Production of spin-$\frac{1}{2}$ particles in inhomogeneous cosmologies

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The production of spin-$\frac{1}{2}$ particles by small gravitational inhomogeneities is discussed by using a perturbative approach based on the evaluation of the scattering matrix. We compute the production of massive and massless particles by linear gravitational inhomogeneities in flat spacetime and the production of massless particles in an expanding universe described by the spatially flat Friedmann-Robertson-Walker models with small inhomogeneities. As in the case of scalar particles the total pair-creation probability is given in terms of geometric invariants of the spacetime.

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I. INTRODUCTION

The present Universe in its large structure seems to be isotropic and spatially homogeneous [1,2] and therefore it can be described by the Friedmann-Robertson-Walker (FRW) models. However, there are reasons to believe that it has not been so in all its evolution and that anisotropies and inhomogeneities may have played an important role in the early Universe. For one, the FRW models are unstable against gravitational perturbations as we go back in time and approach the cosmological singularity [3]. If there is at present a background of gravitational waves these may have originated through inhomogeneities produced by gravitons created near the Planck era in a FRW model [4] or even earlier at the quantum gravity era [5], or may be the consequence of purely classical inhomogeneities at an early time [6–8]. Such inhomogeneities involve all scales including wavelengths larger than the particle horizon. On the other hand perhaps less severe forms of gravitational inhomogeneities have appeared along the Universe evolution, to seed galaxy formation [9], or as a consequence of the formation and evolution of topological defects as the early Universe underwent several phase transitions [10,11].

We know that gravitational inhomogeneities are semiclassically unstable; quantum field theory in curved spacetime (and order-of-magnitude estimates) predicts that variations in the gravitational field with frequencies of order 10^{-20} sec can produce particles of the order of the electron mass [12,13]. This means that inhomogeneities in the early Universe may be the source of relativistic particles. In fact, Zeldovich speculated [14] that inhomogeneities and anisotropies in the early Universe were damped by particle creation and that an initially inhomogeneous Universe became, as a consequence, isotropically and spatially homogeneous. Quantitative arguments for this, when the spacetime was slightly anisotropic, but spatially homogeneous, were given by Zeldovich and Starobinsky [15], Hu and Parker [16], Hartle and Hu [17] and Birrell and Davies [18]. This lent some support to Misner's chaotic cosmology program [19]. However such a mechanism cannot explain the isotropy and homogeneity of the Universe, entirely, because the size of the present Universe is much larger than the comoving horizon corresponding to the epoch when particle creation was significant. Thus it is now generally believed that the present homogeneity and isotropy can only be explained by some kind of inflationary scenario.

The purpose of this paper is to discuss the quantum effects produced by gravitational inhomogeneities, in particular to compute the total particle creation and its possible cosmological significance. The most efficient and standard method to compute particle production by gravitational fields is by means of the Bogoliubov coefficients [4,20,21]. That is, one quantizes the particle field on a given background spacetime by first finding the complete set of mode solutions of the Klein-Gordon equation (for a scalar field) or the Dirac equation (for spin-$\frac{1}{2}$ fields) on such background. Then the coefficients of the fields in terms of the mode solutions are promoted into operators and the respective commutation or anticommutation relations are imposed among them. Provided that one can identify two asymptotic regions of the spacetime with asymptotic timelike Killing fields, one at the far future and one at the far past, it is possible to write down expressions for the physical "in" and "out" vacua of the theory. The Bogoliubov coefficients relate the "in" and "out" operators and give directly the number of particles created, i.e., the number of "out" particles contained in the "in" vacua.

In cosmology, for spatially homogeneous models the above procedure is well suited even if no exact solutions to the field equations can be found. The reason is that the time-dependent part of the modes can be separated and the field equations for the corresponding function reduce to an ordinary differential equation with time-dependent coefficients. The problem is then reduced to solve that equation and relate the "in" modes with the "out" modes. Usually there is no natural 'in" vacuum due to the cosmological initial singularity; one way to define one is by artificially matching the Universe to flat space at some time near the initial singularity [4,15,22]. Exact solutions can only be found for some selected spacetimes; in other cases one may use an approximation or numerical techniques to the approximate Bogoliubov coefficients. Thus, for instance, Zeldovich and Starobinsky [15] and Birrell and Davies [18] give an approximation scheme to find particle creation in an expanding universe with small anisotropies.

For inhomogeneous spacetimes the time dependence of the modes cannot be separated and we end up with partial differential equations generally not well suited to an approximation scheme compatible with the physically relevant boundary conditions. However, the above procedure has been used recently in the large momentum approximation [23]. For small inhomogeneities the simplest method is to use a perturbative technique, based on the S matrix, in which the gravitational inhomogeneities are considered as tensorial external fields that couple to the quantum fields on a given homogeneous background. Then the problem becomes very similar to the computation of electron-positron pair creation by external time-dependent electric fields [24]. Sexl and Urbantke [12] and Zeldovich and Starobinsky [25] were the first to compute pair creation by external linear inhomogeneities on a Minkowski background. The method was later extended to conformally flat expanding backgrounds for the case of quantum scalar fields [26,27]. In this case if the particles are conformally coupled the background does not create particles [22]; they are a consequence of the loss of conformal invariance due to the inhomogeneities. An important feature of this computation is that the total pair-creation probability is given in terms of geometric invariants of the gravitational field and that these can be written, after using Einstein's equations, in terms of the stress tensor which produces the gravitational field. This has led to interesting practical applications which have allowed one to compute the total pair-creation probability of scalar particles produced by the formation and evolution of cosmic strings [28] and other topological defects [29,30] even when the gravitational field of these objects is not explicitly known.

In this paper we want to extend the above results on the creation of scalar particles by gravitational inhomogeneities to the more realistic case of spin-$\frac{1}{2}$ particles. The use of scalar fields considerably simplifies the computations and generally one expects that the results can be extrapolated to higher spin fields but for some numerical factors which will take into account the degrees of freedom of the field (this usually suffices in cosmological applications when only order-of-magnitude estimates are relevant).

Here we have to make use of the usual techniques for dealing with spinor fields in curved spacetime [13,31–33]; as we shall see these are needed in order to separate the interaction part of the action even when the background is flat. When the background is an expanding universe we will take it to be conformally flat, and will assume that the field is conformally coupled (massless), since otherwise the background itself may create particles [33–35] and the effect due to the inhomogeneities would be negligible. As we may expect the final result for the total pair creation probability can be expressed in terms of geometric invariants of the gravitational field, similarly as electron-positron pair creation by time-dependent electric fields is given in terms of invariants of the electromagnetic field [24].

The plan of the paper and a summary of the main results are the following. In Sec. II we start with a short review of the formalism for treating spin-$\frac{1}{2}$ fields in a curved background and describe the perturbative approach to be used in the paper.

In Sec. III we apply the above techniques to compute pair creation of spin-$\frac{1}{2}$ particles in flat spacetime by linear gravitational inhomogeneities. We first deal, in Sec. III A, with the production of massless particles; the results differ from the case of massless scalar particles by a factor of 6 and agree with Refs. [12,25]. The case of massive particles is treated separately, in Sec. III B; the final result differs from the case of scalar particles by a global factor and other numerical factors involving the mass terms.

Finally in Sec. IV we deal with small inhomogeneities on an expanding universe described by the spatially flat FRW model. In this case the computation is only relevant for massless particles, since those are conformally coupled and are not created by the background. The final result may be compared to the case of massless particles on a flat background. The case of neutrinos, massless particles with left polarization, is treated apart. The case of massive particles is also briefly discussed. Finally we conclude with a discussion of some cosmological applications.

