Heavy quark hadronic Lagrangian for *s*-wave quarkonium

A. Pineda and J. Soto

Departament d'Estructura i Constituents de la Materia and Institut de Física d'Altes Energies, Universitat de Barcelona, Diagonal 647, 08028 Barcelona, Catalonia, Spain

(Received 20 January 1995)

We use heavy quark effective theory (HQET) techniques to parametrize certain nonperturbative effects related to quantum fluctuations that put both heavy quarks and antiquarks in quarkonium almost on shell. The large off-shell momentum contributions are calculated using Coulomb-type states. The almost on-shell momentum contributions are evaluated using an effective "chiral" Lagrangian which incorporates the relevant symmetries of the HQET for quarks and antiquarks. The cutoff dependence of both contributions matches perfectly. The decay constants and the matrix elements of bilinear currents at zero recoil are calculated. Their leading nonperturbative contributions are parametrized by a single constant and turn out to be $O(\alpha^2/\Lambda_{QCD}a_n)$, a_n being the Bohr radius and α the strong coupling constant, times the nonperturbative contribution coming from the multipole expansion (gluon condensate). We discuss the physical applications to Υ, J/ψ, and B_c systems.

PACS number(s): 12.39.Hg, 12.39.Pn, 13.25.Gv

I. INTRODUCTION

The so-called heavy quark effective theory (HQET) [1–5] has become a standard tool to study the properties of hadrons containing a single heavy quark (see [6] for reviews). The hadron momentum is essentially the momentum of the heavy quark which may then be considered almost on shell. The dynamics becomes independent of the spin and the mass of the heavy quark giving rise to the so-called Isgur-Wise symmetries [1,2]. The relevant modes are momentum fluctuations of the order of Λ_{QCD} which are described by the HQET [3–5]. One cannot actually carry out reliable perturbative calculations at that scale, but one can certainly use the Isgur-Wise symmetries to obtain relations between physical observables.

For hadrons containing two heavy quarks or more the HQET is not believed to be a suitable approximation, the reason being that a system of two heavy quarks is mainly governed by the perturbative Coulomb-type interaction. The relevant modes are momentum fluctuations of the order of the inverse Bohr radius, which is flavor dependent, and not of the order of Λ_{QCD}. Still, if one is interested in subleading nonperturbative contributions related to the "on shellness" of the heavy quarks, the HQET may provide some useful information. Irrespective of the above, the HQET has already been used in phenomenological approaches to two heavy quark systems [7].

We shall argue that new nonperturbative contributions to the quarkonium decay constants and to the matrix elements of heavy-heavy currents between quarkonia states can be described by a suitably modified HQET. The well-known nonperturbative contributions arising from the multipole expansion [8,9] are $O(\Lambda_{QCD}a_n/\alpha^2)$, a_n being the Bohr radius and α the strong coupling constant, times the contributions we find. (However, the multipole expansion gives indeed the leading nonperturbative corrections to the energy spectrum.) The key observation is that when the heavy quarks are almost on shell the nonperturbative effects must be important. In that regime the multipole expansion breaks down, but it is precisely there that HQET techniques become applicable.

In Ref. [10] it was pointed out that when fields describing both heavy quarks and heavy antiquarks with the same velocity are included in the HQET Lagrangian, the latter has extra symmetries beyond the well-known flavor and spin symmetries [1,2]. In Ref. [11] the extra symmetries were thoroughly analyzed (see [12] for related elaborations). It was shown that they are spontaneously broken down to the spin and flavor symmetries, even if the gluons are switched off. The Goldstone modes turn out to be two particle states with the quantum numbers of *s*-wave quarkonia. Translating these findings into phenomenologically useful statements was the original motivation of this work.

The main hypothesis in what follows is that whenever we have a heavy quark field we may split it into two momentum regimes. The momentum regime where the heavy quark is almost on shell, and the momentum regime where the heavy quark is off shell. The main observation is that the HQET should always be a good approximation for a heavy quark in the almost on-shell momentum regime of QCD [10,12], no matter whether the heavy quark is accompanied by another heavy quark in the hadron or not. What makes a hadron containing a single heavy quark qualitatively different from a hadron containing, say, two heavy quarks are the large off-shell momentum effects. In the former the large off-shell momentum effects are small and can be evaluated order by order in QCD perturbation theory [1,5,13,14]. In the latter the large off-shell momentum effects are dominant giving rise to Coulomb-type bound states. However, once this is taken into account there is no *a priori* reason not to use HQET in the almost on-shell momentum regime for systems with two heavy quarks. Then the extra symmetries found in [10,11], which naturally involve quarkonium systems, should be relevant.

0556-2821/96/53(7)/3983(15)/$10.00

Suppose we have two quarks Q and Q' which are sufficiently heavy so that the formalism below can be readily applicable. Let us denote by ψ_Q, η_Q, $Q^*_{Q'}$, and $Q_{Q'}$ the vector $\bar{Q}Q$, pseudoscalar $\bar{Q}Q$, vector QQ', and pseudoscalar QQ' states. Our main results follow.

(i) The fact that the states above can be regarded as Goldstone modes in the on-shell momentum region [11] implies that their masses do not receive any nonperturbative contribution from that momentum region. Consequently, the leading nonperturbative correction comes from the multipole expansion [8,9]. This allows us to extract m_Q in a model-independent way from m_{ψ_Q}, and hence fix the parameter $\bar{\Lambda}$ relating m_Q with the mass of the $\bar{Q}q$ systems [6].

(ii) The leading new nonperturbative effects in the decay constants f_{ψ_Q}, f_{η_Q}, $f_{Q^*_{Q'}}$, and $f_{Q_{Q'}}$ are given in terms of a single nonperturbative parameter f_H.

(iii) The leading new nonperturbative effects in the matrix elements of bilinear heavy quark currents at zero recoil are given in terms of the same nonperturbative parameter f_H. In particular, this implies that the semileptonic decays ($m_Q > m_{Q'}$)

$$\psi_Q, \eta_Q \to Q^*_{Q'}, Q_{Q'},$$

$$Q^*_{Q'}, Q_{Q'} \to \psi_{Q'}, \eta_{Q'}$$

at zero recoil are known in terms of f_{ψ_Q}, f_{η_Q}, $f_{Q^*_{Q'}}$, and $f_{Q_{Q'}}$.

We organize this paper as follows. In Sec. II we perform some short distance calculations in the kinematical region we are interested in. In Sec. III we summarize the main results of Ref. [11] and match the results from Sec. II with the HQET. In Sec. IV we construct a hadronic effective Lagrangian for on-shell modes in quarkonium. In Sec. V we calculate the decay constant. In Sec. VI we calculate the matrix elements of any bilinear heavy quark current between quarkonia states. This is relevant for the study of semileptonic decays at zero recoil. In Sec. VII we briefly discuss the possible use of our formalism for Υ, B_c, B^*_c, J/ψ, and η_c physics. Section VIII is devoted to the conclusions. In Appendix A we show how to include $1/m$ corrections in the hadronic effective Lagrangian for the on-shell modes. A few technical details are relegated to Appendix B.

