SL(3, C) elementary instanton configurations

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Several SL(3,C) self-dual instanton solutions of the Yang-Mills equations are presented which have very striking properties. While one of them is regular and SU(3) inside (outside) a sphere of arbitrarily large (small) radius, another one has all the characteristics of a meron solution (in particular its topological charge is concentrated at one point) although its Pontryagin number equals one. Their continuation to Minkowski space is also studied.

The recent work of Atiyah *et al.*[1] on the construction of all instantons for several gauge groups has been extended by Corrigan *et al.*[2] and by Christ *et al.*[3] Corrigan and collaborators have been able to construct the general solution in the SU(2) and Sp(n) cases, although they did not concern themselves with the SU(n), $n \geq 3$ case. The last-named authors have taken advantage of the hint provided in Ref. 7 and reduced the problem of the construction of O(n), Sp(n), and SU(n) solutions to that of solving a finite-dimensional, nonlinear system of matrix equations. Using more direct methods, Bais and Weldon[4] have obtained a family of SU(3) multi-instantons with cylindrical symmetry, although they recognize that their solutions are much like the ones already found for SU(2), and that it is thus not probable that they can provide deeper insights into the problem of quark confinement. Finally, Meyers *et al.*[5] have dealt with the SU(3) case as well, which presents an obviously great physical interest, and have given explicit criteria to decide whether a particular self-dual SU(3) solution is merely an embedding of an SU(2) one or not.

Instead of focusing on the multi-instanton configurations, whose contribution to the action is essentially proportional to that of a single pseudoparticle,[6] it is physically more interesting to look for new elementary instanton (and meron) solutions. As is known, all the configurations with unit topological charge in the SU(3) case are embeddings of the corresponding ones for SU(2). However, I prove here that this is not the case for the group SL(3, C). In fact, starting with an appropriate ansatz, I derive several nontrivial,[5] self-dual solutions of the Yang-Mills (YM) equations for this group. One of them is SU(3) inside a sphere of arbitrarily large radius and SL(3, C) outside. Through a conformal transformation it gives rise to an alternative solution which is SU(3) outside an arbitrarily small sphere. At the same time, a regular, self-dual SL(3, C) configuration of unit topological charge is also obtained. Finally, a further solution with an allowed singularity (cf. de Alfaro *et al.*[7] and Jackiw *et al.*[8]) is constructed which has all the characteristics of an instanton, but, surprisingly, its topological charge is concentrated at one point as in the case of the meron configuration.[7] Its analytic continuation to Minkowski space has finite energy and action. All the solutions can be combined with the configurations of 't Hooft[9] and Jackiw *et al.*[8] to provide other nontrivial SL(3, C) solutions with higher topological charges.

As in the SU(2) case, when one deals with the gauge group SU(3), it is convenient to define a matrix-valued vector field $A_\mu(x)$ related to the gauge potentials $A_\mu^a(x)$ by $A_\mu = A_\mu^a T^a$, where $T^a = -i\lambda^a/2$ and the λ's are the Gell-Mann matrices. The field-strength tensor is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ and its covariant derivative is $D_\lambda F_{\mu\nu} = \partial_\lambda F_{\mu\nu} + [A_\lambda, F_{\mu\nu}]$. The pure gauge field theory is defined by the Euclidean action

$$S = -\tfrac{1}{8} \int \mathrm{Tr}(F_{\mu\nu} F_{\mu\nu}) d^4x \,; \qquad (1)$$

which leads to the YM equations of motion

$$D_\mu F_{\mu\nu} = 0 \,. \qquad (2)$$

When the field under consideration is (anti) self-dual

$$\tilde{F}_{\mu\nu} = \pm F_{\mu\nu}, \quad \tilde{F}_{\mu\nu} \equiv \tfrac{1}{2} \epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma} \,, \qquad (3)$$

Eqs. (2) are an immediate consequence of the Bianchi identities for the connection $D_\mu \tilde{F}_{\mu\nu} = 0$. A very interesting question is that of the exhaustivity of these solutions (3). A local statement about it has been given recently by Daniel *et al.*,[10] but the general problem of finding all the solutions (self-dual or not) of the equations of motion (2) still remains open.