II. FORMALISM

A. Spin-$\frac{1}{2}$ particles in curved spacetime

We first briefly review the formalism for dealing with spin-$\frac{1}{2}$ fields in a curved spacetime [1,13,20,31–33]. The most natural way to introduce spinors in general relativity is by means of the tetrad formalism, i.e., by using locally inertial coordinates. At each spacetime point x^μ, we introduce a tetrad $e_a{}^\mu(x)$, such that

$$\eta_{ab}=g_{\mu\nu}(x)e_a{}^\mu(x)e_b{}^\nu(x), \tag{2.1}$$

where $g_{\mu\nu}(x)$ is the spacetime metric and $\eta_{ab}=\mathrm{diag}(+1,-1,-1,-1)$ is the Minkowski metric. We may also define the inverse tetrad $e^a{}_\mu(x)$ by

$$e^a{}_\mu(x)e_b{}^\mu(x)=\delta^a_b,\quad e_a{}^\mu(x)e^a{}_\nu(x)=\delta^\mu_\nu\ ; \tag{2.2}$$

then we can also write

$$g_{\mu\nu}(x)=\eta_{ab}e^a{}_\mu(x)e^b{}_\nu(x)\ . \tag{2.3}$$

The inverse metric in terms of the tetrad is $g^{\mu\nu}(x)=\eta^{ab}e_a{}^\mu(x)e_b{}^\nu(x)$. The tetrad may be seen as the Jacobian matrix corresponding to the transformation of inertial local coordinates at each spacetime point into global spacetime coordinates. Thus the tetrad (and the inverse tetrad) transform tensors with Lorentz indices into tensors with spacetime indices at each spacetime point. Note that the latin (Lorentz) indices are lowered and raised with the local Minkowski metric η_{ab} whereas the greek (spacetime) indices are raised and lowered with the metric $g_{\mu\nu}$. Note also from (2.3) that the tetrad is defined only up to an arbitrary Lorentz transformation at each spacetime point.

In flat spacetime we know how a spinor field ψ changes under an infinitesimal Lorentz transformation $\xi^a \to \Lambda^a{}_b \xi^b \simeq \xi^a + \epsilon^a{}_b \xi^b$, where $\epsilon_{ab} = -\epsilon_{ba}$ are the six parameters of the transformation, i.e.,

$$\psi \to S(\Lambda)\psi \simeq \psi + \tfrac{1}{2}\epsilon_{ab} S^{ab}\psi \;, \tag{2.4}$$

where S^{ab} are the generators of the infinitesimal Lorentz transformation on the spinor, which satisfy the SO(3,1) algebra

$$[S_{ab}, S_{cd}] = \eta_{a[c}S_{bd]} + \eta_{b[d}S_{ac]}. \tag{2.5}$$

It is now a simple matter to incorporate spinors into curved spacetime. The transformation (2.4) depends now on the point $\epsilon_{ab}(x)$ and this means that $\partial_a \psi$ will not transform under a Lorentz transformation as in flat spacetime. As is well known to keep the flat space form of the transformation of such derivative we must define a new covariant derivative operator by introducing a connection field $\Gamma_\mu(x)$. If this field transforms under a Lorentz transformation as

$$\Gamma_\mu \to S\Gamma_\mu S^{-1} - (\partial_\mu S)S^{-1} \;, \tag{2.6}$$

and the derivative operator is defined as

$$D_a \equiv e_a{}^\mu \nabla_\mu \equiv e_a{}^\mu(x)(\partial_\mu + \Gamma_\mu) \;, \tag{2.7}$$

then $D_a\psi$ transforms as required, namely,

$$D_a\psi \to \Lambda_a{}^b(x) S(\Lambda(x)) D_b \psi \;. \tag{2.8}$$

The covariant derivative of tensors with spacetime indices is made through the usual Christoffel symbols

$$\Gamma^\mu_{\nu\lambda} = \tfrac{1}{2} g^{\mu\sigma}(g_{\sigma\lambda,\nu} + g_{\nu\sigma,\lambda} - g_{\nu\lambda,\sigma}) \;, \tag{2.9}$$

which guarantee that the metric tensor vanishes under the spacetime covariant derivative $g_{\mu\nu;\sigma} = 0$. In addition, in the tetrad formalism we must define, in order to include covariant derivative over local or Lorentz indices, the spin connection one-form

$$\omega_a{}^b{}_\nu = -e_a{}^\mu{}_{;\nu} e^b{}_\mu \equiv -(\partial_\nu e_a{}^\mu + \Gamma^\mu_{\rho\nu} e_a{}^\rho) e^b{}_\mu \;, \tag{2.10}$$

which, as a consequence of its definition, is antisymmetric $\omega_{(ab)\nu} = 0$. Note that this may be seen as a transformation similar to (2.6) of the spacetime connection by the tetrad which transforms spacetime indices into local Lorentz indices. Since we have two types of indices and, in addition, we can deal with spinor fields, we may generalize the covariant derivative on such objects as follows. Let $\Psi^{a\mu}$ be a spinor field, we define [32]

$$\nabla_\nu \Psi^{a\mu} = \partial_\nu \Psi^{a\mu} + \Gamma^\mu_{\sigma\nu}\Psi^{a\sigma} + \omega^a{}_{b\nu}\Psi^{b\mu} + \Gamma_\nu \Psi^{a\mu} \;. \tag{2.11}$$

Under this definition and using (2.9) and (2.10) it is easy to show that the metric tensor and the tetrad have vanishing covariant derivatives:

$$\nabla_\nu g_{\mu\lambda} = 0, \quad \nabla_\nu e^b{}_\mu = 0 \;. \tag{2.12}$$

Now to determine the connection $\Gamma_\mu(x)$ for a spinor field we can either impose that the covariant derivative of the bispinor Dirac matrices γ^a ($\{\gamma^a, \gamma^b\} = 2\eta^{ab}$) vanishes,

$$\nabla_\nu \gamma^a = 0 \;, \tag{2.13}$$

or alternatively impose that Γ_μ transforms as (2.6) under an infinitesimal Lorentz transformation. It is easy to check [1] that this last condition is satisfied by the matrix

$$\Gamma_\mu = \tfrac{1}{2} e_a{}^\nu e_{b\nu;\mu} S^{ab} \;. \tag{2.14}$$

For spin-$\frac{1}{2}$ fields the generator of infinitesimal Lorentz transformation is given by the commutator of Dirac matrices [36]:

$$S^{ab} \equiv \tfrac{1}{4}[\gamma^a, \gamma^b] \;. \tag{2.15}$$

One can now prove (2.13) by taking into account that, if Λ^a is a bispinor,

$$D_a \Lambda^b = e_a{}^\mu(\partial_\mu \Lambda^b + \omega^b{}_{c\mu}\Lambda^c + \tfrac{1}{2}[S^{cd}, \Lambda^b]\omega_{cd\mu}) \;, \tag{2.16}$$

where we have used that, as usual, $\bar\psi \equiv \psi^\dagger \gamma^0$ and $\bar\psi \overleftarrow{D}_a = \bar\psi(\overleftarrow{\partial}_\mu - \Gamma_\mu) e_a{}^\mu$ since $\gamma^0 \Gamma^\dagger_\mu \gamma^0 = -\Gamma_\mu$.