II. SHORT DISTANCE CONTRIBUTIONS IN THE ON-SHELL MOMENTUM REGIME

As mentioned in the introduction, what makes a $\bar{Q}Q$ system qualitatively different from a $\bar{Q}q$ system is the short distance contributions. In a $\bar{Q}q$ system these are well understood. They amount to Wilson coefficients in the currents and in the operators of the HQET Lagrangian, with anomalous dimensions which are computable in the loop expansion of QCD. For a $\bar{Q}Q$ system the short distance contributions cannot be accounted for by just anomalous dimensions in Wilson coefficients. Indeed, the anomalous dimension of a current containing a heavy quark field and a heavy antiquark field with the same velocity becomes imaginary and infinite [15]. For large m_Q, the two quarks in a $\bar{Q}Q$ system appear to be very close. Due to asymptotic freedom the system can be understood in a first approximation as a Coulomb-type bound state. In perturbation theory this is equivalent to summing up an infinite set of diagrams (ladder approximation) whose kernel is the tree level one gluon exchange (see [16] for a review).

We shall assume that the dominant short distance contribution to heavy quarkonia is the existence of Coulomb-type bound states. Typically we shall be interested in Green functions of the kind

$$G_\Gamma(p_1,p_2) := \int d^4x_1 d^4x_2 e^{ip_1x_1+ip_2x_2} \times\langle 0|T[\bar{Q}^a\Gamma Q^b(0)\bar{Q}^{bi_1}_{\alpha_1}(x_1)Q^{ai_2}_{\alpha_2}(x_2)]|0\rangle, \tag{2.1}$$

for the range of momentum

$$p_1 = -m_b v - k_1, \quad p_2 = -m_a v - k_2, \tag{2.2}$$

k_1 and k_2 being small.

Since the quarks are very massive, for the range of momentum (2.2) the leading contribution to (2.1) is only given by the ordering

$$G_\Gamma(p_1,p_2) = \int d^4x_1 d^4x_2 e^{ip_1x_1+ip_2x_2}\theta[-\max(x_1^0,x_2^0)] \times\langle 0|\bar{Q}^a\Gamma Q^b(0)T\{\bar{Q}^{bi_1}_{\alpha_1}(x_1)Q^{ai_2}_{\alpha_2}(x_2)\}|0\rangle. \tag{2.3}$$

We insert the identity between the current and the fields and we approximate it by the vacuum plus the Coulomb-type states (the states above threshold shall not give contribution when we sit in the relevant pole). We treat then the fields as being free:

$$1 \simeq |0\rangle\left\langle 0\right| + \sum_{n,s}\int \frac{d^3\vec{P}_n}{(2\pi)^3 2P_n^0}\left|s,\vec{P}_n = m_{ab,n}\vec{v}\right\rangle \times\langle s,\vec{P}_n = m_{ab,n}\vec{v}|. \tag{2.4}$$

The Coulomb state in the center of mass (c.m.) frame reads

$$|s,\vec{P}_n = m_{ab,n}\vec{v}\rangle = \frac{1}{\sqrt{N_c}}\frac{m_{ab}^{(3/2)}}{m_{ab,n}}v^0\int\frac{d^3\vec{k}}{(2\pi)^3}\tilde{\psi}_{ab,n}(\vec{k})\frac{1}{\sqrt{2p_1^0 2p_2^0}} \times\sum_{\alpha,\beta}\bar{u}^\alpha(p_1)\Gamma_s v^\beta(p_2)a^\dagger_\alpha(p_1)b^\dagger_\beta(p_2)|0\rangle, \tag{2.5}$$

where

$$\vec{p}_1=m_a\vec{v}+\vec{k}+\frac{\vec{k}\cdot\vec{v}}{1+v^0}\,\vec{v},\quad \vec{p}_2=m_b\vec{v}-\vec{k}-\frac{\vec{k}\cdot\vec{v}}{1+v^0}\,\vec{v},$$

$$p_1^0=m_a v^0+\vec{k}\cdot\vec{v},\quad p_2^0=m_b v^0-\vec{k}\cdot\vec{v},$$

$$m_{ab}:=m_a+m_b,\quad m_{ab,n}:=m_{ab}-E_{ab,n},$$

$$\Gamma_s=i\gamma_5 p_-,i\not{e}^i p_-,$$

$$v^2=1,\quad p_\pm:=\frac{1\pm\not{v}}{2},\quad e^i\cdot v=0. \tag{2.6}$$

$E_{ab,n}$, $\psi_{ab,n}(\vec{x})$, and $\tilde{\psi}_{ab,n}(\vec{k})$ are the energy, the coordinate space wave function and the momentum space wave function of a Coulomb-type state with principal quantum number n. v is the bound state four-vector velocity. $a_\alpha^\dagger(p_1)$ and $b_\beta^\dagger(p_2)$ are creation operators of particles and antiparticles, respectively. $u^\alpha(p_1)$ and $v^\beta(p_2)$ are spinors normalized in such a way that in the large m limit the following holds:

$$\sum_\alpha u^\alpha(p_1)\bar{u}^\alpha(p_1)=p_+,\quad \sum_\alpha v^\alpha(p_1)\bar{v}^\alpha(p_1)=-p_-. \tag{2.7}$$

Choosing the momenta as in (2.6) is crucial in order to take into account that the c.m. of the bound state moves with a fixed velocity v with respect to the laboratory frame [17]. Equation (2.5) has the usual relativistic normalization:

$$\langle s,\vec{P}_n=m_{ab,n}\vec{v}|r,\vec{P}_m=m_{ab,m}\vec{v}\rangle =2m_{ab,n}v^0(2\pi)^3\delta^{(3)}(m_{ab,n}(\vec{v}-\vec{v}))\delta_{nm}\delta_{rs}. \tag{2.8}$$

We have to consider the following kind of matrix elements:

$$\begin{aligned}\langle s,m_{ab,n}\vec{v}|Q^a_{\alpha_2}(x_2)\bar{Q}^b_{\alpha_1}(x_1)|0\rangle &=e^{im_{ab,n}v\cdot X}\langle s,m_{ab,n}\vec{v}|Q^a_{\alpha_2}(x_2-X)\bar{Q}^b_{\alpha_1}(x_1-X)|0\rangle\\ &=e^{im_{ab,n}v\cdot X}\frac{m_{ab}^{(3/2)}}{m_{ab,n}}(\overline{\Gamma_s})_{\alpha_2\alpha_1}\int\frac{d^3\vec{k}}{(2\pi)^3}\tilde{\psi}^*_{ab,n}(\vec{k})\\ &\quad\times\exp\left\{i\left[k\cdot\vec{v}x^0-\vec{x}\left(\vec{k}+\frac{\vec{k}\cdot\vec{v}}{1+v^0}\,\vec{v}\right)\right]\right\},\end{aligned}$$

$$X=\frac{m_a x_1+m_b x_2}{m_{ab}},\quad x=x_1-x_2, \tag{2.9}$$

where it is essential to extract the c.m. dependence in the fields *before* using the explicit expression (2.5) for the calculation of (2.9). As mentioned above the states $|s,m_{ab,n}\vec{v}\rangle$ have the explicit expression (2.5) only in the c.m. frame [16,17]. Factors of the kind $m_{ab}/m_{ab,n}$ appearing in several expressions above have been approximated to 1 in the rest of the paper. Finally, performing the x_1,x_2 integral and taking into account that