I will consider self-dual fields only. My ansatz for the gauge vector potential $A_\mu(x)$ is

$$A_\mu(x)=\sigma_{\mu\alpha}a_\alpha(x)\quad(\mu=0,\ldots,3;\ \alpha=0,\ldots,8),\tag{4}$$

where the matrices $\sigma_{\mu\alpha}$ are antisymmetric and defined by

$$\sigma_{0j}=-i\lambda_j/2,\quad \sigma_{0u}=-i\lambda_u/4,\quad \sigma_{ja}=[\lambda_j,\lambda_a]/4$$
$$(j=1,2,3;\ u=4,\ldots,8;\ a=1,\ldots,8).\tag{5}$$

The self-duality condition $\tilde F_{\mu\nu}=F_{\mu\nu}$ provides the following equations for the potentials $a_\alpha(x)$:

$$\begin{aligned}
&\partial_\mu a_\mu+a_\mu a_\mu+\tfrac14(a_4{}^2+a_5{}^2+a_6{}^2+a_7{}^2)=0,\\
&f_{\mu\nu}\equiv\partial_\mu a_\nu-\partial_\nu a_\mu=\tilde f_{\mu\nu}\quad(\mu,\nu=0,1,2,3),\\
&\partial_1 a_8=\partial_2 a_8=\partial_3 a_8=0,\\
&D_0a_4+D_1a_7+D_2a_6+D_3a_5-\frac{\sqrt3}{2}a_5a_8=0,\\
&D_0a_5-D_1a_6+D_2a_7-D_3a_4+\frac{\sqrt3}{2}a_4a_8=0,\\
&D_0a_6+D_1a_5-D_2a_4-D_3a_7-\frac{\sqrt3}{2}a_7a_8=0,\\
&D_0a_7-D_1a_4-D_2a_5+D_3a_6+\frac{\sqrt3}{2}a_6a_8=0,
\end{aligned}\tag{6}$$

where $D_\mu=2\partial_\mu+a_\mu$. To find the general solution of this system of partial differential equations is not easy. But let us consider the restriction

$$(2\partial_\mu+a_\mu)a_u=0\quad(u=4,\ldots,7),\ a_8=0,\tag{7}$$

which, together with the assumption that the a_μ's can be written in terms of a scalar superpotential $\rho(x)$, $a_\mu(x)=\partial_\mu\rho(x)/\rho(x)$, reduces the set of Eqs. (6) to

$$a_u=\alpha_u\rho^{-1/2},\quad \Box\rho+\tfrac14(\alpha_4{}^2+\alpha_5{}^2+\alpha_6{}^2+\alpha_7{}^2)=0\tag{8}$$

(the α_u are arbitrary constants) which have the solution

$$\rho(x)=\rho_0(x)-\frac{\alpha^2}{32}[(x-a)^2+b],\quad \alpha^2\equiv\sum_{u=4}^{7}\alpha_u{}^2,\tag{9}$$

where a and b are constants and $\rho_0(x)$ is the general solution of the homogeneous equation

$$\Box\rho=0.\tag{10}$$

Equation (10) already appeared in the SU(2) case, where it led to an explicit $(5n+4)$-parameter solution[8] in the n-instanton case

$$\rho_0(x)=\sum_{l=1}^{n+1}\frac{\beta_l{}^2}{(x-y_l)^2}.\tag{11}$$

Notice that corresponding to any of these solutions (11) there is one (9) for the SU(3) gauge group. Moreover, this correspondence is not a trivial embedding, as we shall see in a moment.

Let us concentrate in the non-SU(2) part of (9),

$$\rho(x)=-\frac{\alpha^2}{32}(x^2+b),\tag{12}$$

where, without loss of generality, I have set $a=0$. We have

$$\begin{aligned}
&a_\mu=\frac{2x_\mu}{x^2+b},\\
&a_u=\pm\frac{4\sqrt2\,\alpha_u}{\alpha}[-(x^2+b)]^{-1/2}\quad(u=4,\ldots,7),\\
&a_8=0.
\end{aligned}\tag{13}$$

When b is positive, say $b=a^2$, we obtain an everywhere regular solution, but one (at least) of the a_u must be pure imaginary. When $b=0$ an (acceptable[8]) singularity at $x=0$ appears. Finally, if $b=-a^2$ the solution is real in the hypersphere $x^2<a^2$ and has a singularity at its surface. The gauge potentials are given by

$$A_\mu(x)=\frac{2\sigma_{\mu\nu}x_\nu}{x^2+b}\mp\frac{4\sqrt2\,i}{\alpha(x^2+b)^{1/2}}(\alpha_4\sigma_{\mu4}+\alpha_5\sigma_{\mu5}+\alpha_6\sigma_{\mu6}+\alpha_7\sigma_{\mu7}).\tag{14}$$

These solutions are nontrivial for all values of b, in the sense that they are not embeddings of SU(2) or SU(2)×U(1) configurations. In fact, for example, in the case $b\neq0$ and for $x=0$, F_{01} is already diagonal while the potential A_μ does not have any of the possible blocklike structures which would indicate[5] that it was an embedding of some subgroup solution. One can also argue in a direct way. A similar reasoning shows that this is also true in the case $b=0$. At the same time, this argument excludes the possibility that A_μ can be converted into an SU(3) potential by means of a convenient gauge transformation, because then we would obtain a nontrivial SU(3) one-instanton configuration.

