$\lambda\phi^4$ theory in a time-dependent space-time

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(Received 31 March 1986)

We study the triviality problem of $\lambda\phi^4$ theory in a time-dependent space-time within a variational Gaussian approach. A unique, precarious ($\lambda_B \sim -1/\ln\Lambda$) phase makes sense. As a consequence, λ_R is compelled to lie in the range $0 \geq \lambda_R > \lambda_{R_{\min}}$, where $\lambda_{R_{\min}}$ depends on the curvature.

I. INTRODUCTION

Recently the problem of the existence of $\lambda\phi^4$ theory has been studied, we believe, for the first time in a curved space-time.[1] The reason is the accumulating evidence for the triviality of $\lambda\phi^4$ theory in Minkowski space,[2] only counterbalanced by the possible existence of a precarious phase, that is, a phase which does not lead to a sound regularized theory, but nevertheless yields an interacting renormalized theory with a ground state.[3] This precariousness is due to the fact that the bare coupling constant is negative in this phase. More rigorously, this does not seem to spoil the construction of Euclidean $\lambda\phi^4$ theory, but its continuation to Minkowski space is not understood.[4] Now, the question addressed in Ref. 1 is about the role played by curvature in the triviality and interaction issue of $\lambda\phi^4$ theory. Naively one could argue that the existence and interaction are determined by the continuum (regularized to renormalized) limit and thus are only ultraviolet sensitive. When the distance goes to zero, curvature becomes meaningless. However, the issues studied do not depend only on the leading UV divergences, but also on subleading contributions, and these depend on finite distances. In the language of perturbation theory this implies, e.g., that overlapping divergences are curvature sensitive. In Ref. 1 the space-time chosen was a static Robertson-Walker one. The main reason for this choice was the existence of relatively simple solutions of the Klein-Gordon equation in this space-time, an essential ingredient to the variational approach followed. The penalty for this simplicity was a space-time which does not differ very much from Minkowski space-time. Its high degree of symmetry and its static character make important qualitative changes unlikely. Here a similar study has been performed for the simplest time-dependent space-time. We have not given up symmetry; on the contrary, the group of isometries is maximal, but now there is an explicit time dependence in the metric. Of course, the choice has been again constrained by the requirement of the existence of not-too-difficult solutions to the Klein-Gordon equation. Therefore, the question raised, and answered, within the approximations used, is as follows: How relevant is the static character of the curved space-time for the existence issue of $\lambda\phi^4$ theory?

We quickly recall the conclusions of Ref. 1. For a static Robertson-Walker space-time the only interacting phase with ground state is precarious, exactly as for Minkowski space-time. However, the range of allowed values of the renormalized coupling constant depends on the curvature. For a closed space-time the range shrinks to zero as the curvature becomes large. For an open space-time the range becomes unbounded, allowing arbitrarily large negative values, as the curvature increases. These results are a reflection of asymptotic freedom, as for negative coupling $\lambda\phi^4$ theory is asymptotically free.[5]

We will now study the same issues in a time-dependent space-time and so analyze the role that time dependence plays with respect to interaction, precariousness, and existence of the ground state.

Let us briefly review here the main ideas behind our approach to the study of these problems. Recall that there are two equivalent definitions of the effective potential of a field theory.[6] One corresponds to the lowest expectation value of the Hamiltonian when the states are constrained to lead to a fixed field expectation value, i.e.,

$$\begin{aligned} &V(\phi_0) \equiv \min\langle\psi\,|\,H\,|\,\psi\rangle\ , \\ &\phi_0 = \langle\psi\,|\,\phi\,|\,\psi\rangle\ . \end{aligned} \tag{1.1}$$

This is the definition that will allow us to compute an approximate effective potential density with the help of a variational approach, as is clear from Eq. (1.1). We will make a free-field ansatz which is known as the Gaussian approximation[7,3] and which leads to a nonperturbative upper bound:

$$v_G(\phi_0) \geq v(\phi_0)\ , \tag{1.2}$$

where $v(\phi_0)$ is the density corresponding to $V(\phi_0)$.

As a first step one requires the knowledge of free-field solutions of the scalar fields (and their corresponding vacuum states as trial states) in the variational approach followed.[7,3] This leads in Sec. III to a parameter-dependent bound on the regularized energy density. Its minimization gives $v_G(\phi_0)$.

The other definition of the effective potential is as a generator of all the proper Green's functions at zero external momenta. Based on this definition one can introduce renormalized masses and coupling constants as second-and fourth-order derivatives of the effective potential. This allows one to renormalize and write in an explicitly finite form the effective potential. This is done in Sec. IV and leads to the final expression for the Gaussian effective potential. As we will see, a finite bounded effective potential requires a very specific behavior of the bare parameters of the theory as functions of the ultraviolet cutoff Λ. This analysis is performed also in Sec. IV. The bounds on the renormalized coupling constant follow from it. Section V is devoted to comments and conclusions.