$$\sum_s(\Gamma_s)_{\alpha_2\alpha_4}(\overline{\Gamma_s})_{\alpha_1\alpha_3}=-2(p_+)_{\alpha_2\alpha_3}(p_-)_{\alpha_1\alpha_4}, \tag{2.10}$$

we obtain

$$\begin{aligned}G_\Gamma(p_1,p_2)=\sum_n\tilde{\psi}^*_{ab,n}(0)\psi_{ab,n}(0)&\times(p_-\Gamma p_+)_{\alpha_2\alpha_1}\delta_{i_1i_2}\frac{1}{v\cdot k_2+\dfrac{m_a}{m_{ab}}E_{ab,n}+i\epsilon}\\ &\times\frac{1}{v\cdot k_1+\dfrac{m_b}{m_{ab}}E_{ab,n}+i\epsilon}.\end{aligned} \tag{2.11}$$

In the last expression we approximated $\tilde{\psi}_{ab,n}(e^i\cdot k)\simeq\tilde{\psi}_{ab,n}(0)$ {we neglect $O[(n|e^i\cdot k|/m\alpha)^2]$}. In (2.11) there is a sum over an infinite number of poles. Each term in the sum corresponds to a Coulomb-type bound state. At the hadronic level we want to describe only one of those states. This is achieved by tuning the external momenta to sit on the relevant pole. Suppose we are interested in $\psi_Q(n)$ state. Then we take

$$k_1=k_1'-\frac{m_b}{m_{ab}}E_{ab,n}v,\quad k_2=k_2'-\frac{m_a}{m_{ab}}E_{ab,n}v, \tag{2.12}$$

so that in the limit $k_i'\to 0$ $(i=1,2)$ we obtain

$$G_\Gamma(p_1,p_2)=\tilde{\psi}^*_{ab,n}(0)\psi_{ab,n}(0)\times(p_-\Gamma p_+)_{\alpha_2\alpha_1}\delta_{i_1i_2}\frac{1}{v\cdot k_2'+i\epsilon}\frac{1}{v\cdot k_1'+i\epsilon}. \tag{2.13}$$

Notice from (2.2) and (2.12) that we must subtract from the momentum of the quark $[m_a-(m_a/m_{ab})E_{ab,n}]v$ in order to get an expression suitable to be reproduced in the HQET. This may be interpreted as if integrating out off-shell short-distance degrees of freedom produces an effective mass for the almost on-shell modes of a heavy quark inside quarkonium. This effective mass depends on the precise bound state the quark is in. We are almost on shell when $v\cdot k_i'$, $e^j\cdot k_i'\sim\Lambda_{\rm QCD}$ $(i=1,2)$.

This restricts the validity of our approximation to the case $E_{ab,n}\sim\mu_{ab}\alpha^2/n^2\gg\Lambda_{\rm QCD}$ (μ_{ab} is the reduced mass), otherwise momentum fluctuations of the order of $\Lambda_{\rm QCD}$ would take us from one pole to another. Notice also that for arbitrarily large but fixed μ_{ab} there is always an n where this approximation fails. Therefore we shall always be dealing with a finite number of low-lying energy levels.

Consider the four-point function

$$\begin{aligned}G(p_1,p_2,p_3,p_4)&:=\int d^4x_1d^4x_2d^4x_3d^4x_4e^{ip_1x_1+ip_2x_2+ip_3x_3+ip_4x_4}\\ &\quad\times\langle 0|T\{Q^{bi_1}_{\alpha_1}(x_1)Q^{ai_2}_{\alpha_2}(x_2)\bar{Q}^{ai_3}_{\alpha_3}(x_3)\bar{Q}^{bi_4}_{\alpha_4}(x_4)\}|0\rangle.\end{aligned} \tag{2.14}$$

For the momenta

$$p_1=-\left(m_b-\frac{m_b}{m_{ab}}E_{ab,n}\right)v-k_1',$$

$$p_2=\left(m_a-\frac{m_a}{m_{ab}}E_{ab,n}\right)v+k_2',$$

$$p_3=-\left(m_a-\frac{m_a}{m_{ab}}E_{ab,n}\right)v-k_3',$$

$$p_4=\left(m_b-\frac{m_b}{m_{ab}}E_{ab,n}\right)v+k_4' \qquad (2.15)$$

($k_i'\to 0$, $i=1,\ldots,4$). Working in the same way we obtain

$$\begin{aligned}G(p_1,p_2,p_3,p_4)&=(2\pi)^4\delta^{(4)}(-k_1'+k_2'-k_3'+k_4')\\&\times\frac{i}{2N_c}\sum_{\Gamma_n=i\gamma_5 p_-,i\not\epsilon^i p_-}(\Gamma_n)_{\alpha_2\alpha_4}(\overline{\Gamma}_n)_{\alpha_1\alpha_3}\\&\times\delta_{i_1i_3}\delta_{i_2i_4}\tilde{\psi}^*_{ab,n}(0)\tilde{\psi}_{ab,n}(0)\frac{1}{v\cdot k_3'+i\epsilon}\frac{1}{v\cdot k_1'+i\epsilon}\\&\times\left(\frac{1}{v\cdot k_2'+i\epsilon}+\frac{1}{v\cdot k_4'+i\epsilon}\right).\end{aligned} \qquad (2.16)$$

We shall see in the next section that (2.13) and (2.16) can be reproduced (with suitable changes) by a HQET for quarks and antiquarks.

III. HQET FOR QUARKS AND ANTIQUARKS

The Lagrangian of the HQET for quarks and antiquarks moving at the same velocity v_μ ($v_\mu v^\mu=1$) reads [4]

$$L_v=i\bar{h}_v\not v v_\mu D^\mu h_v=i\bar{h}_v^+ v\cdot Dh_v^+-i\bar{h}_v^- v\cdot Dh_v^-, \qquad (3.1)$$

where $h_v=h_v^++h_v^-$ and $h_v^\pm=(1\pm\not v)/2h_v$. h_v^+ contains annihilation operators of quarks with small momentum about mv_μ and h_v^- contains creation operators of antiquarks again with small momentum about mv_μ. D_μ is the covariant derivative containing the gluon field.

The quark and antiquark sector of (3.1) are independently invariant under the well-known spin and flavour symmetry [1,2,4]

$$h_v^\pm\to e^{i\epsilon_\pm^i S_i^\pm}h_v^\pm \quad\text{and}\quad \bar{h}_v^\pm\to\bar{h}_v^\pm e^{-i\epsilon_\pm^i S_i^\pm}, \qquad (3.2)$$

where $S_i^\pm=i\epsilon_{ijk}[\not e_j,\not e_k](1\pm\not v)/2$, with e_j^μ, $j=1,2,3$ being an orthonormal set of space-like vectors orthogonal to v_μ, and

$$h_v^\pm\to e^{i\theta_\pm}h_v^\pm \quad\text{and}\quad \bar{h}_v^\pm\to\bar{h}_v^\pm e^{-i\theta_\pm}. \qquad (3.3)$$

$\epsilon_\pm^i$ and $\theta_\pm$ are arbitrary real numbers corresponding to the parameters of the transformations.

