k_{-}k_a|T_{\mu}^{\mu}(u=u_f)|k_{-}k_a\rangle_{\xi_c}$ will be infinite on a subspace of dimension $r=n-k$ and zero elsewhere. Therefore as remarked earlier one expects that back reaction will be important when the gravitational wave propagates in the presence of quantum particles.

ACKNOWLEDGMENTS

E.V. would like to thank Gary Horowitz, Robert Wald, Alex Vilenkin, and Ulvi Yurtsever for discussions. He is also indebted to the Dirección General de Investigación Científica y Técnica of Spain for a fellowship. This work has also been partially supported by a CICYT grant.

[1]D. Amati and C. Klimcik, Phys. Lett. B **210**, 92 (1988); **219**, 443 (1989).
[2]G. T. Horowitz and A. R. Steif, Phys. Rev. Lett. **64**, 260 (1990).
[3]G. T. Horowitz and A. R. Steif, Phys. Rev. D **42**, 1950 (1990).
[4]J. Ehlers and W. Kundt, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962), p. 49.
[5]D. Kramer, H. Stephani, M. MacCullum, and E. Herlt, *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge, England, 1980).
[6]O. R. Baldwin and G. B. Jeffrey, Proc. R. Soc. London **A111**, 95 (1926).
[7]E. Verdaguer, Nuovo Cimento **100B**, 787 (1987).
[8]R. Penrose, Rev. Mod. Phys. **37**, 215 (1965).
[9]P. Szekeres, Nature (London) **228**, 1183 (1970).
[10]K. Kahn and R. Penrose, Nature (London) **229**, 185 (1971).
[11]S. Chandrasekhar and B. Xanthopoulos, Proc. R. Soc. London **A408**, 175 (1986).
[12]P. D. D'Eath, in *Sources of Gravitational Radiation,* edited by L. Smarr (Cambridge University Press, Cambridge, England, 1979).
[13]V. Ferrari, P. Pendenza, and G. Veneziano, Gen. Relativ. Gravit. **20**, 1185 (1988).
[14]U. Yurtsever, Phys. Rev. D **40**, 360 (1989).
[15]B. J. Carr and E. Verdaguer, Phys. Rev. D **28**, 2995 (1983).
[16]D. Garfinkle, Phys. Rev. D **41**, 1112 (1990).
[17]D. Garfinkle and T. Vachaspati, Phys. Rev. D **42**, 1960 (1990).
[18]G. W. Gibbons, Commun. Math. Phys. **45**, 191 (1975).
[19]S. Deser, J. Phys. A **8**, 1972 (1975).
[20]J. Schwinger, Phys. Rev. **82**, 664 (1951).
[21]C. Klimcik, Phys. Lett. B **208**, 373 (1988).
[22]S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, England, 1973).
[23]G. 't Hooft, Phys. Lett. B **198**, 61 (1987).
[24]P. C. Aichelburg and R. V. Sexl, Gen. Relativ. Gravit. **2**, 303 (1971).
[25]T. Dray and G. 't Hooft, Nucl. Phys. **B253**, 173 (1985); Class. Quantum Grav. **3**, 825 (1986).
[26]G. Veneziano, Mod. Phys. Lett. A **2**, 899 (1987).
[27]C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
[28]N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982).
[29]A. Vilenkin, Nuovo Cimento **44A**, 441 (1977).
[30]W. Pauli and F. Villars, Rev. Mod. Phys. **21**, 434 (1949).