Finally, the Dirac action in flat space,

$$S = \int d^4 y \left[\frac{i}{2} \bar\psi \gamma^a \overleftrightarrow{\partial}_a \psi - m \bar\psi \psi \right] ,$$

where y^a are global inertial coordinates, can be easily expressed in curved spacetime, by using the covariant derivative $D_a = e_a{}^\mu \nabla_\mu$, as

$$S = \int d^4 x \, e \left[\frac{i}{2} \bar\psi \gamma^a \overleftrightarrow{D}_a \psi - m \bar\psi\psi \right] , \tag{2.17}$$

where $e \equiv \det(e^a{}_\mu) = (-\det g_{\mu\nu})^{1/2}$. If we now introduce the so-called Dirac matrices in curved spacetime $\gamma^\mu(x) = \gamma^a e_a{}^\mu(x)$, which satisfy $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, and neglect a surface term we may write

$$S = \int d^4 x \, e [i \bar\psi \gamma^\mu(x)(\partial_\mu + \Gamma_\mu)\psi - m\bar\psi\psi] \;. \tag{2.18}$$

By varying the action with respect to the field $\bar\psi$ we obtain the Dirac equation in curved spacetime:

$$[i\gamma^\mu(x)\partial_\mu + i\gamma^\mu(x)\Gamma_\mu(x) - m]\psi(x) = 0 \;. \tag{2.19}$$

Note that a coupling of the spinor fields with the curvature such as $R\bar\psi\psi$ cannot be introduced unless we have extra matter; since the above quantity has dimensions of L^{-5} we need an extra field ξ, with dimensions of L, to form a density term: $\xi R \bar\psi\psi$. An action with such term leads to a nonrenormalizable theory; however, at the tree level one may consider such type of coupling (as in Fermi theory).

B. Perturbative approach

Let us now describe the perturbative approach, which is based in the interaction picture and the S matrix. S-matrix methods on curved backgrounds have been intensively used in recent years in connection with the analysis of self-interacting and mutually interacting fields in inflationary cosmology [37–40].

As a first step we must write down the interaction Lagrangian that describes the interaction of the inhomogeneous gravitational field with the quantum spinorial field. Let us separate the spacetime metric $g_{\mu\nu}$ into an inhomogeneous background part $\mathring{g}_{\mu\nu}$, and a small inhomogeneous field, $\delta g_{\mu\nu}$, as

$$g_{\mu\nu}=\mathring{g}_{\mu\nu}+\delta g_{\mu\nu} \ , \tag{2.20}$$

then the Dirac action (2.18) can be separated into a "free" action and an "interaction" part:

$$S=\mathring{S}+\delta S=\mathring{S}+\int d^4x \left[\frac{\delta S}{\delta g_{\mu\nu}} \right]_0 \delta g_{\mu\nu} \ .$$

Note that since the stress tensor of the spinor field can be deduced from the action as

$$T^{\mu\nu}=\frac{2}{\sqrt{g}}\frac{\delta S}{\delta g_{\mu\nu}}$$

we may write the interaction term as

$$\frac{\sqrt{\mathring{g}}}{2}\mathring{T}^{\mu\nu}\delta g_{\mu\nu} \ ,$$

i.e., the external inhomogeneous field couples linearly with the stress tensor of the spinor field, as one might expect. The Lagrangian density defined as $S=\int d^4x\mathcal{L}$ may be written as

$$\mathcal{L}=\mathcal{L}_0+\mathcal{L}_I+O(\delta g^2) \ ,$$

where $\mathcal{L}_0$ is considered as the free Lagrangian density and $\mathcal{L}_I$ as the interaction Lagrangian density. The interaction is linear in the inhomogeneities (unquantized external field) and quadratic in the spinor field. We can now compute particle production in perturbation theory provided that the interaction vanishes in the "in" and "out" regions. This may be imposed by assuming that the inhomogeneities vanish at least adiabatically in these regions (symbolically, $\eta \to \pm\infty$):

$$\lim_{\eta\to\pm\infty} \delta g_{\mu\nu}=0 \ . \tag{2.21}$$

In the interaction picture the states are described by the free modes, i.e., modes that are solutions of the field equation corresponding to the Lagrangian density $\mathcal{L}_0$.

When there is no particle creation by the background the "in" and "out" vacuum can be identified. This is the case for instance when the background is flat $\mathring{g}_{\mu\nu}=\eta_{\mu\nu}$, or when it is conformally flat and we consider conformally coupled particles, i.e., massless fermions. Such are the cases we shall consider in this paper, but the technique can also be implemented when the background does create particles (see Refs. [26,27]); then one must distinguish the "in" vacuum $|0,\text{in}\rangle$ from the "out" vacuum $|\bar{0},\text{out}\rangle$.

Let us now consider a physical state $|\Psi\rangle$; in the interaction picture it evolves in time according to the Schrödinger equation

$$H_I|\Psi\rangle=i\frac{\partial|\Psi\rangle}{\partial\eta} \ , \tag{2.22}$$

where H_I is the interaction Hamiltonian constructed from the interaction Hamiltonian density $H_I=\int d^3x\,\mathcal{H}_I$. This equation can be formally solved by means of the unitary S-matrix operator connecting the "in" and "out" states:

$$S=T\exp\left[-i\int d^4x\,\mathcal{H}_I\right] \ , \tag{2.23}$$

where T stands for the time-ordering operator. If we start with $|0,\text{in}\rangle$, i.e., no "in" particles, the final evolution of this state as a consequence of the interaction is

$$|0,\text{out}\rangle=S|0,\text{in}\rangle \ . \tag{2.24}$$

Now we can expand S perturbatively; up to first order in $\delta g_{\mu\nu}$ it becomes $S\simeq 1-iT\int d^4x\,\mathcal{H}_I$. For a nonderivative coupling we have

$$\mathcal{H}_I=-\mathcal{L}_I \ . \tag{2.25}$$

In our case, however, the interaction Lagrangian contains derivative terms and we must be careful. However, it is not hard to see using path-integral methods [41] that since there is no ordering problem at the quantum level in the Hamiltonian operator (because there is no mixing of canonical conjugate variables in the classical action) the substitution (2.25) still holds. Thus, we may write

$$S=1+iT\int d^4x\,\mathcal{L}_I\equiv 1+S^{(1)} \ . \tag{2.26}$$

Since $\mathcal{L}_I$ is quadratic in the field and its derivatives, to first order the particles are produced in pairs. The probability amplitude for pair creation is given by the S-matrix element

$$S^{(1)}_{\alpha\beta}\equiv\langle\alpha\beta|S^{(1)}|0\rangle=iT\int d^4x\langle\alpha\beta|\mathcal{L}_I|0\rangle \ , \tag{2.27}$$

where α and β are the necessary quantum numbers to describe the particles. Note that the state $|\alpha\beta\rangle$ indicates two-particle states in the same Fock space as $|0\rangle$. When the background creates particles these states do not correspond to physical states in the "out" region and one needs the Bogoliubov transformation to relate the two types of states [27]. When the background does not create particles then $|\alpha\beta\rangle$ correspond to physical particle states and the total pair-creation probability is

$$W^{(1)}=\sum_{\alpha\beta}|S^{(1)}_{\alpha\beta}|^2 \ , \tag{2.28}$$

which is quadratic in the inhomogeneities. Our next task will be to compute this probability for Dirac particles.