The Lagrangian (3.1) is also invariant under the following set of transformations:

$$h_v\to e^{i\gamma_5\epsilon}h_v,\quad \bar{h}_v\to\bar{h}_v e^{i\gamma_5\epsilon}, \qquad (3.4)$$

$$h_v\to e^{\gamma_5\not v\epsilon}h_v,\quad \bar{h}_v\to\bar{h}_v e^{\gamma_5\not v\epsilon}, \qquad (3.5)$$

$$h_v\to e^{\epsilon^i\not e_i}h_v,\quad \bar{h}_v\to\bar{h}_v e^{\epsilon^i\not e_i}, \qquad (3.6)$$

$$h_v\to e^{i\epsilon^i\not e_i\not v}h_v,\quad \bar{h}_v\to\bar{h}_v e^{i\epsilon^i\not e_i\not v}. \qquad (3.7)$$

The whole set of transformations (3.2)–(3.7) corresponds to a U(4) symmetry for a single flavor. For N_{hf} heavy flavors they correspond to a U($4N_{hf}$) group. In the latter case h_v must be considered a vector in flavor space and the parameters of the transformations (3.2)–(3.7) as Hermitian matrices in that space.

When the gluons are switched off it is easy to prove that the U($4N_{hf}$) symmetry breaks down spontaneously to U($2N_{hf}$)$\otimes$U($2N_{hf}$) (see [11]). The following currents correspond to the broken generators

$$j_{5\pm}^{ab}:=\bar{h}_v^a i\gamma_5 p_\pm h_v^b \quad\text{and}\quad j_{5\pm}^{ab^i}:=\bar{h}_v^a i\not e_i p_\pm h_v^b, \qquad (3.8)$$

$a,b,c,\ldots=1,\ldots,N_{hf}$ are flavor indices. They transform according to two four-dimensional irreducible representations of U($2N_{hf}$)$\otimes$U($2N_{hf}$). In what follows we are going to assume that the situation above is not modified when soft gluons are switched on. The currents (3.8) have the quantum numbers of pseudoscalar and vector quarkonium respectively. The heavy quark and antiquark fields interact with soft gluons according to the Lagrangian (3.1). For soft gluons, perturbation theory cannot be reliably applied. However, one can use effective Lagrangian techniques, which fully exploit the symmetries above, to parametrize the nonperturbative contributions in this region. This shall be done in Sec. IV.

For further purposes let us carry out some leading order perturbative calculations. Consider first

$$\begin{aligned}G_{\Gamma\Gamma'}(k)&=\int d^4x e^{-ikx}\langle 0|T\{\bar{h}_v^{a-}\Gamma h_v^{b+}(0)\bar{h}_v^{b+}\Gamma' h_v^{a-}(x)\}|0\rangle\\&=-iN_c\frac{\mu^3}{6\pi^2}\,\mathrm{tr}(p_+\Gamma' p_-\Gamma)\frac{1}{v\cdot k+i\epsilon},\end{aligned} \qquad (3.9)$$

where μ is an ultraviolet symmetric cutoff in three-momentum (see [11] for more details). Consider also

$$\begin{aligned}G_{\Gamma\Gamma'\Gamma''}(k_1',k_2')&=\int d^4x_1 d^4x_2 e^{ik_1'x_1-ik_2'x_2}\\&\times\langle 0|T[\bar{h}_v^{a-}\Gamma'' h_v^{b+}(x_1)\bar{h}_v^{b+}\Gamma h_v^{c+}(0)\\&\times\bar{h}_v^{c+}\Gamma' h_v^{a-}(x_2)]|0\rangle\\&=N_c\frac{\mu^3}{6\pi^2}\,\mathrm{tr}(p_-\Gamma'' p_+\Gamma p_+\Gamma')\\&\times\frac{1}{v\cdot k_1'+i\epsilon}\frac{1}{v\cdot k_2'+i\epsilon}.\end{aligned} \qquad (3.10)$$

The flavor indices (a,b,c) are not summed up unless otherwise indicated. Color indices are not explicitly displayed in the color singlet currents. Otherwise they will be denoted by $i_1, i_2, \ldots = 1, \ldots, N_c$, with N_c being the number of colors. We shall drop the subscript v from h_v and change the superscript $\pm$ into subscripts in the following.

The strong cutoff dependence of (3.9)–(3.10) is puzzling. We shall see later on that it cancels against suitable short distance contributions.

As claimed before, it is easy to see that (2.13) is reproduced by the following HQET Green function at the tree level:

$$G_\Gamma(k_1',k_2') = \int d^4x_1 d^4x_2 e^{-ik_1'x_1 - ik_2'x_2} \times \langle 0|T\{C_\Gamma \bar{h}^a \Gamma h^b(0) \bar{h}^{bi_1}_{+\alpha_1}(x_1) h^{ai_2}_{-\alpha_2}(x_2)\}|0\rangle \tag{3.11}$$

with C_Γ being a Wilson coefficient,

$$C_\Gamma = \tilde{\psi}^*_{ab,n}(0)\psi_{ab,n}(0). \tag{3.12}$$

Analogously, (2.16) is reproduced in the HQET by[1]

$$G(k_1',k_2',k_3',k_4') = \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 e^{-ik_1'x_1 + ik_2'x_2 - ik_3'x_3 + ik_4'x_4} \times \left\langle 0 \left| T\left\{ h^{bi_1}_{-\alpha_1}(x_1) h^{ai_2}_{+\alpha_2}(x_2) \bar{h}^{ai_3}_{+\alpha_3}(x_3) \bar{h}^{bi_4}_{-\alpha_4}(x_4) \times i\int d^4y \left(-\frac{1}{2N_c} \tilde{\psi}^*_{ab,n}(0)\tilde{\psi}_{ab,n}(0) \right) \sum_{\Gamma_n} \bar{h}^a \Gamma_n h^b(y) iv\cdot\partial(\bar{h}^b \bar{\Gamma}_n h^a(y)) \right\} \right| 0 \right\rangle. \tag{3.13}$$

IV. EFFECTIVE HADRONIC LAGRANGIAN FOR THE ON-SHELL CONTRIBUTIONS OF s-WAVE QUARKONIA

We have seen that for the on-shell kinematical regime certain correlators can be reproduced in the HQET. We shall see in Secs. V and VI that the contributions from this region to the decay constants and matrix elements reduce to the evaluation of heavy quark-antiquark currents in the HQET. For the range of momentum we are interested in these Green functions cannot reliably be evaluated in perturbation theory. We shall use in this section effective Lagrangian techniques, very similar to those used in chiral perturbation theory, to parametrize the nonperturbative contribution.

There are well-known rules [18] (see also [19]) to construct phenomenological Lagrangians for Goldstone bosons associated to the symmetry breaking of a group G down to a subgroup H for relativistic theories. These rules need two slight modifications to become applicable to our case.