II. FREE FIELDS

There are many coordinate descriptions of a de Sitter space-time.[8] For our purposes, and because in other coordinates the difficulties of the computations seem insurmountable, the most useful coordinates are those which lead to the following line element:

$$ds^2=dt^2-e^{2t/\alpha}\sum_{i=1}^{3}(dx^i)^2 . \tag{2.1}$$

This coordinate description covers only half the de Sitter manifold. The fact that covering half the de Sitter space does not take into account its global topological aspects limits this study somewhat. All of our work therefore refers to half the de Sitter space, which has a topology R^4 instead of $R\times S^3$. In terms of the conformal time

$$\eta=-\alpha e^{-t/\alpha}, \quad -\infty<\eta<0 , \tag{2.2}$$

the line element becomes

$$ds^2=\left[\frac{\alpha}{\eta}\right]^2\left[d\eta^2-\sum_{i=1}^{3}(dx^i)^2\right] \tag{2.3}$$

which is conformal with a time-dependent conformal factor to half the Minkowski space-time. (Allowing η to range over all real numbers covers the other half.) All the novelty therefore lies in the time dependence of the conformal factor.

The Klein-Gordon equation is

$$\left[\frac{\eta^2}{\alpha^2}\frac{\partial^2}{\partial\eta^2}-\frac{2\eta}{\alpha^2}\frac{\partial}{\partial\eta}-\frac{\eta^2}{\alpha^2}\Delta+m^2+\xi R(\eta)\right]\phi(x)=0 , \tag{2.4}$$

where

$$\Delta\equiv\frac{\partial^2}{\partial(x^1)^2}+\frac{\partial^2}{\partial(x^2)^2}+\frac{\partial^3}{\partial(x^3)^2} , \tag{2.5}$$

and the scalar curvature is given by

$$R(\eta)=\frac{12}{\alpha^2} . \tag{2.6}$$

Notice that it is constant. Notice also the last term in (2.4) which is due to the coupling to curvature.

Free fields, solutions of the Klein-Gordon equation, can be expanded in modes as usual:

$$\phi(x)=\int d^3k[u_{\mathbf{k}}(x)a(\mathbf{k})+u_{\mathbf{k}}^*(x)a^\dagger(\mathbf{k})] . \tag{2.7}$$

Separating the trivial spatial dependence from the time dependence one can write

$$u_{\mathbf{k}}(x)=(2\pi)^{-3/2}e^{i\mathbf{k}\cdot\mathbf{x}}\frac{\eta}{\alpha}\tau_k(\eta) \tag{2.8}$$

which reduces the Klein-Gordon equation to

$$\frac{d^2\tau_k}{d\eta^2}+\left[k^2+\frac{\alpha^2}{\eta^2}\left[m^2+(\xi-\tfrac{1}{6})\frac{12}{\alpha^2}\right]\right]\tau_k=0 . \tag{2.9}$$

Of course, the particular combination of solutions to this equation which we use will define the positive-frequency modes and thus the vacuum. A reasonable choice is the adiabatic vacuum[8] which corresponds to the combination which satisfies

$$\tau_k(\eta)\underset{k,\eta\to\infty}{\sim}\frac{1}{\sqrt{2k}}e^{-ik\eta} . \tag{2.10}$$

It is given by

$$\tau_k(\eta)=\tfrac{1}{2}(\pi\eta)^{1/2}e^{-i(\nu+1/2)\pi/2}H_\nu^{(2)}(k\eta) , \tag{2.11}$$

where $H^{(2)}$ is the second Hankel function and

$$\nu^2\equiv\tfrac{9}{4}-\alpha^2m^2-12\xi . \tag{2.12}$$

Notice from Eq. (2.9) that the conformal limit corresponds to $m=0$, $\xi=\frac{1}{6}$, i.e., to $\nu^2=\frac{1}{4}$. No mass scales are left.

III. THE ENERGY DENSITY

The energy-momentum tensor for $\lambda\phi^4$ theory in a curved space-time is given by

$$T_{\mu\nu}=-\tfrac{1}{2}g_{\mu\nu}[\partial^\alpha\phi\partial_\alpha\phi-(m^2+\xi R)\phi^2-2\lambda\phi^4] +\partial_\mu\phi\partial_\nu\phi-\xi(R_{\mu\nu}\phi^2-g_{\mu\nu}\phi^{2;\alpha}{}_{\alpha}+\phi^2{}_{;\mu\nu}) , \tag{3.1}$$

which is obtained from the Lagrangian

$$\mathcal{L}=\tfrac{1}{2}(-g)^{1/2}[\partial^\alpha\phi\partial_\alpha\phi-(m^2+\xi R)\phi^2-2\lambda\phi^4] , \tag{3.2}$$

where $g=\det(g_{\mu\nu})$ and $R_{\mu\nu}$ is the Ricci tensor. For our space-time the energy density reads

$$T_{00}=\frac{1}{2}\left[\frac{\partial\phi}{\partial\eta}\right]^2-(2\xi-\tfrac{1}{2})\sum_{i=1}^{3}\left[\frac{\partial\phi}{\partial x^i}\right]^2 +\frac{\alpha^2}{2\eta^2}\left[m^2+\frac{6\xi}{\alpha^2}\right]+\frac{\lambda\alpha^2}{\eta^2}\phi^4+\frac{2\xi\alpha^2}{\eta^2}\phi\Box\phi -2\xi\phi\frac{\partial^2\phi}{\partial\eta^2}-\frac{2\xi}{\eta}\phi\frac{\partial\phi}{\partial\eta} , \tag{3.3}$$

where the d'Alembertian is given by

$$\Box\equiv\frac{\eta^4}{\alpha^4}\left[\frac{\partial}{\partial\eta}\left[\frac{\alpha^2}{\eta^2}\frac{\partial}{\partial\eta}\right]-\sum_{i=1}^{3}\frac{\partial}{\partial x^i}\left[\frac{\alpha^2}{\eta^2}\frac{\partial}{\partial x^i}\right]\right] . \tag{3.4}$$