III. FLAT SPACE WITH GRAVITATIONAL INHOMOGENEITIES

In flat space there is a timelike Killing vector over all spacetime; there is no ambiguity in the definition of the positive-frequency modes; the vacuum is unique and Poincaré invariant. We start with the Dirac action (2.18) and assume that the spacetime is described by

$$g_{\mu\nu}(x)=\eta_{\mu\nu}+h_{\mu\nu}(x) \ , \tag{3.1}$$

where $h_{\mu\nu}$ is a symmetric tensor which represents the gravitational inhomogeneities. To define the interaction Lagrangian we first introduce the tetrad. According to (2.3) the tetrad is not unique since it is given up to an arbitrary Lorentz transformation at each spacetime point. A convenient tetrad is

$$e^{a\mu}(x)=\eta^{a\mu}-\tfrac{1}{2}h^{a\mu}(x)+O(h^2)\ , \tag{3.2}$$

and its inverse is $e_{a\mu}(x)=\eta_{a\mu}+\frac{1}{2}h_{a\mu}(x)+O(h^2)$. The connection (2.14) is now

$$\Gamma_\mu=\tfrac{1}{8}h_\mu{}^{b,c}[\gamma_b,\gamma_c]+O(h^2)\ , \tag{3.3}$$

and the Dirac matrices are

$$\gamma^\mu(x)=(\eta^{a\mu}-\tfrac{1}{2}h^{a\mu})\gamma_a+O(h^2)\ . \tag{3.4}$$

A. Massless particles

We will first compute the case of massless particles. There is no problem in doing the massive case and then taking the mass equal to zero in the final result. However, we treat this case separately, and with some detail, because a few partial results of this section will be later used when dealing with an expanding background in Sec. IV.

Substituting the above expressions in the Dirac action and taking $m=0$ we can separate the Lagrangian into a "free" term and an "interaction" term, respectively,

$$\mathcal{L}_0=i\bar\psi\eta^{a\mu}\gamma_a\partial_\mu\psi\ , \tag{3.5}$$

$$\mathcal{L}_I=\frac{i}{2}(2\bar\psi\eta^{a\mu}\gamma_a\Gamma_\mu^{(1)}\psi-\bar\psi h^{a\mu}\gamma_a\partial_\mu\psi+h\bar\psi\eta^{a\mu}\gamma_a\partial_\mu\psi)\ , \tag{3.6}$$

where $h=h^\mu{}_\mu=h^{\mu\nu}\eta_{\mu\nu}$, and we have expanded the connection to first order in the inhomogeneities $\Gamma_\mu=\Gamma_\mu^{(0)}+\Gamma_\mu^{(1)}+O(h^2)$ as

$$\Gamma_\mu^{(0)}=0,\quad \Gamma_\mu^{(1)}=\tfrac{1}{8}h_\mu{}^{b,c}[\gamma_b,\gamma_c]\ . \tag{3.7}$$

The equation of motion of the field $\psi(x)$ is simply the Dirac equation in flat space,

$$i\eta^{a\mu}\gamma_a\partial_\mu\psi\equiv i\partial\!\!\!/\psi(x)=0\ , \tag{3.8}$$

which has well-known solutions ([24,36]) in terms of creation and annihilation operator such as

$$\psi(x)=\sum_\lambda\int\frac{d^3\mathbf{p}}{(2\pi)^3 2p^0}[a_\mathbf{p}(\lambda)u_\mathbf{p}(\lambda)e^{-ip\cdot x}+b_\mathbf{p}^\dagger(\lambda)v_\mathbf{p}(\lambda)e^{ip\cdot x}]\ , \tag{3.9}$$

where $\sum_\lambda$ indicates sum over polarizations λ, and the Lorentz-invariant integral sums over all momenta. The operators $a_\mathbf{p}(\lambda)$ and $b_\mathbf{p}^\dagger(\lambda)$ are annihilator of particles and creator of antiparticles, respectively, which satisfy the anticommutation rules

$$\{a_\mathbf{p}(\lambda),a_{\mathbf{p}'}^\dagger(\lambda')\}=\{b_\mathbf{p}(\lambda),b_{\mathbf{p}'}^\dagger(\lambda')\}=(2\pi)^3 2p^0\delta(\mathbf{p}-\mathbf{p}')\delta_{\lambda\lambda'}\ , \tag{3.10}$$

and the polarization vectors satisfy the completeness relations

$$\sum_\lambda v_\mathbf{p}(\lambda)\bar v_\mathbf{p}(\lambda)=\sum_\lambda u_\mathbf{p}(\lambda)\bar u_\mathbf{p}(\lambda)=p\!\!\!/. \tag{3.11}$$

The probability amplitude of pair creation (2.27) can be written, after use of Wick's theorem for fermionic fields, as

$$\begin{aligned}S_{\alpha\beta}^{(1)}&=i\int d^4x\,\langle\alpha,\beta|T\mathcal{L}_I|0\rangle\\&=i\int d^4x\,\langle\mathbf{p},\lambda;\mathbf{p}',\lambda'|T\mathcal{L}_I|0\rangle\\&=\frac{i\tilde h^{ab}(p'+p)}{8}\bar u_{\mathbf{p}'}(\lambda')\{(p'+p)^c\gamma_a[\gamma_b,\gamma_c]-4p^c\eta_{ab}\gamma_c+4p_b\gamma_a\}v_\mathbf{p}(\lambda)\ ,\end{aligned} \tag{3.12}$$

where we have defined the Fourier transform of the inhomogeneities as

$$\begin{aligned}\tilde h^{ab}(p)&\equiv\int d^4x\,e^{ip\cdot x}h^{ab}(x)\ ,\\ h^{ab}(x)&\equiv\frac{1}{(2\pi)^4}\int d^4p\,e^{-ip\cdot x}\tilde h^{ab}(p)\ .\end{aligned} \tag{3.13}$$

Making use of the definitions

$$\begin{aligned}\Pi_{ab}&\equiv q^c\gamma_a[\gamma_b,\gamma_c]-4\eta_{ab}p\!\!\!/+4\gamma_a p_b\ ,\\ \hat\Pi_{mn}&\equiv\gamma_0\Pi_{mn}^\dagger\gamma_0=q^r[\gamma_r,\gamma_n]\gamma_m-4\eta_{mn}p\!\!\!/+4\gamma_m p_n\ ,\end{aligned} \tag{3.14}$$

the total pair-creation probability for massless particles $W_0^{(1)}$ is given by

$$\begin{aligned}W_0^{(1)}&=\sum_{\alpha,\beta}|S_{\alpha\beta}^{(1)}|^2\\&=\sum_{\lambda\lambda'}\int\frac{d^3\mathbf{p}}{(2\pi)^3 2p^0}\int\frac{d^3\mathbf{p}'}{(2\pi)^3 2p^{0'}}\left|\langle\mathbf{p},\lambda;\mathbf{p}',\lambda'|i\int d^4x\,T\mathcal{L}_I|0\rangle\right|^2\\&=\int\frac{d^4q}{(4\pi)^6}\tilde h^{ab}(q)\tilde h^{mn}(-q)\int\frac{d^3\mathbf{p}}{2p^0}\int\frac{d^3\mathbf{p}'}{2p^{0'}}\delta(q-p'-p)\mathrm{Tr}(p\!\!\!/'\Pi_{ab}p\!\!\!/\hat\Pi_{mn})\ ,\end{aligned} \tag{3.15}$$