The self-duality of $F_{\mu\nu}$ for any value of b implies that in all three cases which I have distinguished before, the Euclidean action (1) is proportional to the topological charge or Pontryagin index

$$q=-\frac{1}{16\pi^2}\int\mathrm{Tr}(\tilde F_{\mu\nu}F_{\mu\nu})d^4x,\tag{15}$$

i.e., $S=2\pi^2q$. Actually, the correct way to calculate q (valid also when $F_{\mu\nu}$ has singularities) is to use the expression

$$q_r=-\frac{1}{8\pi^2}\epsilon_{\mu\nu\rho\sigma}\int_r\mathrm{Tr}[A_\nu(\partial_\rho A_\sigma+\tfrac23A_\rho A_\sigma)]d\sigma_\mu,\quad q=\lim_{r\to+\infty}q_r\tag{16}$$

where the integration is performed over a sphere in E_4 of radius r. One gets in our case

$$\mathrm{Tr}(F_{\mu\nu}F_{\mu\nu}) = -\frac{96b^2}{(x^2+b)^4},$$

$$\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}[A_\nu(\partial_\rho A_\sigma + \tfrac{2}{3}A_\rho A_\sigma)] = -\frac{12x_\mu}{(x^2+b)^2} + \frac{8x^2x_\mu}{(x^2+b)^3}. \quad (17)$$

In the case $b>0$, using indistinctly (15) or (16) we obtain

$$q_r = \frac{r^4(r^2+3b)}{(r^2+b)^3}, \qquad q=1 \quad (18)$$

which gives the distribution and the total value of the topological charge in the Euclidean space. The value $q=1$ is typical for an instanton solution. Now when $b=0$ and owing to the singularity of $F_{\mu\nu}$ at $x=0$, we find ourselves in the situation described by de Alfaro *et al.*[7] for their meron solution. Making use of (15) we obtain $q=0$, but (16) tells us that $q_r=1$, for any value of r. The situation in this case is very curious because, on the one hand, we still have a self-dual instanton but, on the other, instead of being spread over all space (as in the case $b>0$) its topological charge is concentrated at the origin, as is known to be typical for the meron solution in SU(2).[7]

When $b=-a^2$ we still obtain from (16) the charge distribution (18), but now it is only proportional to the action for $r<a$. Owing to the singularity on the surface $x^2=a^2$, the Euclidean action (1) is no longer finite, while the total topological charge [given by (16)—notice that it has positive and negative contributions which cancel out] still has the value 1. By means of a conformal transformation consisting of an inversion $y_\mu = -x_\mu/x^2$ followed by a space reflection $z=I_s y$, where $I_s(y_0,y_1,y_2,y_3) = (y_0,-y_1,-y_2,-y_3)$, one obtains a new solution of the self-duality equations $\tilde{F}_{\mu\nu}=F_{\mu\nu}$ given by the vector potential

$$A'_\mu(x) = \Lambda^{-1}_{\mu\nu}\frac{1}{x^2}\left[\frac{2\sigma_{\nu\rho}x_\rho}{1-a^2x^2} \pm \frac{4\sqrt{2}}{\alpha} r_{\nu\rho}(I_s x) \times \left(\frac{x^2}{a^2x^2-1}\right)^{1/2}\tau_\rho\right], \qquad \tau_\rho \equiv \sum_{u=4}^{7}\sigma_{\rho u}\alpha_u \quad (19)$$

where $r_{\mu\nu}(x) = 2x_\mu x_\nu/x^2 - \delta_{\mu\nu}$ and $\Lambda_{\mu\nu}$ is a matrix representation of I_s. The new field strength tensor is

$$F'_{\mu\nu}(x) = \Lambda^{-1}_{\mu\mu'}\Lambda^{-1}_{\nu\nu'}z^4 r_{\mu'\rho}(z) r_{\nu'\sigma}(z) F_{\rho\sigma}(z), \qquad z=z(x). \quad (20)$$