Our trial fields will be of the form

$$\phi(x)=\phi_0+\phi_\Omega(x) , \tag{3.5}$$

where ϕ_0 is a constant background field and $\phi_\Omega(x)$ a free quantum field of mass Ω, as given by the equations in Sec. II. The variational parameters will be ϕ_0 and Ω. The ground state corresponding to $\phi_\Omega(x)$ is $|0\rangle_\Omega$. It satisfies

$$a(\mathbf{k})|0\rangle_\Omega=0 . \tag{3.6}$$

The energy density of the true ground state of the theory will be bounded from above according to

$$\epsilon_{\text{true}} \le {}_\Omega\langle 0|T_{00}|0\rangle_\Omega \equiv \epsilon(\phi_0,\Omega,\eta) . \tag{3.7}$$

The computation of the right-hand side of Eq. (3.7) is not difficult. It gives

$$\epsilon(\phi_0,\Omega,\eta)\equiv\left[\frac{\alpha}{\eta}\right]^2\epsilon(\phi_0,\eta) \tag{3.8}$$

with

$$\begin{aligned}\epsilon(\phi_0,\Omega)=&I_1(\Omega^2)+\tfrac{1}{2}(m^2-\Omega^2)I_0(\Omega^2)+\tfrac{1}{2}m^2\phi_0^2\\&+\lambda\phi_0^4+6\lambda\phi_0^2I_0(\Omega^2)+3\lambda I_0^2(\Omega^2) ,\end{aligned} \tag{3.9}$$

where Ω^2 and m^2 have been redefined according to

$$\begin{aligned}\Omega^2&\equiv\Omega^2+12\frac{\xi}{\alpha^2} ,\\ m^2&\equiv m^2+6\frac{\xi}{\alpha^2} ,\end{aligned} \tag{3.10}$$

and $I_1(\Omega^2)$ and $I_0(\Omega^2)$ are given by

$$\begin{aligned}I_0(\Omega^2)&\equiv\frac{1}{8\pi\alpha^2}\int_0^\infty dx\,x^2\,|H_\nu^{(2)}(x)|^2e^{\pi\,\text{Im}\nu} ,\\ I_1(\Omega^2)&\equiv\frac{1}{8\pi\alpha^4}\int_0^\infty dx\,x^2(x^2+\alpha^2\Omega^2)\,|H_\nu^{(2)}(x)|^2e^{\pi\,\text{Im}\nu} ,\end{aligned} \tag{3.11}$$

and $\nu^2=\frac{9}{4}-\alpha^2\Omega^2$. Notice that the whole time dependence factorizes according to Eq. (3.8), which is very likely due to the high degree of symmetry of the de Sitter space. This makes the space-time analyzed not so different from the previously studied static Robertson-Walker space-time. In fact Eq. (3.9) is the same for both, only the integrals I_1 and I_0 differ: they are much more complicated in this case. All relevant properties of these integrals are given in the Appendix.

Let us recall that the point $\nu^2=\frac{1}{4}$ corresponds to the conformal limit of our theory. This ν value coincides with the appearance of IR singularities in I_1 and I_0. Moreover, beyond $|\text{Re}\nu|=\frac{1}{2}$ the Gaussian approach is meaningless. This situation usually takes place at $\Omega^2=0$, that is to say when no intrinsic mass scale enters into the approximation. In the present case, nonflat space-time changes this value to $\nu^2=\frac{1}{4}$ $(\alpha^2\Omega^2=2)$ since curvature plays also the role of a mass. As a consequence the range of variation of $\alpha^2\Omega^2$ is from 2 to ∞.

IV. RENORMALIZATION AND ANALYSIS

A. Renormalization

The expression (3.9) for the energy density is plagued with infinities which should be absorbed by renormalization of the mass and the coupling constant and by the subtraction of the zero-point energy. In case this renormalization procedure allows different valid behaviors for m_B and λ_B we will talk of "phases" of the theory.

Following Barnes and Ghandour[7] we take the renormalization scheme defined by

$$\begin{aligned}m_R{}^2&=\left.\frac{\partial^2\epsilon(\phi_0,\Omega(\phi_0))}{\partial\phi_0{}^2}\right|_{\phi_0=\bar\phi_0} ,\\ \lambda_R&=\frac{1}{24}\left.\frac{\partial^4\epsilon(\phi_0,\Omega(\phi_0))}{\partial\phi_0{}^4}\right|_{\phi_0=\bar\phi_0} ,\end{aligned} \tag{4.1}$$

where $\bar\phi_0$ stands for the subtraction point. The simplest obvious choice is $\bar\phi_0=0$. The physics does not depend on this particular choice, thanks to renormalization-group invariance.