where we have introduced a Dirac delta function for the total momentum and used the completeness relations (3.11). Computing the traces we can write the above expression in terms of well-known phase-space integrals (see the Appendix),

$$W_0^{(1)}=\int\frac{d^4p}{(4\pi)^6}\tilde{h}^{ab}(q)\tilde{h}^{mn}(-q)$$
$$\times\{64[I_{a/mbn}+I_{m/abn}]-32[q_a(I_{m/bn}+I_{b/mn})+q_m(I_{n/ab}+I_{a/nb})+q^2\eta_{am}I_{bn}]$$
$$+16[q^2\eta_{mn}(I_{ab}+I_{a/b})+q^2\eta_{ab}(I_{mn}+I_{m/n})+q_aq_m(I_{n/b}+I_{b/n})$$
$$+q^2q_m(\eta_{an}I_b-\eta_{ab}I_n)+q^2q_a(\eta_{bm}I_n-\eta_{nm}I_b)]$$
$$-8[q^2q_m\eta_{ab}(I_n+I_{/n})+q^2q_a\eta_{nm}(I_b+I_{/b})-q^2(q_aq_b\eta_{nm}-q_aq_m\eta_{bn}+q_mq_n\eta_{ab})I]\}\ ,$$

and after a long but straightforward computation we can write the total probability as

$$W_0^{(1)}=\frac{8}{15}\int\frac{d^4q}{(4\pi)^6}I(q)\tilde{h}^{ab}(q)\tilde{h}^{mn}(-q)q^4(4P_{am}P_{bn}-P_{ab}P_{mn}-P_{an}P_{mb})\ ,$$

where we have defined the projector $P_{ab}=\eta_{ab}-q_aq_b/q^2$. Now the Fourier transform of the Riemann tensor components at this linear order can be written [26,27], in terms of the above projector, as

$$|\tilde{R}(q)|^2\equiv\tilde{R}(q)\tilde{R}(-q)=q^4P_{ab}P_{mn}\tilde{h}^{ab}(q)\tilde{h}^{mn}(-q)\ ,$$
$$|\tilde{R}^{abcd}(q)|^2\equiv\tilde{R}^{abcd}(q)\tilde{R}_{abcd}(-q)=q^4P_{an}P_{bm}\tilde{h}^{ab}(q)\tilde{h}^{mn}(-q)\ , \tag{3.16}$$

and the above integral simplifies to

$$W_0^{(1)}=\frac{1}{960\pi}\int\frac{d^4q}{(2\pi)^4}\theta(q^0)\theta(q^2)\times[3|\tilde{R}^{abcd}(q)|^2-|\tilde{R}(q)|^2]\ . \tag{3.17}$$

We can now use the following geometric relations. The definition of the Weyl tensor C_{abcd} in terms of the Riemann tensor implies that, in four dimensions,

$$C_{abcd}C^{abcd}=R_{abcd}R^{abcd}-2R_{ab}R^{ab}+\tfrac{1}{3}R^2\ . \tag{3.18}$$

The integral of (3.18),

$$\int d^4x\sqrt{|g|}C_{abcd}C^{abcd}\ ,$$

is invariant under conformal transformations. Furthermore the four-dimensional analogue of the Gauss-Bonnet invariant in two dimensions,

$$K\equiv\int d^4x\sqrt{|g|}(R_{abcd}R^{abcd}-4R_{ab}R^{ab}+R^2)\ , \tag{3.19}$$

is a topological invariant, independent of the geometry.

In considering the metric $g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$ due to the boundary conditions (2.21) imposed on $h_{\mu\nu}$, one finds that $K=0$. This translates into the following relation between the Fourier components of the Riemann tensor,

$$|\tilde{R}^{abcd}(q)|^2-4|\tilde{R}^{ab}(q)|^2+|\tilde{R}(q)|^2=0\ , \tag{3.20}$$

which, together with the corresponding relation (3.18), leads to the following alternative expression of (3.17), which is manifestly conformally invariant:

$$W_0^{(1)}=\frac{1}{160\pi}\int\frac{d^4q}{(2\pi)^4}\theta(q^0)\theta(q^2)|\tilde{C}^{abcd}(q)|^2\ . \tag{3.21}$$

This result is in agreement with that of Ref. [25] and it differs by a factor of 6 with that for scalar particles.

B. Massive particles

In this section we will study the case of massive particles on a flat spacetime with inhomogeneities. Following the preceding subsection if we introduce the massive term the free Lagrangian and the dynamical equation for the free field ψ are

$$\mathcal{L}_0=i\bar{\psi}\eta^{a\mu}\gamma_a\partial_\mu\psi-m\bar{\psi}\psi\ , \tag{3.22}$$

$$(i\not{\partial}-m)\psi=0\ , \tag{3.23}$$

and the completeness relations for the polarization vectors are

$$\sum_\lambda u_{\mathbf{p}}(\lambda)\bar{u}_{\mathbf{p}}(\lambda)=(\not{p}+m)\ ,\qquad \sum_\lambda v_{\mathbf{p}}(\lambda)\bar{v}_{\mathbf{p}}(\lambda)=(\not{p}-m)\ . \tag{3.24}$$

The new interaction Lagrangian

$$\mathcal{L}_I=\frac{i}{2}(2\bar{\psi}\eta^{a\mu}\gamma_a\Gamma_\mu^{(1)}\psi-\bar{\psi}h^{a\mu}\gamma_a\partial_\mu\psi+h\bar{\psi}\eta^{a\mu}\gamma_a\partial_\mu\psi+imh\bar{\psi}\psi) \tag{3.25}$$

gives, after a short computation, the probability amplitude for pair creation,

$$S_{\alpha\beta}^{(1)}=\frac{i\tilde{h}^{ab}(p'+p)}{8}\bar{u}_{\mathbf{p}'}(\lambda')\{(p'+p)^c\gamma_a[\gamma_b,\gamma_c]-4p^c\eta_{ab}\gamma_c+4p_b\gamma_a-4m\eta_{ab}\}v_{\mathbf{p}}(\lambda)\ , \tag{3.26}$$

and the total pair-creation probability (2.28), for massive spin-$\frac{1}{2}$ particles $W_m^{(1)}$, is

$$W_m^{(1)}=\int \frac{d^4q}{(4\pi)^6}\tilde{h}^{ab}(q)\tilde{h}^{mn}(-q) \times\int\frac{d^3\mathbf{p}}{2p^0}\int\frac{d^3\mathbf{p}'}{2p^{0'}}\delta(q-p'-p) \times \mathrm{Tr}[(\not{p}'+m)(\Pi_{ab}-4m\eta_{ab}) \times(\not{p}-m)(\hat{\Pi}_{mn}-4m\eta_{mn})] .$$

Following the steps of the preceding subsection this probability can be expressed in terms of geometric quantities as

$$W_m^{(1)}=\frac{1}{160\pi}\int\frac{d^4q}{(2\pi)^4}\theta(q^0)\theta(q^2-4m^2)\left[1-\frac{4m^2}{q^2}\right]^{3/2} \times\left[\left[1+\frac{8}{3}\frac{m^2}{q^2}\right]|\tilde{C}^{abcd}(q)|^2 +\frac{10}{9}\frac{m^2}{q^2}|\tilde{R}(q)|^2\right] . \tag{3.27}$$

Note that we can now write the Fourier transform of the Weyl components in terms of the Ricci components and the Ricci scalar by using (3.18) and (3.20), and then we can use the Einstein equations $R_{\mu\nu}=8\pi G(T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T)$ to write the total probability of pair production in terms of the energy-momentum tensor of the gravitational source as

$$W_m^{(1)}=\frac{(8\pi G)^2}{80\pi}\int\frac{d^4q}{(2\pi)^4}\theta(q^0)\theta(q^2-4m^2)\left[1-\frac{4m^2}{q^2}\right]^{3/2} \times\left[\left[1+\frac{8}{3}\frac{m^2}{q^2}\right]|\tilde{T}^{ab}(q)|^2 -\frac{1}{3}\left[1+\frac{m^2}{q^2}\right]|\tilde{T}(q)|^2\right] . \tag{3.28}$$

When comparing with the corresponding expression for scalar particles [27] there is not only a global factor of 6 but also the mass-dependent terms acquire different coefficients so that we could not have guessed the above result by extrapolating the scalar expression with simple arguments.