This new solution of the YM equations is regular and SU(3) outside the sphere $x^2 \leq 1/a^2$. Taking the constant $a>1$, the two open charts $R_1 = \{x\in E_4 | x^2<a^2\}$ and $R_2 = \{x\in E_4 | x^2>1/a^2\}$ constitute a covering of E_4. The solutions $A_\mu(x)$ and $A'_\mu(x)$ are regular in R_1 and R_2, respectively, but, unfortunately, in the intersection they are not related by a gauge transformation. It is easy to see that the topological charge distribution of the new solution is given by

$$q'\left(x^2 \geq \frac{1}{r^2}\right) = q(x^2 \leq r^2) = \frac{r^4(r^2-3a^2)}{(r^2-a^2)^2}. \quad (21)$$

Notice that $q'(x^2 \geq 3/a^2)=1$ and that a can be made arbitrarily small.

Finally, let us study the behavior of (14) in the Minkowski space. In the case $b\neq 0$ both the Minkowskian action and the topological charge vanish. With respect to the energy one encounters the same problems as in the case of the pseudoparticle solution of Belavin *et al.*[11]: Singularities develop and no direct information can be provided.

The singular case deserves more attention. In what follows I will apply the procedure developed in Ref. 7. Let us go back to Eq. (14) with $b=0$. A conformal transformation takes the singularities of the solution to two arbitrary points u and v,

$$A'_\mu(s) = \eta_{\mu\rho}(s)\left[4\sigma_{\rho\lambda}s_\lambda \mp \frac{8i}{\alpha}(2s^2)^{1/2}\tau_\rho\right],$$

$$F'_{\mu\nu}(s) = \eta_{\mu\rho}(s)\eta_{\nu\sigma}(s)\left\{32[s^2\sigma_{\rho\sigma} + (s_\rho\sigma_{\sigma\lambda} - s_\sigma\sigma_{\rho\lambda})s_\lambda] + \frac{16i}{\alpha}(2s^2)^{1/2}(\pm(s_\rho\tau_\sigma - s_\sigma\tau_\rho) - 2s_\lambda([\sigma_{\rho\lambda},\tau_\sigma] - [\sigma_{\sigma\lambda},\tau_\rho])) - \frac{64}{\alpha^2}2s^2[\tau_\rho,\tau_\sigma]\right\}(\mu,\nu,\lambda,\rho = 0,1,2,3), \quad (22)$$

where

$$s_\lambda \equiv \frac{1}{2}\left[\frac{(x-u)_\lambda}{(x-u)^2} + \frac{(x-v)_\lambda}{(x-v)^2}\right],$$
$$\tau_\mu \equiv \sum_{u=4}^{7} \sigma_{\mu u}\alpha_u, \quad \eta_{\mu\nu} \equiv \frac{\partial}{\partial x_\nu}\left(\frac{s_\mu}{s^2}\right). \tag{23}$$

Without loss of generality we can set $-u_\mu = v_\mu = a_\mu$ and orient a_μ along the time direction $a_\mu = (1,0,0,0)$. The analytic continuation to Minkowski space amounts now to setting $x_0 \to ix_0$. In this way a perfectly regular solution in the physical space is obtained.

As an immediate consequence of the self-duality of the solution, its Minkowskian action and topological charge are zero, as in the former cases. The energy-momentum tensor is given by

$$\theta_{\mu\nu} = -\tfrac{1}{2}\,\mathrm{Tr}(F_{\mu\rho}F^{\rho}{}_{\nu} - \tfrac{1}{4}F_{\rho\sigma}F^{\rho\sigma}g_{\mu\nu}) \tag{24}$$

and, again making use of self-duality,

$$\theta_{00} = -\tfrac{1}{2}\,\mathrm{Tr}(F_{01}{}^2 + F_{02}{}^2 + F_{03}{}^2). \tag{25}$$

After some calculations it is not difficult to see that the total energy is

$$E = \int d^3x\, \theta_{00}(\vec{x}) = 0. \tag{26}$$

We see that to this Euclidean singular instanton solution corresponds, in Minkowski space, a regular one whose associated energy and action remain finite.

The exact significance of these solutions, with their unavoidable imaginary part, is not completely clear. Nevertheless I hope that their striking properties [so different from the usual ones known to us from explicit SU(2) instanton configurations] may prove useful in order to throw some light into the problem of finding the appropriate pseudoparticle configurations which can actually lead to the comprehension of quark confinement.

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