The explicit form of (4.1) is obtained from (3.9):

$$m_R{}^2=m_B{}^2+12\lambda_BI_0(\Omega_0{}^2)+(\Omega_0{}^2)'\left[\left.\frac{\partial\epsilon}{d\Omega^2}\right|_{\phi_0=0}\right] , \tag{4.2}$$

$$\begin{aligned}\lambda_R=&\lambda_B+(\Omega_0{}^2)'3\lambda_BI_0'(\Omega_0{}^2)+\frac{(\Omega_0{}^2)'^2}{24}\{3I_1''(\Omega_0{}^2)+\tfrac{3}{2}[m_B{}^2-\Omega_0{}^2+12\lambda_BI_0(\Omega_0{}^2)]I_0''(\Omega_0{}^2)-3I_0'(\Omega_0{}^2)+18\lambda_BI_0'^2(\Omega_0{}^2)\}\\&+\frac{(\Omega_0{}^2)''}{24}\left[\left.\frac{\partial\epsilon}{\partial\Omega^2}\right|_{\phi_0=0}\right] ,\end{aligned} \tag{4.3}$$

where

$$\Omega_0{}^2\equiv\Omega^2(\phi_0\equiv0),\quad (\Omega_0{}^2)'\equiv\left.\frac{\partial^2\Omega^2}{\partial\phi_0{}^2}\right|_{\phi_0=0},\quad (\Omega_0{}^2)''\equiv\left.\frac{\partial^4\Omega^2}{\partial\phi_0{}^4}\right|_{\phi_0=0},\quad I_n'(\Omega_0{}^2)=\left.\frac{\partial I_n(\Omega^2)}{\partial\Omega^2}\right|_{\Omega^2=\Omega_0{}^2} . \tag{4.4}$$

These equations are to be applied to the subtracted energy density

$$\begin{aligned}v_G(\phi_0)\equiv&\,\epsilon(\phi_0,\Omega(\phi_0))-\epsilon(0,\Omega_0)\\=&\,I_1(\Omega^2)-I_1(\Omega_0{}^2)+\tfrac{1}{2}[m_B{}^2-\Omega^2+12\lambda_B\phi_0{}^2+6\lambda_BI_0(\Omega^2)]I_0(\Omega^2)\\&+\tfrac{1}{2}m_B{}^2\phi_0{}^2+\lambda_B\phi_0{}^4-\tfrac{1}{2}[m_B{}^2-\Omega_0{}^2+6\lambda_BI_0(\Omega_0{}^2)]I_0(\Omega_0{}^2) .\end{aligned} \tag{4.5}$$

Notice that quartic UV divergences have been removed by subtraction in (4.5) as is usual in ordinary $\lambda\phi^4$. Quadratic and logarithmic ones ought to be absorbed by $m_B{}^2$ and λ_B, respectively. Nevertheless this is a nonobvious statement. In order to find out every acceptable phase we have to perform a careful analysis of Eqs. (4.2), (4.3), and (4.5).

B. Analysis

We are looking for an effective potential in terms of $\phi_0{}^2$. Thus, Ω should be considered as a function of $\phi_0{}^2$. In order to obtain an optimal upper bound for the ground-state energy we introduce the implicit definition

$$\frac{\partial\epsilon(\phi_0,\Omega)}{\partial\Omega}=0 . \tag{4.6}$$

In particular, when $\phi_0=0$ we obtain

$$I_1'(\Omega_0{}^2)-\tfrac{1}{2}I_0(\Omega_0{}^2)+\tfrac{1}{2}(m_B{}^2-\Omega_0{}^2)I_0'(\Omega_0{}^2)+6\lambda_B I_0(\Omega_0{}^2)I_0'(\Omega_0{}^2)=0 . \tag{4.7}$$

This equation can have a solution, $\Omega_0{}^2=\bar{\Omega}_0{}^2$, or not. However, we must also study the end points $\alpha^2\Omega_0{}^2=2$ and $\alpha^2\Omega_0{}^2=\infty$ which could provide the operative minimum of $\epsilon(0,\Omega_0)$. In this later case, since $\partial\epsilon(\phi_0,\Omega)/\partial\Omega\,|_{\phi_0=0}\neq0$, the derivatives satisfy $(\Omega_0{}^2)'=(\Omega_0{}^2)''=0$ just by continuity. We conclude that, whatever the situation is, the last terms in (4.2) and (4.3) vanish, and thus

$$m_R{}^2=m_B{}^2+12\lambda_B I_0(\Omega_0{}^2) , \tag{4.8}$$

$$\lambda_R=\lambda_B+(\Omega_0{}^2)'3\lambda_B I_0'(\Omega_0{}^2)+\frac{(\Omega_0{}^2)'^2}{24}\{3I_1''(\Omega_0{}^2)+\tfrac{3}{2}[m_B{}^2-\Omega_0{}^2+12\lambda_B I_0(\Omega_0{}^2)]I_0''(\Omega_0{}^2)-3I_0'(\Omega_0{}^2)+18\lambda_B I_0'^2(\Omega_0{}^2)\} . \tag{4.9}$$

Next we analyze any possible value of $\alpha^2\bar{\Omega}_0{}^2$.

Case 1. $\alpha^2\bar{\Omega}_0{}^2>2$. This case corresponds to the solution of $\partial\epsilon(0,\Omega_0)/\partial\Omega_0{}^2\,|_{\Omega_0{}^2=\bar{\Omega}_0{}^2}=0$. To leading order in $\ln\Lambda^2$ this restriction provides a unique solution

$$\alpha^2\bar{\Omega}_0{}^2=\alpha^2 m_R{}^2+1 \tag{4.10}$$

which must satisfy

$$\left.\frac{\partial^2\epsilon(0,\Omega_0)}{\partial(\Omega_0{}^2)^2}\right|_{\Omega_0{}^2=\bar{\Omega}_0{}^2}>0 \tag{4.11}$$

in order to be a minimum. Therefore neither $\alpha^2\Omega_0{}^2=2$ nor $\Omega_0=\infty$ are operative.