IV. COSMOLOGICAL MODELS WITH INHOMOGENEITIES

A. Massless particles

In this section we will consider a massless spin-$\frac{1}{2}$ matter field described by the Dirac action

$$S=\int d^4x\, ei\bar{\psi}\gamma^\mu(x)(\partial_\mu+\Gamma_\mu)\psi \tag{4.1}$$

on a spacetime with a conformally flat FRW metric

$$g_{\mu\nu}(x)=a^2(\eta)[\eta_{\mu\nu}+h_{\mu\nu}(x)] , \tag{4.2}$$

where the conformal factor $a(\eta)$ depends only on the conformal time η.

Now we choose the expressions for the tetrads,

$$e^b{}_\nu=a(\eta)(\eta^b{}_\nu+\tfrac{1}{2}h^b{}_\nu) , \quad e^{a\nu}=a^{-1}(\eta)(\eta^{a\nu}-\tfrac{1}{2}h^{a\nu}) , \tag{4.3}$$

and the connection can be written at the zero and the first orders (the latter being linear in the inhomogeneities), respectively, as

$$\Gamma_\mu^{(0)}=\frac{a_{,\lambda}}{4a}\eta^{b\lambda}\eta^a{}_\mu[\gamma_a,\gamma_b] , \tag{4.4}$$

$$\Gamma_\mu^{(1)}=\frac{1}{8}[\gamma_a,\gamma_b]\left[h_\mu{}^{a,b}+\eta^b{}_\mu h^{a\lambda}\frac{a_{,\lambda}}{a}+\eta^{b\lambda}h_\mu{}^a\frac{a_{,\lambda}}{a}\right] . \tag{4.5}$$

Then using the Dirac action (4.1) we can separate the "free" and the "interaction" Lagrangians as

$$\mathcal{L}_0=ia^3(\bar{\psi}\eta^{a\mu}\gamma_a\partial_\mu\psi+\bar{\psi}\eta^{a\mu}\gamma_a\Gamma_\mu^{(0)}\psi) , \tag{4.6}$$

$$\mathcal{L}_I=\frac{ia^3}{2}(2\bar{\psi}\eta^{a\mu}\gamma_a\Gamma_\mu^{(1)}\psi-\bar{\psi}h^{a\mu}\gamma_a\partial_\mu\psi+h\bar{\psi}\eta^{a\mu}\gamma_a\partial_\mu\psi -\bar{\psi}h^{a\mu}\gamma_a\Gamma_\mu^{(0)}\psi+h\bar{\psi}\eta^{a\mu}\gamma_a\Gamma_\mu^{(0)}\psi) . \tag{4.7}$$

The first gives the following dynamical equation for the spinorial free field:

$$\left[\frac{\partial}{\partial\eta}+\frac{3}{2}\frac{\dot{a}}{a}+\boldsymbol{\alpha}\cdot\nabla\right]\psi=0 , \tag{4.8}$$

where $\alpha_i\equiv\gamma_0\gamma_i$. It is easy to see that the solution of this equation can be written as

$$\psi(x)=\sum_\lambda\int\frac{d^3\mathbf{p}}{(2\pi)^3 2p^0}\frac{1}{a^{3/2}(\eta)} \times[a_{\mathbf{p}}(\lambda)u_{\mathbf{p}}(\lambda)e^{-ip\cdot x} +b_{\mathbf{p}}^\dagger(\lambda)v_{\mathbf{p}}(\lambda)e^{ip\cdot x}] , \tag{4.9}$$

where the creation and annihilation operators satisfy the standard anticommutation rules (3.10) and the polarization vectors the completeness relations (3.11). Since the background is conformally flat and the field massless, this operator differs from the operator in flat space by the factor $a^{3/2}(\eta)$ only. Then the probability amplitude computed from the interaction Lagrangian (4.7),

$$S_{\alpha\beta}^{(1)}=\frac{i\tilde{h}^{ab}(p'+p)}{8}\bar{u}_{\mathbf{p}'}(\lambda') \times\{(p'+p)^c\gamma_a[\gamma_b,\gamma_c]-4p^c\eta_{ab}\gamma_c +4p_b\gamma_a\}v_{\mathbf{p}}(\lambda) , \tag{4.10}$$

coincides with that for the flat background (3.12); i.e., the dependence on the conformal factor has disappeared. Thus the total pair-creation probability follows the steps of the computation in Sec. III, with the result

$$W^{(1)}=\frac{1}{160\pi}\int\frac{d^4q}{(2\pi)^4}\theta(q^0)\theta(q^2)|\tilde{C}^{abcd}(q)|^2 , \tag{4.11}$$

where here $\tilde{C}_{abcd}(q)$ is the Fourier transform of the Weyl tensor for the cosmological metric (4.2). We have used that the above expression is conformally invariant, as follows from the discussion at the end of Sec. III A; of course, the result (4.11) is the same as that for the metric (3.1). Therefore the total pair-creation probability for massless conformally coupled particles due to small gravitational inhomogeneities follows from the total pair-creation probability in flat space for massless (conformally coupled) spinors in the presence of small inhomogeneities.

B. Neutrinos

We have just seen that the probability of the production of spin-$\frac{1}{2}$ particles in a FRW universe is given by the equation (4.11). We will show in this section that, as one should expect, this probability must be divided by 2 in the case of polarized spin-$\frac{1}{2}$ massless particles (neutrinos). This condition can be expressed, mathematically, requiring that the matter field satisfy the equation

$$(1-\gamma_5)\psi=0 , \tag{4.12}$$

as well as the Dirac equation (4.8). Now the fields have polarization vectors with completeness relations and normalization conditions given by

$$\begin{aligned}\sum_\lambda u_{\mathrm{p}}(\lambda)\bar{u}_{\mathrm{p}}(\lambda)&=u_{\mathrm{p}}\bar{u}_{\mathrm{p}}=\not{p}\left[\frac{1-\gamma_5}{2}\right] ,\\ \sum_\lambda v_{\mathrm{p}}(\lambda)\bar{v}_{\mathrm{p}}(\lambda)&=v_{\mathrm{p}}\bar{v}_{\mathrm{p}}=\not{p}\left[\frac{1-\gamma_5}{2}\right] ,\end{aligned} \tag{4.13}$$

$$u_{\mathrm{p}}^\dagger u_{\mathrm{p}}=v_{\mathrm{p}}^\dagger v_{\mathrm{p}}=2p^0 . \tag{4.14}$$