(i)The HQET is formally relativistic only after assigning transformation properties to the fixed velocity v^μ. We must take into account that the velocity v^μ as well as the $e^{\,i}_{\mu}$ can also be used to build up relativistic invariant terms.

(ii) The HQET is not only globally $U(4N_{hf})$ invariant, but locally $U(4N_{hf})$ gauge invariant under transformations which only depend on the components $x^i := x^\mu e^{\,i}_{\mu}$. We shall also require the phenomenological Lagrangian to be local gauge invariant under the corresponding transformations.

With the above modifications (i) and (ii) we shall apply the rules [18] to the case $G = U(4N_{hf})$, $H = U(2N_{hf}) \otimes U(2N_{hf})$. Let us first associate with the currents (3.8) fields in the phenomenological Lagrangian which have the same transformation properties under H:

$$H^{ab} \to \bar{h}^a i\gamma_5 p_+ h^b, \quad H^{ab^i} \to \bar{h}^a i\not{e}^i p_+ h^b,$$

$$H^{ba*} \to \bar{h}^b i\gamma_5 p_- h^a, \quad H^{ba^{i*}} \to -\bar{h}^b i\not{e}^i p_- h^a. \tag{4.1}$$

We build up the following object

$$H = i\gamma_5 p_- H - i\not{e}_i p_- H^i + i\gamma_5 p_+ H^\dagger + i\not{e}_i p_+ H^{i^\dagger},$$

$$\bar{H} := \gamma^0 H^\dagger \gamma^0 = H, \tag{4.2}$$

where we use matrix notation for H^{ab} and H^{ab^i}. H transforms under the unbroken subgroup as

$$H \to hHh^{-1}, \quad h \in U(2N_{hf}) \otimes U(2N_{hf}). \tag{4.3}$$

We assign nonlinear transformations under the full group $U(4N_{hf})$ in the standard manner [18]:

$$g(\theta)e^H =: e^{H'} h(H,\theta),$$

[1]One may be tempted to include (3.13) as a perturbation in the HQET Lagrangian. This is not quite correct. The Green function

$$G(k_1',k_2',k_3',k_4') = \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 e^{-ik_1'x_1 + ik_2'x_2 - ik_3'x_3 + ik_4'x_4} \times \langle 0|T\{h^{bi_1}_{-\alpha_1}(x_1) h^{ai_2}_{+\alpha_2}(x_2) \bar{h}^{ai_3}_{+\alpha_3}(x_3) \bar{h}^{bi_4}_{-\alpha_4}(x_4)\}|0\rangle$$

gives a nonzero contribution in the HQET which does not correspond to (2.14)–(2.16). It is (3.13) which gives the leading contribution to (2.14) in the HQET and hence the last term in (3.13) must not be included in the Lagrangian. This means that unlike in the case of heavy-light systems, the short distance effects here cannot always be accounted for by only modifications of the currents and the Lagrangian, as we may have naively expected. We have to content ourselves by identifying for a given Green function, the Green function in the HQET that gives the same result.

$$g \in \mathrm{U}(4N_{hf}), \quad h \in \mathrm{U}(2N_{hf}) \otimes \mathrm{U}(2N_{hf}),$$

$$e^{H} \in \mathrm{U}(4N_{hf})/\mathrm{U}(2N_{hf}) \otimes \mathrm{U}(2N_{hf}), \tag{4.4}$$

where H' is the transformed field. Then

$$e^{H} \rightarrow e^{H'} = g e^{H} h^{-1} = h e^{H} \bar{g}, \tag{4.5}$$

where $\bar{g} = \gamma^0 g^\dagger \gamma^0$. The following property holds:

$$\not{v} e^{H} = e^{-H} \not{v}, \tag{4.6}$$

which implies that

$$S := e^{2H} \not{v} = \not{v} e^{-2H}, \quad S^2 = 1, \quad e^{2H} \not{v} \rightarrow g e^{2H} \not{v} g^{-1}. \tag{4.7}$$

Because of the local gauge symmetry we can only build the following connection and covariant tensor:

$$V := \frac{1}{2} \left(e^{-H} v \cdot \partial e^{H} + e^{H} v \cdot e^{-H} \right),$$

$$V \rightarrow h V h^{-1} + h v \cdot \partial h^{-1}, \quad \not{v} V = V \not{v},$$

$$A := \frac{1}{2} \left(e^{-H} v \cdot \partial e^{H} - e^{H} v \cdot \partial e^{-H} \right), \quad A \rightarrow h A h^{-1},$$

$$\not{v} A = -A \not{v}, \quad v \cdot \partial S = e^{H} A e^{H} \not{v}. \tag{4.8}$$

Notice that any derivative with respect to $x^i := e^{\,i}_{\mu} x^{\mu}$ acting on functions of x^i which are not scalars will not be covariant under the local transformations.

The $u(4N_{hf})$ algebra and the HQET Lagrangian are invariant under the discrete symmetry

$$e^{i}_{\mu} \rightarrow -e^{i}_{\mu}, \quad v_{\mu} \rightarrow -v_{\mu}, \tag{4.9}$$

which is reminiscent of charge conjugation. They are also invariant under the SO(3) transformations $e^{\,i}_{\mu} \rightarrow R^{\,i}_{j} e^{\,j}_{\mu}$ and, of course, under Lorentz transformations if we assign $v_{\mu} \rightarrow \Lambda_{\mu}{}^{\nu} v_{\nu}$, $e^{\,i}_{\mu} \rightarrow \Lambda_{\mu}{}^{\nu} e^{\,i}_{\nu}$. All these symmetries should also be implemented in the effective Lagrangian.

We can start at this point the construction of the effective Lagrangian, order by order in derivatives, using the objects defined above. At first order it turns out that there is no invariant term. Still, there is a term which is invariant up to a total derivative. It reads

$$\mathrm{Tr}(\not{v} V) \simeq -4\, \mathrm{tr}(H^\dagger v \cdot \partial H + H^{i\dagger} v \cdot \partial H^i) + \cdots ,$$

$$\mathrm{Tr}(\not{v} V) \rightarrow \mathrm{Tr}(\not{v} V) + \mathrm{Tr}(\not{v} h v \cdot \partial h^{-1}). \tag{4.10}$$

Tr means trace over flavor and Dirac indices whereas tr means trace over flavor indices only. We keep tr for trace over Dirac indices only. It is not difficult to prove that $\mathrm{Tr}(\not{v} h v \cdot \partial h^{-1})$ is indeed a total derivative. This is analogous to the case of the Heisenberg ferromagnet where the leading-order term in the effective Lagrangian for the Goldstone mode is also invariant up to a total derivative [20]. Then at leading order the long-distance properties of heavy quarkonia are governed by a single constant. At next-to-leading order we have the term

$$\mathrm{Tr}(AA) \simeq -4\, \mathrm{tr}(v \cdot \partial H^\dagger v \cdot \partial H + v \cdot \partial H^{i\dagger} v \cdot \partial H^i) + \cdots . \tag{4.11}$$

Terms containing x^i derivatives start appearing at sixth order. Notice that there is no vertex involving an odd number of fields. This holds at any order in derivatives and it is a consequence of the separate conservation of the number of heavy quarks and antiquarks.