The general restriction (4.6) allows by continuity a simple calculation of $(\Omega_0{}^2)'$:

$$(\Omega_0{}^2)'=-\frac{12\lambda_B I_0'(\Omega_0{}^2)}{I_1''(\Omega_0{}^2)-I_0'(\Omega_0{}^2)+\frac{1}{2}(m_R{}^2-\Omega_0{}^2)I_0''(\Omega_0{}^2)+6\lambda_B I_0'^2(\Omega_0{}^2)} . \tag{4.12}$$

Substituting into (4.8) we obtain the compact expression

$$\lambda_R=\lambda_B\frac{[I_1''(\Omega_0{}^2)-I_0'(\Omega_0{}^2)]I_0'(\Omega_0{}^2)+[\frac{1}{2}I_0(\Omega_0{}^2)-I_1'(\Omega_0{}^2)]I_0''(\Omega_0{}^2)-12\lambda_B I_0'^3(\Omega_0{}^2)}{[I_1''(\Omega_0{}^2)-I_0'(\Omega_0{}^2)]I_0'(\Omega_0{}^2)+[\frac{1}{2}I_0(\Omega_0{}^2)-I_1'(\Omega_0{}^2)]I_0''(\Omega_0{}^2)+6\lambda_B I_0'^3(\Omega_0{}^2)} . \tag{4.13}$$

Notice that only $\ln\Lambda^2$ divergences are still involved. Nor does the mass play any role. Equation (4.13) contains the information about the possible phases of the theory in the sense that only λ_B behavior which renders λ_R finite are acceptable. The inversion of (4.13) provides two very interesting cases since renormalizability is then guaranteed.

(i) The negative infinitesimal phase. One of the two solutions of (4.13) is

$$\lambda_B=-\frac{1}{6}\frac{8\pi^2}{\ln\Lambda^2}\left[1+\frac{1}{\ln\Lambda^2}\left[\frac{4\pi^2}{\lambda_R}-\frac{2}{\alpha^4}[S_0''(\Omega_0)-2S_1''(\Omega_0)]\right]+O\left[\frac{1}{(\ln\Lambda^2)^2}\right]\right] . \tag{4.14}$$

Restriction (4.11) implies $\lambda_R<0$.

When $\phi_0{}^2>0$ we consider two possibilities: one possibility is $\alpha^2\Omega^2=2$, therefore,

$$v_G(\phi_0)=\frac{1}{16\pi^2\alpha^4}\left[8\pi^2\alpha^2\phi_0{}^2+\frac{S_0''(\bar{\Omega}_0)}{4\alpha^4}(2-\alpha^2\bar{\Omega}_0{}^2)^3-\frac{\pi^2}{\lambda_R}(2-\alpha^2\bar{\Omega}_0{}^2)^2\right] . \tag{4.15}$$

The other is $\Omega=\bar{\Omega}$, $\bar{\Omega}$ being given by

$$\left.\frac{\partial\epsilon(\phi_0,\Omega)}{\partial\Omega^2}\right|_{\Omega^2=\bar{\Omega}^2}=0 .$$

Taking into account the λ_B behavior this restriction reads

$$\left[\frac{\alpha^2}{2}[S_0(\bar\Omega)-S_0(\bar\Omega_0)]+S_0'(\bar\Omega)\left[\frac{\alpha^2\bar\Omega^2}{2}-1\right]+S_0'(\bar\Omega_0)\left[1+\frac{\alpha^2\bar\Omega_0^2}{2}-\alpha^2\bar\Omega^2\right]+8\pi^2\alpha^4\phi_0^2-\frac{\alpha^4}{2}(\bar\Omega^2-\bar\Omega_0^2)\frac{4\pi^2}{\lambda_R}+S_0''(\bar\Omega_0)\left[1-\frac{\alpha^2\bar\Omega_0^2}{2}\right](\bar\Omega^2-\bar\Omega_0^2)\right]=0 . \tag{4.16}$$

In this case the subtracted energy density turns out to be

$$v_G(\phi_0)=\frac{1}{16\pi^2\alpha^4}\left[8\pi^2\alpha^2\phi_0^2(\alpha^2\bar\Omega^2-1)+[S_0(\bar\Omega)-S_0(\bar\Omega_0)]\left[\frac{\alpha^2\bar\Omega^2}{2}-1\right]+S_0'(\bar\Omega_0)(\bar\Omega^2-\bar\Omega_0^2)\left[1-\frac{\alpha^2\bar\Omega^2}{2}\right]+S_0''(\bar\Omega_0)\frac{(\bar\Omega^2-\bar\Omega_0^2)}{2}\left[1-\frac{\alpha^2\bar\Omega_0^2}{2}\right]-\frac{\alpha^4\pi^2}{\lambda_R}(\bar\Omega^2-\bar\Omega_0^2)^2\right] . \tag{4.17}$$

The limit $\alpha^2\Omega^2\rightarrow\infty$ leads to $\epsilon\rightarrow+\infty$, so it is not operative.