The total probability for neutrino pair production, $W_\nu^{(1)}$, is then given by

$$\begin{aligned}W_\nu^{(1)}&=\int\frac{d^4q}{(4\pi)^6}\tilde{h}^{ab}(q)\tilde{h}^{mn}(-q)\int\frac{d^3\mathrm{p}}{2p^0}\int\frac{d^3\mathrm{p}'}{2p^{0'}}\delta(q-p'-p)\mathrm{Tr}\left[\left[\frac{1-\gamma_5}{2}\right]\not{p}'\Pi_{ab}\not{p}\hat{\Pi}_{mn}\right]\\ &=\frac{W_0^{(1)}}{2}+\frac{1}{2}\int\frac{d^4q}{(4\pi)^6}\tilde{h}^{ab}(q)\tilde{h}^{mn}(-q)\int\frac{d^3\mathrm{p}}{2p^0}\int\frac{d^3\mathrm{p}'}{2p^{0'}}\delta(q-p'-p)\mathrm{Tr}[\gamma_5\not{p}'\Pi_{ab}\not{p}\hat{\Pi}_{mn}] ,\end{aligned}$$

where we have used (3.15). It is not difficult to see that the last trace is identically zero; thus,

$$W_\nu^{(1)}=\frac{W_0^{(1)}}{2}=\frac{1}{320\pi}\int\frac{d^4q}{(2\pi)^4}\theta(q^0)\theta(q^2)|\tilde{C}^{abcd}(q)|^2 . \tag{4.15}$$

C. Massive particles

The creation of massive particles in a cosmological background with small inhomogeneities constitutes a different situation. Now the background does create particles and the contribution of the inhomogeneities is physically less relevant. However, for completeness we shall consider how one can handle that case. In principle the method for dealing with the inhomogeneities is similar to the previous case, i.e., a perturbative approach, but with the difference that now the "in" and "out" vacua differ. That is, the "free" modes define different "in" and "out" particles.

Let us assume that $\{u_i(x),v_i(x)\}$ are a complete set of mode solutions of the Dirac equation (2.19) with the normalization

$$(\psi_1,\psi_2)=i\int_\Sigma\bar{\psi}_1\gamma^\mu\psi_2\sqrt{-g_\Sigma}n_\mu d\Sigma .$$

Here ψ_1 and ψ_2 are solutions of (2.19), the integral is over a spacelike Cauchy hypersurface Σ, and n_μ is the vector orthogonal to it (we assume that the spacetime is globally hyperbolic). The field operator can be written in terms of these modes as

$$\psi(x)=\sum_i[a_iu_i(x)+b_i^\dagger v_i(x)] ,$$

where $\sum_i$ is shorthand notation to indicate an integral over momenta and sum over polarizations; see (3.9). The a_i and b_i are the annihilation operators of particles and antiparticles, respectively, and satisfy (3.10).

Let us assume that we have two sets of such complete solutions, $\{u_i^{\mathrm{in}}(x),v_j^{\mathrm{in}}(x)\}$ and $\{u_i^{\mathrm{out}}(x),v_j^{\mathrm{out}}(x)\}$, which may define physical particles in the "in" and "out" regions, respectively. Since each are a complete set of modes we may write one set in terms of the other:

$$u_i^{\mathrm{in}}(x)=\sum_j[\alpha_{ij}u_j^{\mathrm{out}}(x)+\beta_{ij}v_j^{\mathrm{out}}(x)],$$

$$v_i^{\mathrm{in}}(x)=\sum_j[\gamma_{ij}u_j^{\mathrm{out}}(x)+\eta_{ij}v_j^{\mathrm{out}}(x)] .$$

The coefficients α,β,γ,η are called Bogoliubov coefficients and verify $AA^T=1$ where A is the matrix

$$A=\begin{bmatrix}\alpha & \beta\\ \gamma & \eta\end{bmatrix} .$$

The "in" and "out" creation operators of particles and antiparticles are related by

$$\begin{aligned}a_j^{\mathrm{out}}&=\sum_i(\alpha_{ij}a_i^{\mathrm{in}}+\gamma_{ij}b_i^{\mathrm{in}\dagger}) ,\\ b_j^{\mathrm{out}\dagger}&=\sum_i(\beta_{ij}a_i^{\mathrm{in}}+\eta_{ij}b_i^{\mathrm{in}\dagger}) .\end{aligned} \tag{4.16}$$

The expectation value of the number of "out" particles of type j in the "in" vacuum (i.e., particles created) is given by $N_j^{(P)}=\langle 0,\text{in}|a_j^{\text{out}\dagger}a_j^{\text{out}}|0,\text{in}\rangle=\sum_i|\gamma_{ij}|^2$, where we have used the above expressions and that $a_i^{\text{in}}|0,\text{in}\rangle=b_i^{\text{in}}|0,\text{in}\rangle=0$ for all i. Similarly, the number of "out" antiparticles in the "in" vacuum is $N_j^{(A)}=\langle 0,\text{in}|b_j^{\text{out}\dagger}b_j^{\text{out}}|0,\text{in}\rangle=\sum_i|\beta_{ij}|^2$. Thus the total number of particles created is $N=\sum_{i,j}(|\gamma_{ij}|^2+|\beta_{ij}|^2)$.

Now let us compute the perturbative effect due to an interaction Lagrangian. In the interaction picture a physical state $|\Psi\rangle$ evolves in time according to the Schrödinger equation (2.22). Let us assume as before that we have no initial particles, i.e.,

$$\lim_{\eta\to-\infty}|\Psi\rangle=|0,\text{in}\rangle\ ;$$

as a consequence of the interaction the state will evolve to a final state:

$$\lim_{\eta\to+\infty}|\Psi\rangle=S|0,\text{in}\rangle=\mathcal{N}\left[|0,\text{in}\rangle+\sum_n\frac{1}{n!}|n,\text{in}\rangle\langle n,\text{in}|S^{(1)}|0,\text{in}\rangle\right], \tag{4.17}$$

where $\mathcal{N}$ is a normalization factor such that $\langle\Psi|\Psi\rangle=1$. If the interaction is quadratic in the quantum field, the first-order contribution will come from the transition amplitude to two-particle states: $\langle r,s;\text{in}|S^{(1)}|0,\text{in}\rangle$. On the other hand, it is easy to see that $\mathcal{N}=1+O(h^2)$ where h indicates the order of the interaction term.

We may now compute the number of "out" particles in the mode j in the final state by using the relations (4.16) and (4.17):

$$N_j=\lim_{\eta\to+\infty}\langle\Psi|a_j^{\text{out}\dagger}a_j^{\text{out}}+b_j^{\text{out}\dagger}b_j^{\text{out}}|\Psi\rangle$$

$$=\sum_i(|\gamma_{ij}|^2+|\beta_{ij}|^2)+\sum_{r,s}\text{Re}[(\alpha_{sj}\gamma_{jr}^*+\beta_{sj}\eta_{jr}^*)\langle r,s;\text{in}|S^{(1)}|0,\text{in}\rangle]\ . \tag{4.18}$$

This equation generalizes Eq. (3.16) of Ref. [27] valid for scalar particles to the case of spinorial particles. Note that whereas the first term contains the background contribution only, the second is due to the combined effect of the interaction and the background. The second term is linear in the S-matrix elements; note that when particles are not created by the background the Bogoliubov coefficients β and γ are zero and the above expression vanishes. In that case particle production will be proportional to the next order of approximation, which is of the order of the square of the S-matrix elements, as seen in (3.15) for instance.