For convenience we normalize the effective Lagrangian as

$$-i\, \frac{f_H^2}{4}\, \mathrm{Tr}(\not{v} V) = i\, \mathrm{tr}(\Pi^\dagger v \cdot \partial \Pi + \Pi^{i\dagger} v \cdot \partial \Pi^i) + \cdots ,$$

$$H = \frac{\Pi}{f_H}, \quad H^i = \frac{\Pi^i}{f_H}. \tag{4.12}$$

f_H^2 is a dimension-3 parameter of the order of Λ^3_{QCD}.

The effective Lagrangian built above makes sense by itself as a toy model. If we ignore the matching with high energies we can withdraw some consequences out of the lowest order Lagrangian. These and the $1/m$ corrections to this toy model are worked out in Appendix B.

Let us next discuss how to represent quark currents in the effective Lagrangian. Consider

$$j_{\Gamma}^{ab} = \bar{h}^{a} \Gamma h^{b}. \tag{4.13}$$

Let us introduce a source a_{Γ}^{ab} for each of these currents and write all possible currents up in the Lagrangian

$$L_v = i \bar{h} \not{v} v_{\mu} D^{\mu} h + \bar{h} \not{v} a h,$$

$$a := \sum_{\Gamma} a_{\Gamma}^{ab} \not{v} \Gamma. \tag{4.14}$$

L is now locally invariant under $\mathrm{U}(4N_{hf})$ if we assign to a the transformation property

$$a \rightarrow g a g^{-1} + g i v \cdot \partial g^{-1}. \tag{4.15}$$

At the hadronic level we may also require local gauge invariance upon the introduction of a. This is easily achieved by changing $v \cdot \partial$ into $v \cdot \partial - i a$ in the definition of V in (4.8). We obtain

$$L = -i\, \frac{f_H^2}{4} \left[\mathrm{Tr}(\not{v} V) - i\, \mathrm{Tr}(a S) \right]. \tag{4.16}$$

Then we may identify

$$\bar{h}^{a} \Gamma h^{b} \rightarrow -\frac{f_H^2}{4}\, \mathrm{Tr}(\Gamma T^{ab} e^{2H}), \tag{4.17}$$

where T^{ab} is the zero matrix in flavor space except for a 1 in row a, column b. It is interesting to observe that the $\mathrm{U}(4N_{hf})$ symmetry is so large that any bilinear current of the kind (4.13) can be written in terms of a generator of the $\mathrm{U}(4N_{hf})$ symmetry. This is the actual reason why the identification (4.17) does not involve any extra unknown parameter. It is analogous to the case of the vector and axial-vector currents in the chiral Lagrangian [21].

Let us next calculate for further convenience the correlators (3.9) and (3.10) in the hadronic effective Lagrangian. For (3.9) we have

$$G_{\Gamma\Gamma'}(k)=\int d^4xe^{-ik\cdot x}\langle 0|T\{\bar{h}^a_-\Gamma h^b_+(0)\bar{h}^b_+\Gamma' h^a_-(x)\}|0\rangle=\int d^4xe^{-ik\cdot x}\left\langle 0\left|T\left\{\left[-\frac{f_H^2}{4}\,\mathrm{Tr}(p_-\Gamma p_+T^{ab}e^{2H(0)})\right]\right.\right.\right.$$

$$\left.\left.\left.\times\left[-\frac{f_H^2}{4}\,\mathrm{Tr}(p_+\Gamma' p_-T^{ba}e^{2H(x)})\right]\right\}\right|0\right\rangle$$

$$\simeq\int d^4xe^{-ik\cdot x}\left\langle 0\left|T\left(\left\{-\frac{f_H^2}{4}\mathrm{Tr}[p_-\Gamma p_+T^{ab}2H(0)]\right\}\left\{-\frac{f_H^2}{4}\,\mathrm{Tr}[p_+\Gamma' p_-T^{ba}2H(x)]\right\}\right)\right|0\right\rangle$$

$$=-i\,\frac{f_H^2}{2}\,\mathrm{tr}(p_+\Gamma' p_-\Gamma)\,\frac{1}{v\cdot k+i\epsilon}. \tag{4.18}$$

For (3.10) we have

$$G_{\Gamma\Gamma'\Gamma''}(k_1',k_2')=\int d^4x_1d^4x_2e^{ik_1'x_1-ik_2'x_2}\langle 0|T\{\bar{h}^a_-\Gamma'' h^b_+(x_1)\bar{h}^b_+\Gamma h^c_+(0)\bar{h}^c_+\Gamma' h^a_-(x_2)\}|0\rangle$$

$$=\int d^4x_1d^4x_2e^{ik_1'x_1-ik_2'x_2}\left\langle 0\left|T\left\{\left[-\frac{f_H^2}{4}\,\mathrm{Tr}(p_-\Gamma'' p_+T^{ab}e^{2H(x_1)})\right]\left[-\frac{f_H^2}{4}\,\mathrm{Tr}(p_+\Gamma p_+T^{bc}e^{2H(0)})\right]\right.\right.\right.$$

$$\left.\left.\left.\times\left[-\frac{f_H^2}{4}\,\mathrm{Tr}(p_+\Gamma' p_-T^{ca}e^{2H(x_2)})\right]\right\}\right|0\right\rangle$$

$$\simeq\int d^4x_1d^4x_2e^{ik_1'x_1-ik_2'x_2}\left\langle 0\left|T\left[\left(-\frac{f_H^2}{4}\,\mathrm{Tr}[p_-\Gamma'' p_+T^{ab}2H(x_1)]\right)\left(-\frac{f_H^2}{4}\,\mathrm{Tr}[p_+\Gamma p_+T^{bc}2H^2(0)]\right)\right.\right.\right.$$

$$\left.\left.\left.\times\left(-\frac{f_H^2}{4}\,\mathrm{Tr}[p_+\Gamma' p_-T^{ca}2H(x_2)]\right)\right]\right|0\right\rangle$$

$$=\frac{f_H^2}{2}\,\mathrm{tr}(p_-\Gamma'' p_-\Gamma p_+\Gamma')\,\frac{1}{v\cdot k_1'+i\epsilon}\,\frac{1}{v\cdot k_2'+i\epsilon}. \tag{4.19}$$

Notice at this point that we may obtain (3.9) and (3.10) from (4.18) and (4.19) by taking $f_H^2/2\rightarrow N_c\mu^3/6\pi^2$. Hence f_H^2 at the hadronic level plays the role of the cut-off μ at quark level. Observe also that the dependence on the Γ matrices in (4.18) and (4.19) is explicit. All decay constants and matrix elements of bilinear currents are given in terms of the only nonperturbative parameter f_H. This is a direct consequence of the U($4N_{hf}$) symmetry being spontaneously broken down to U($2N_{hf}$)$\otimes$U($2N_{hf}$).