We conclude that this infinitesimal phase makes sense. When plotting (4.15) and (4.17) we observe that $\bar\Omega=\bar\Omega_0$, $\phi_0=0$ provide the absolute minimum of v. The shape of the energy density is smooth until a certain ϕ_0 value where (4.15) becomes the absolute minimum. (See Fig. 1.) Then a parabolic effective potential sets in. When the height of this later curve at the origin is just zero this phase no longer makes sense. In such a case

$$\frac{S_0''(\bar\Omega_0)}{4\alpha^2}(2-\alpha^2\bar\Omega_0^2)-\frac{\pi^2}{\lambda_R}=0 \tag{4.18}$$

which corresponds to a relation between λ_R and m_R plotted in Fig. 2.

(ii) The constant phase. The second branch of (4.13) leads to

$$\lambda_B=-\frac{\lambda_R}{2} . \tag{4.19}$$

In order to have a minimum we are forced to consider (4.11):

$$\lambda_B>0\Longrightarrow\lambda_R<0 . \tag{4.20}$$

We consider $\lambda_B=\text{const}>0$. Then the energy density contains logarithmic divergences:

$$\epsilon(\phi_0,\Omega)=\frac{1}{8\pi^2\alpha^4}\frac{\ln\Lambda^2}{8}[\alpha^4(\Omega^4-\bar\Omega_0^4)-2\alpha^2(\Omega^2-\bar\Omega_0^2)]+\tfrac{1}{2}m_R^2\phi_0^2+\lambda_B\phi_0^4+\tfrac{1}{2}(m_R^2+12\lambda_B\phi_0^2)\frac{1}{8\pi^2\alpha^2}\left[-\frac{\alpha^2}{2}(\ln\Lambda^2)(\Omega^2-\bar\Omega_0^2)\right]+\frac{3\lambda_B}{(8\pi^2\alpha^2)^2}(\ln\Lambda^2)^2\frac{\alpha^4}{4}(\Omega^2-\bar\Omega_0^2)^2+\epsilon(0,\bar\Omega_0) . \tag{4.21}$$

We only need to consider leading behavior in what follows. The minimization procedure to obtain $\bar\Omega$ leads to

$$\bar\Omega^2=\bar\Omega_0^2=\frac{16\pi^2\phi_0^2}{\ln\Lambda^2}+O\left[\frac{1}{(\ln\Lambda^2)^2}\right] \tag{4.22}$$

and therefore

$$v_G(\phi_0)=-2\lambda_B\phi_0^4 . \tag{4.23}$$

The theory is unbounded from below.

(iii) The other phases. Any other behavior of λ_B leads to either a trivial theory or an unbounded one. Let us, for instance, present the case

$$\lambda_B\sim-\frac{4\pi^2}{3}\frac{1}{\ln\Lambda^2}a,\quad a<1 . \tag{4.24}$$

Restriction (4.11) is satisfied but

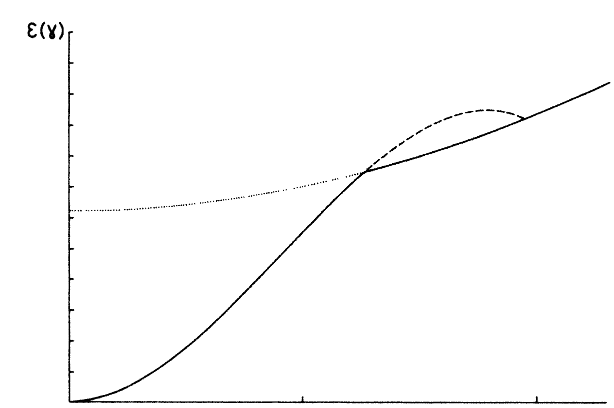

FIG. 1. Effective potential v in terms of $\gamma\equiv\phi_0^2$. The dotted line corresponds to $\alpha^2\Omega^2=2$ whereas the dashed line corresponds to the solution to (4.16). The solid curve jumps from one case to the other in order to keep the absolute minimum for $\epsilon(\gamma)$. The curves shown correspond to $\lambda_R=-1$ and $\alpha^2\bar\Omega_0^2=5$.

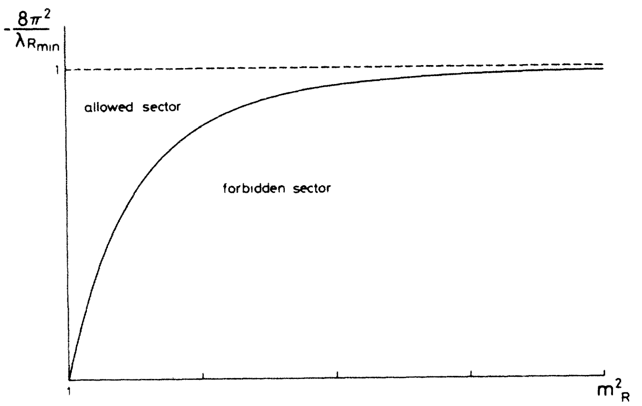

FIG. 2. Relation between $\lambda_{R\,\min}$ and $m_R{}^2$ provided by (4.18). Acceptable λ_R's lie in the upper sector (mass in units of α).

$$\lambda_R = -\frac{4\pi^2}{3\ln\Lambda^2} a \frac{-\frac{1}{2}-a}{-1-a} , \tag{4.25}$$

so triviality is found at best.