In practice given an arbitrary background the computation of (4.18) will be difficult if we do not know the exact modes of the free Lagrangian, because the exact modes are required to compute the scattering matrix. As explained in Ref. [27] given an expansion background factor $a(\eta)$ in (4.2) one may look for another expansion background $a_b(\eta)$ in which the modes can be solved exactly (for instance a step expansion [27] or an expansion law of a radiation-filled universe [34]) and such that $\lim_{\eta\to-\infty}a_b=a_1$ and $\lim_{\eta\to+\infty}a_b=a_2$ (for a bounded expansion). If the deviation from the true background $(a^2/a_b^2)-1$ is small one can write this term in the interaction Lagrangian and the scattering matrix $S^{(1)}$ will contain contributions due to the small inhomogeneities and due to the deviation from $a_b(\eta)$. Then the computation can be carried out explicitly.

From the cosmological viewpoint, however, the physically relevant contribution is the first term in (4.18) since the perturbative term is just a small addition to that of the background. An expression such as (4.18) is more interesting when it is used to compute the effect of self-interaction or mutually interacting fields [37–40] since then there are interaction processes which are not induced by the background expansion

V. CONCLUSIONS

We have seen that the total pair-creation probability by inhomogeneities depends essentially on the Fourier transform of the stress tensor which is the source of the inhomogeneities; see (3.28). From this it is clear that static sources do not create particles at this perturbative order. It is also clear that gravitational waves, i.e., linear perturbations of the gravitational field which are solutions of Einstein's equation in vacuum, do not create particles. This means that such type of inhomogeneities (static and gravitational waves) will not be damped by particle creation in its cosmological evolution. Note also that exact gravitational plane waves do not create particles [42,43]; however, the nonlinear superpositions of exact plane waves, as represented by exact colliding plane-wave spacetimes, for instance, may be the source of particles [44,45].

As we know, the cosmological importance of the particles created will be greater if these are produced earlier in the Universe. In this sense it is tempting to extrapolate the results (4.11) or (4.15) all the way back to Planck time. Multiplying the total pair-creation probability by the frequency of a particle and integrating over frequencies we get the total energy of the particles produced. The energy density of the particles created, ρ, is then given by $\rho a^4=(2\pi)^{-3}\int d\omega\,\omega W^{(1)}$. According to the formulas of the preceding section, in order of magnitude the energy density of particles created is given by

$\rho(x) \sim C_{abcd}(x)C^{abcd}(x)$. It is plausible that at the Planck time we have inhomogeneities which may have been originated from quantum metric fluctuations at the Planck era. These may translate into inhomogeneities with a typical Weyl tensor scale of $C \sim t_P^{-2}$, where t_P is the Planck time (the only time scale available). That would produce an energy density of the order of the Planck values, i.e., 10^{93} g/cm^3, which corresponds to a temperature of $T \sim 10^{32}$ K, assuming thermalization of the particles. Assuming entropy conservation in a comoving volume this temperature would redshift to a present temperature of a few degrees Kelvin (the present background temperature) [4]. Note that particles created by a similar mechanism after the Planck time have negligible influence as compared to the ones created at the Planck time; the reason is that the energy density of the latter evolves with time as $\rho(t) \sim 3/32\pi G t^2$ whereas at this same time the energy density of the particles created goes as $\rho_c(t) \sim 1/t^4$; the ratio of the two is $\rho(t)/\rho_c(t) \sim t_P^2/t^2$, which will be increasingly smaller. Note also that particle and entropy production by gravitational inhomogeneities differs from that of a homogeneous expansion in that here all kinds of particles, conformally coupled or not, are created. Thus, in principle, gravitational inhomogeneities could be the source of matter and entropy in the Universe.

If we move now to safer ground, away from the Planck time where back-reaction effects may be less important, the earliest effects that we reasonably understand and may compute are at the grand unification time, i.e., at 10^{-34} sec, when the Universe had a temperature of $T \sim 10^{16}$ GeV. At this time quantum fields underwent phase transitions which may have resulted in topological defects [10,11]. When a topological defect forms, a sudden change in the gravitational field takes place, resulting in gravitational inhomogeneities which may create high-energy particles. The procedure explained in this paper is quite appropriate for the computation of such effects since gravitational fields involved are very well described by the linear approximation [28–30]; in fact, for dimensional reasons their gravitational effects in the spacetime metric will be of order $GT^2 \sim 10^{-6}$, which is much smaller than one.

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APPENDIX

Here we write the phase-space integrals that are needed in order to compute the final results for pair creation in the main text. It is not difficult to prove by induction, and taking into account that the integration must be a linear combination of the flat metric and polynomial tensors constructed with the total momentum q_a, that

$$I_{i_1 - i_m / i_{m+1} - i_n} \equiv \int \frac{d^3\mathbf{p}}{2p^0} \frac{d^3p'}{2p^{0'}} \delta(q - p' - p) p'_{i_1} - p'_{i_m} p_{i_{m+1}} - p_{i_n}$$
$$= \frac{I(q)}{2^n} \sum_{j=0}^{[n/2]} \frac{(4m^2 - q^2)^j}{(2j+1)!!}$$
$$\times \left[\sum_C (-1)^A q_{i_1} - \hat{q}_{i_{a_1}} - \hat{q}_{i_{a_2}} \cdots \hat{q}_{i_{a_{2j-1}}} - \hat{q}_{i_{a_{2j}}} - q_{i_n} \left[\sum_P P_{i_{a_1} i_{a_2}} \cdots P_{i_{a_{2j-1}} i_{a_{2j}}} \right] \right] ,$$

where

$$I(q) = \int \frac{d^2\mathbf{p}}{2p^0} \frac{d^3\mathbf{p}'}{2p^{0'}} \delta(q - p' - p) = \frac{\pi}{2} \left[1 - \frac{4m^2}{q^2} \right]^{1/2} \theta(q^0) \theta(q^2 - 4m^2) .$$

Here the caret in $\hat{q}_a$ means that the four-momentum q_a is absent; $\sum_C$ means the sum over the number of ways (order independent) of choosing $n - 2j$ vectors from the set of n vectors q; $\sum_P$ means the sum over the permutations of $2j$ indices which give different tensorial terms; and A is the number of i_a indices between i_{m+1} and i_n. Every element of the series has $(2j-1)!! C_n^{n-2j}$ terms if $j \geq 1$ and only one if $j = 0$. A few examples of the integrals which we need are

$$I_a = I_{/a} = \frac{I(q)}{2} q_a ,$$

$$I_{ab} = \frac{I(q)}{4} \left[q_a q_b + \frac{4m^2 - q^2}{3} P_{ab} \right] ,$$

$$I_{a/b} = \frac{I(q)}{4} \left[q_a q_b - \frac{4m^2 - q^2}{3} P_{ab} \right] ,$$

$$I_{a/bc} = \frac{I(q)}{8} \left[q_a q_b q_c + \frac{4m^2 - q^2}{3} (q_a P_{bc} - q_b P_{ac} - q_c P_{ab}) \right] ,$$

$$I_{a/bcd}=\frac{I(q)}{16}\left[q_aq_bq_cq_d+\frac{4m^2-q^2}{3}(q_aq_bP_{cd}+q_aq_cP_{bd}+q_aq_dP_{bc}-q_bq_cP_{ad}-q_bq_dP_{ac}-q_cq_dP_{ab})\right.$$
$$\left.-\frac{(4m^2-q^2)^2}{15}(P_{ab}P_{cd}+P_{ac}P_{bd}+P_{ad}P_{bc})\right].$$

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