V. EXAMPLE: THE DECAY CONSTANT, f_Υ

A. Separating and evaluating off-shell and on-shell contributions

Consider the current–current correlator

$$G_\Gamma(p):=\int d^4xe^{ipx}\langle 0|T\{\bar{Q}^a\Gamma Q^b(0)\bar{Q}^b\bar{\Gamma}Q^a(x)\}|0\rangle.$$

$$p=-m_{ab,n}v-k,\quad k\rightarrow 0. \tag{5.1}$$

We separate

$$\bar{Q}^a\Gamma Q^b=(Q^a\Gamma Q^b)_{\mathrm{on}}+(\bar{Q}^a\Gamma Q^b)_{\mathrm{off}}, \tag{5.2}$$

where $(\bar{Q}^a\Gamma Q^b)_{\mathrm{on}}$ and $(\bar{Q}^a\Gamma Q^b)_{\mathrm{off}}$ means that both heavy quark fields in the current have momenta almost on shell and off shell, respectively. Our goal is to obtain a representation in terms of the HQET of any Green function containing an $(\bar{Q}^a\Gamma Q^b)_{\mathrm{on}}$. In order to enforce "on shellness" it is convenient to make the substitution

$$\int d^4x[\bar{Q}^b\bar{\Gamma}Q^a(x)]_{\mathrm{on}}e^{ipx}$$
$$\rightarrow\int d^4x_1\bar{Q}^{bi_1}_{\alpha_1}(x_1)e^{ip_1x_1}$$
$$\times\int d^4x_2Q^{ai_2}_{\alpha_2}(x_2)e^{ip_2x_2}(\bar{\Gamma})_{\alpha_1\alpha_2}\delta_{i_1i_2}, \tag{5.3}$$

$$p_1=-\left(m_a-\frac{m_a}{m_{ab}}\,E_{ab,n}\right)v-k_1', \tag{5.4}$$

$$p_2=-\left(m_b-\frac{m_b}{m_{ab}}\,E_{ab,n}\right)v-k_2',$$

$$k=k_1'+k_2',$$

$$k_1',k_2'\to 0$$

and see whether the new Green function admits a representation in terms of the HQET. This is nothing but the calculations carried out above. Then we undo (5.3) by putting the fields depending on x_1 and x_2 in the HQET at the same point x. We have [from (2.1), (2.13), and (3.11)]

$$\int d^4x\, e^{ipx}\langle 0|T\{[\bar{Q}^a\Gamma Q^b(0)]_{\rm on}[\bar{Q}^b\bar{\Gamma}Q^a(x)]_{\rm on}\}|0\rangle$$
$$=\int d^4x e^{-ikx}\left\langle 0\left|T\left\{\bar{h}_-^a\Gamma h_+^b(0)\bar{h}_+^b\bar{\Gamma}h_-^a(x)i\int d^4y\left(-\frac{1}{2N_c}\tilde{\psi}^*_{ab,n}(0)\tilde{\psi}_{ab,n}(0)\right)\right.\right.\right.$$
$$\left.\left.\left.\times\sum_{\Gamma_n=i\gamma_5 p_-,i\epsilon^i p_-}\bar{h}^b\Gamma_n h^a(y)iv\cdot\partial[\bar{h}^a\bar{\Gamma}_n h^b(y)]\right\}\right|0\right\rangle. \tag{5.6}$$

The contribution involving only off shell quarks has the familiar form

$$\int d^4xe^{ipx}\langle 0|T\{[\bar{Q}^a\Gamma Q^b(0)]_{\rm off}[\bar{Q}^b\bar{\Gamma}Q^a(x)]_{\rm off}\}|0\rangle$$
$$=-iN_c{\rm tr}(\Gamma p_+\bar{\Gamma}p_-)|\psi_{ab,n}(0)|^2\frac{1}{v\cdot k+i\epsilon}. \tag{5.7}$$

The expressions (5.5) and (5.6) correspond to corrections $O(\Lambda^3_{\rm QCD}a^3_{ab,n})$ and $O(\Lambda^6_{\rm QCD}a^6_{ab,n})$ respectively to the leading result (5.7); $a_{ab,n}\sim n/(\alpha\mu_{ab})$ is the Bohr radius. Since we are only interested in the leading nonperturbative corrections we shall neglect (5.6) in the following. Let us only remark that the hadronization of the four-quark operator in (5.6) introduces new parameters. This is because it is not a generator of the U$(4N_{hf})$ symmetry as the currents of the kind (4.17) are.

The right-hand side of (5.5) can be hadronized and calculated using the effective Lagrangian discussed in Sec. IV. From (4.18) we obtain

$$\int d^4xe^{ipx}\langle 0|T\{[\bar{Q}^a\Gamma Q^b(0)]_{\rm off}[\bar{Q}^b\bar{\Gamma}Q^a(x)]_{\rm on}\}|0\rangle$$
$$=\frac{-i}{2}{\rm tr}(p_-\Gamma p_+\bar{\Gamma})\tilde{\psi}^*_{ab,n}(0)\psi_{ab,n}(0)f_H^2\frac{1}{v\cdot k+i\epsilon}. \tag{5.8}$$

Notice that the result is spin independent and the flavor dependence resides only in the wave function, which is known. We finally obtain

$$\int d^4xe^{ipx}\langle 0|T\{[\bar{Q}^a\Gamma Q^b(0)]_{\rm off}[\bar{Q}^b\bar{\Gamma}Q^a(x)]_{\rm on}\}|0\rangle$$
$$=\int d^4xe^{-ikx}\langle 0|T[C_\Gamma\bar{h}_-^a\Gamma h_+^b(0)\bar{h}_+^b\bar{\Gamma}h_-^a(x)]|0\rangle. \tag{5.5}$$

Analogously, using (2.14), (2.16), and (3.13) we have

$$|f_{\psi_Q(n)}|^2=4m_{ab,n}\left\{N_c|\psi_{ab,n}(0)|^2+\frac{1}{2}[\tilde{\psi}^*_{ab,n}(0)\psi_{ab,n}(0)\right.$$
$$+\psi^*_{ab,n}(0)\tilde{\psi}_{ab,n}(0)f_H^2,$$
$$|f_{\eta_Q(n)}|=\frac{|f_{\psi_Q(n)}|}{m_{ab,n}}. \tag{5.9}$$

Notice that the nonperturbative correction we find to the decay constant is $O(\Lambda^3_{\rm QCD}a^3_{ab,n})$ and hence presumably more important than the correction arising from the multipole expansion which is $O((\Lambda_{\rm QCD}a_{ab,n})^4/\alpha^2)$ [8,9] [we count the gluon condensate as $O(\Lambda^4_{\rm QCD})$].