Case 2. $\alpha^2\Omega_0{}^2=2$, $\partial\epsilon/\partial\Omega_0{}^2\,|_{\alpha^2\Omega_0{}^2=2}>0$. This situation appears when

$$\alpha^2 m_R{}^2 < 1 . \tag{4.26}$$

By continuity we find

$$(\Omega_0{}^2)'=(\Omega_0{}^2)''=0 \Longrightarrow \lambda_R=\lambda_B . \tag{4.27}$$

There is no coupling constant renormalization. In a neighborhood of $\phi_0=0$, the effective potential remains ($\alpha^2\Omega^2=2$)

$$v_G(\phi_0)=\tfrac{1}{2}m_R{}^2\phi_0{}^2+\lambda_B\phi_0{}^4 \tag{4.28}$$

which forces

$$\lambda_B>0 \Longrightarrow \lambda_R>0 . \tag{4.29}$$

In the case $\lambda_B \sim 1/\ln\Lambda^2$ we get triviality but if $\lambda_B \sim$const there is a certain ϕ_0 value beyond which

$$\overline{\Omega}{}^2-\Omega_0{}^2=\frac{16\pi^2\phi_0{}^2}{\ln\Lambda^2}+O\left[\frac{1}{(\ln\Lambda^2)^2}\right] . \tag{4.30}$$

This situation is similar to that of Eqs. (4.22) and (4.23) and leads to an unbounded theory.

Case 3. $\alpha^2\overline{\Omega}_0{}^2=2$, $\partial\epsilon(0,\Omega_0)/\partial\Omega_0{}^2\,|_{\Omega_0{}^2=\overline{\Omega}_0{}^2}=0$. This case can be considered as a limit of cases 1 and 2. No new physics can appear.

V. CONCLUSIONS

Our analysis of $\lambda\phi^4$ in a simple time-dependent background space-time shows that only one phase makes sense. λ_B must vanish as the cutoff tends to infinity like $\sim -1/\ln\Lambda^2$. This corresponds to a precarious phase. This concept was introduced by Stevenson[3] and states that any regularized version of the theory remains unbounded from below. Nevertheless there is a metastable spectrum which becomes stable as $\Lambda\to\infty$.

However, this precarious phase only exists for λ_R's in a range determined by m_R (Fig. 2):

$$0 \ge \lambda_R \ge \frac{4\alpha^4\pi^2}{S_0''(\overline{\Omega}_0)(2-\alpha^2\overline{\Omega}_0{}^2)} \equiv \lambda_{R\,\min}$$

(let us recall that $\overline{\Omega}_0{}^2=m_R{}^2+1/\alpha^2$). Two special cases are

$$m_R{}^2\to 1 \text{ (conformal limit) } \Longrightarrow 0\ge\lambda_R\ge-\infty ,$$

$$m_R{}^2\to\infty \Longrightarrow 0\ge\lambda_R\ge-8\pi^2 .$$

This agrees with intuition since $m_R{}^2 \gg 1$ is equivalent to neglecting curvature and $-8\pi^2$ corresponds to the lowest acceptable λ_R in flat $\lambda\phi^4$. On the contrary $m_R{}^2\to 1$ corresponds to large curvature which, from the study of Ref. 1, is expected to lead for an open space-time to an unbounded λ_R range. This is indeed what we find.

The inclusion of a time dependence has minor but subtle consequences. It seems to us that triviality is deeply connected with the degree of symmetry of the space-time. Since our metric has ten Killing vectors no drastic qualitative changes appear. Nevertheless the allowed $\lambda_{R\,\min}$ range goes from $-\infty$ to $-8\pi^2$ just as it happens in a static open space-time. We have found that somehow a flat space-time, scaled with a factor which increases as the time flows, behaves as a static RW universe with positive three-curvature. It is conceivable that for the complete de Sitter space-time two opposite tendencies compete: on the one hand, at any fixed time a closed universe forces $\lambda_{R\,\min}\to 0$ as its curvature becomes large, on the other hand, time evolution enlarges the range of $\lambda_{R\,\min}$. All the same, the only alternative to triviality is the precarious phase, as usual for $\lambda\phi^4$.

ACKNOWLEDGMENT

This work has been financially supported by Comisión Asesora de Investigación Científica y Técnica, Spain (Plan Movilizador de Altas Energias).

APPENDIX

In this appendix we will use standard notations borrowed from Ref. 9. Let us consider the integrals appearing in (3.11):

$$I_0(\Omega^2)\equiv\frac{1}{8\pi\alpha^2}\int_0^\infty dx\, x^2\,|H_\nu^{(2)}(x)|^2 e^{\pi\,\mathrm{Im}\nu} ,$$
$$I_1(\Omega^2)\equiv\frac{1}{8\pi\alpha^4}\int_0^\infty dx\, x^2(x^2+\alpha^2\Omega^2)\,|H_\nu^{(2)}(x)|^2 e^{\pi\,\mathrm{Im}\nu} , \tag{A1}$$

where $\nu^2=\frac{9}{4}-\alpha^2\Omega^2$. The range of variation of ν is only limited by the requirement of normalizability. Consequently we must consider real as well as pure imaginary values for ν. In addition, UV divergences appear in (A1) and therefore we need to introduce a regularization procedure. The usual symmetric cutoff $|x|<\Lambda$ leads to a rather involved output. As a matter of convenience we will work with the integral

$$I(\Omega^2,\Lambda)\equiv\int_0^\infty dx\, H_\nu^{(2)}(x)[H_\nu^{(2)}(x)]^*\cos\frac{x}{\Lambda} \quad \text{(A2)}$$

which provides the regularized expressions $I_0(\Omega^2,\Lambda)$ and $I_1(\Omega^2,\Lambda)$ by simple derivations with respect to Λ. A further trick consists in writing[10]

$$I(\Omega^2,\Lambda)=\lim_{\epsilon\to 0+}\int_0^\infty dx\, H_\nu^{(2)}(x(1+i\epsilon))\times[H_\nu^{(2)}(x(1-i\epsilon))]^*\cos\frac{x}{\Lambda}\,. \quad \text{(A3)}$$

Hankel functions satisfy

$$K_\nu(z)=\frac{\pi i}{2}e^{i\nu\pi/2}H_\nu^{(1)}(ze^{i\pi/2}),\quad -\pi<\arg z\le\pi/2\,,$$
$$K_\nu(z)=-\frac{\pi i}{2}e^{-i\nu\pi/2}H_\nu^{(2)}(ze^{-i\pi/2}),\quad -\pi/2<\arg z\le\pi\,, \quad \text{(A4)}$$

where K is the modified Bessel function. Therefore, for real ν, (A3) is transformed into

$$I(\Omega^2,\Lambda)=\frac{4}{\pi^2}\lim_{\epsilon\to 0+}\int_0^\infty dx\, K_\nu(x(\epsilon+i))\times K_\nu(x(\epsilon-i))\cos\frac{x}{\Lambda}\,. \quad \text{(A5)}$$

This integral is a special case of [11]

$$\int_0^\infty dx\, K_\nu(\alpha x)K_\nu(\beta x)\cos\frac{x}{\Lambda}=\frac{\pi^2}{4}(\alpha\beta)^{-1/2}\sec(\pi\nu)P_{\nu-1/2}\left[\frac{\alpha^2+\beta^2+1/\Lambda^2}{2\alpha\beta}\right], \quad \text{(A6)}$$

$$|\mathrm{Re}\nu|<\tfrac{1}{2},\quad \mathrm{Re}(\alpha+\beta)>0,\quad \Lambda>0\,,$$

where P corresponds to Legendre functions. Furthermore,

$$P_\mu(z)=F\left[-\mu,\mu+1;1;\frac{1-z}{2}\right],\quad |1-z|<2 \quad \text{(A7)}$$

so that

$$I(\Omega^2,\Lambda)=\sec(\pi\nu)F\left[\tfrac{1}{2}-\nu,\tfrac{1}{2}+\nu;1;1-\frac{1}{4\Lambda^2}\right]. \quad \text{(A8)}$$

An expansion of hypergeometric functions is provided by[9]

$$F(a,b;a+b,z)=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(n!)^2}[2\psi(n+1)-\psi(a+n)-\psi(b+n)-\ln(1-z)](1-z)^n\,,$$
$$|\arg(1-z)|<\pi,\quad |1-z|<1\,, \quad \text{(A9)}$$

where ψ is the ψ function and $(a)_n=a(a+1)\cdots(a+n-1)$. Our final expression for $I(\Omega^2,\Lambda)$ is

$$I(\Omega^2,\Lambda)=\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{(\frac{1}{2}-\nu)_n(\frac{1}{2}+\nu)_n}{(n!)^2}[2\psi(n+1)-\psi(n+\tfrac{1}{2}-\nu)-\psi(n+\tfrac{1}{2}+\nu)+\ln 4\Lambda^2]\frac{1}{(4\Lambda^2)^n},\quad |\mathrm{Re}\nu|<\tfrac{1}{2}\,. \quad \text{(A10)}$$

Exactly the same result is obtained when $\nu^2<0$.

Our basic integrals are easily derived from (A10).

$$I_0(\Omega^2,\Lambda)=\frac{1}{8\pi\alpha^2}[-2\Lambda^2-\tfrac{1}{2}(\alpha^2\Omega^2-2)\ln\Lambda^2+S_0(\Omega)]\,,$$
$$I_1(\Omega^2,\Lambda)=\frac{1}{8\pi\alpha^4}[12\Lambda^4+\Lambda^2(-2-\alpha^2\Omega^2)-\tfrac{1}{8}\alpha^2\Omega^2(-2+\alpha^2\Omega^2)\ln\Lambda^2+S_1(\Omega)]\,, \quad \text{(A11)}$$

where $S_0(\Omega)$ and $S_1(\Omega)$ are the finite parts

$$S_0(\Omega)=\tfrac{1}{2}(\tfrac{1}{4}-\nu^2)[3-2\psi(2)+\psi(\tfrac{3}{2}-\nu)+\psi(\tfrac{3}{2}+\nu)-\ln 4]\,,$$
$$S_1(\Omega)=\tfrac{1}{16}(\tfrac{1}{4}-\nu^2)(\tfrac{9}{4}-\nu^2)[-25+12\psi(3)-6\psi(\tfrac{5}{2}-\nu)-6\psi(\tfrac{5}{2}+\nu)+6\ln 4]$$
$$+\tfrac{1}{2}(\tfrac{1}{4}-\nu^2)\alpha^2\Omega^2[3-2\psi(2)+\psi(\tfrac{3}{2}-\nu)+\psi(\tfrac{3}{2}+\nu)-\ln 4]\,. \quad \text{(A12)}$$

These functions are related by

$$S_1(\Omega)=\tfrac{1}{4}\alpha^2\Omega^2S_0(\Omega)+\tfrac{9}{4}-\alpha^2\Omega^2-\tfrac{1}{16}\alpha^4\Omega^4\,. \quad \text{(A13)}$$

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