B. Cutoff independence

Let us next discuss the important issue of the cutoff independence. Even though we have not written it down explicitly, the introduction of a cutoff to separate almost on-shell momenta from off-shell momenta is necessary. Of course, the final results must not depend on the particular value of the cutoff. At the short-distance end of the calculation, the cutoff must exclude momenta which are almost on shell. This is easily achieved by cutting off small momenta from the wave function

$$\psi_{ab,n}(0)=\int\frac{d^3\vec{k}}{(2\pi)^3}\tilde{\psi}_{ab,n}(\vec{k})\to\int_\mu\frac{d^3\vec{k}}{(2\pi)^3}\tilde{\psi}_{ab,n}(\vec{k})$$
$$=:\psi^{(\mu)}_{ab,n}(0), \tag{5.10}$$

where μ is a symmetric IR cutoff in three-momentum. The wave functions in (5.9) must be understood as the cutoff wave functions (5.10). On the HQET side the cutoff must be ultraviolet. It has already been displayed in the leading-order perturbative calculation at quark level in Sec. III. In particular, from (3.9) we obtain

$$\int d^4x e^{ipx}\langle 0|T\{[\bar{Q}^a\Gamma Q^b(0)]_{\rm off}[\bar{Q}^b\bar{\Gamma}Q^a(x)]_{\rm on}\}|0\rangle = \frac{-i}{2} N \,{\rm tr}(p_-\Gamma p_+\bar{\Gamma})\tilde{\psi}^*_{ab,n}(0)\psi_{ab,n}(0)\left(\frac{\mu^3}{6\pi^2}\right)\frac{1}{v\cdot k+i\epsilon}. \quad (5.11)$$

This strong cutoff dependence, however, is totally compensated by (5.10). Indeed, once (5.10) is used we have

$$\frac{d}{d\mu}|\psi^{(\mu)}_{ab,n}(0)|^2 = -\frac{\mu^2}{2\pi^2}[\tilde{\psi}^*_{ab,n}(\mu)\psi^{(\mu)}_{ab,n}(0) + \psi^{(\mu)*}_{ab,n}(0)\tilde{\psi}_{ab,n}(\mu)] = -\frac{\mu^2}{2\pi^2}[\tilde{\psi}^*_{ab,n}(0)\psi_{ab,n}(0) + \psi^*_{ab,n}(0)\tilde{\psi}_{ab,n}(0) + O((\mu a_{ab,n})^2)],$$

$$\frac{d}{d\mu}\left[\tilde{\psi}^*_{ab,n}(0)\psi^{(\mu)}_{ab,n}(0)\left(\frac{\mu^3}{6\pi^2}\right)\right] = \tilde{\psi}^*_{ab,n}(0)\psi_{ab,n}(0)\frac{\mu^2}{2\pi^2}[1+O((\mu a_{ab,n})^2)],$$

$$\frac{d}{d\mu}\left[\psi^{(\mu)*}_{ab,n}(0)\tilde{\psi}_{ab,n}(0)\left(\frac{\mu^3}{6\pi^2}\right)\right] = \psi^*_{ab,n}(0)\tilde{\psi}_{ab,n}(0)\frac{\mu^2}{2\pi^2}[1+O((\mu a_{ab,n})^2)]. \quad (5.12)$$

Notice that the way in which the cutoff dependence cancels is remarkable. The strong cutoff dependence of (5.11) was first found in [11]. It was not clear at all which short-distance contribution it should cancel against Equation (5.10) gives the solution to that puzzle. It is apparent from (5.8) and (5.11) that f_H in the hadronic theory plays the role of the UV cutoff in the HQET at quark level. From (5.12) it is clear that the cutoff μ must be much smaller than the inverse Bohr radius. Therefore our formalism becomes exact in the following situation:

$$m_a, m_b \gg 1/a_{ab,n} \gg \mu \gg \Lambda_{\rm QCD} \gg k', \qquad \frac{\mu_{ab}\alpha^2(1/a_{ab,n})}{n^2} \gg k'. \quad (5.13)$$

Furthermore, we have to assume that μ can be taken large enough so that we may enter the asymptotic freedom regime from the HQET side. Otherwise the matching we have carried out at tree level would not make much sense.

From the discussion above it should also be clear that (5.9) can be written in a cutoff independent way at $O((\mu a_{ab,n})^3)$ by just replacing

$$f_H^2 \to \bar{f}_H^2 := f_H^2 - \frac{N_c\mu^3}{3\pi^2}, \quad (5.14)$$

where $\bar{f}_H^2$ need not be positive.

C. Physical state normalization

There is still a subtle point which makes Eq. (5.9) with the replacement (5.14) not quite correct. It has to do with the normalization of physical states. It will be clear later on [see Eq. (6.14) below] that the states we obtain by this procedure do not have the standard relativistic normalization that they are supposed to. When we evaluate the Green function (5.1) we insert resolutions of the identity which are approximated by Coulomb-type states. This is all right. However, the low momentum tale of these states is cut off and substituted by a quantity evaluated using the effective hadronic theory. After doing so there is no guarantee that the resolution of the identity we introduced is still properly normalized. This can be fixed up by changing

$$\sum_n \int \frac{d^3\vec{P}_n}{(2\pi)^3 2P_n^0}|n\rangle\langle n| \to \sum_n \int \frac{d^3\vec{P}_n}{(2\pi)^3 2P_n^0}|n\rangle\langle n|^{(\mu)}N_n(\mu,f_H), \quad (5.15)$$

where $|n\rangle\langle n|^{(\mu)}$ symbolizes the cutoff Coulomb states whose low energy tale is evaluated in the hadronic effective theory. We present a heuristic calculation of $N_n(\mu,f_H)$.

We start from the Coulomb-type bound state (2.5) and separate high and low relative momentum according to

$$|\Gamma_n,\vec{P}_n = m_{ab,n}\vec{v}\rangle = |\Gamma_n,\vec{P}_n = m_{ab,n}\vec{v}\rangle^{k>\mu} + |\Gamma_n,\vec{P}_n = m_{ab,n}\vec{v}\rangle^{k<\mu}. \quad (5.16)$$

The high momentum part of the physical state can be well approximated by the Coulomb-type contribution so we may leave it as it stands. However, the low momentum part receives nonperturbative corrections, which we evaluate using the effective hadronic Lagrangian.

We proceed as follows. Since $a_{ab,n}\mu \ll 1$, we can approximate the low momentum region by

$$|\Gamma_n,\vec{P}_n = m_{ab,n}\vec{v}\rangle^{k<\mu} \simeq \frac{v^0}{\sqrt{N_c}}\frac{\tilde{\psi}_{ab,n}(\vec{0})}{\sqrt{2m_a v^0 2m_b v^0}}\frac{m_{ab}^{3/2}}{m_{ab,n}}\int^{k<\mu}\frac{d^3\vec{k}}{(2\pi)^3} \times \sum_{\alpha,\beta}\bar{u}^\alpha(p_1)\Gamma_n v^\beta(p_2)a^\dagger_\alpha(p_1)b^\dagger_\beta(p_2)|0\rangle. \quad (5.17)$$

Observe now that (5.17) is nothing but the integral of a local HQET current:

$$|\Gamma_n,\vec{P}_n = m_{ab,n}\vec{v}\rangle^{k<\mu} \simeq \frac{v^0}{\sqrt{N_c}}\tilde{\psi}_{ab,n}(\vec{0})\frac{m_{ab}^{3/2}}{m_{ab,n}}\int d^3\vec{x}e^{-ikx}\bar{h}^a\Gamma_n h^b(x)|0\rangle, \quad (5.18)$$

where $k\to 0$ and only low momenta are allowed.

At this point, we can hadronize the current [see (4.17)] and calculate the low momentum contribution to $N_n(\mu,f_H)$:

$$
{}^{k<\mu}\langle \Gamma_n, \vec{P}_n = m_{ab,n}\vec{v} | \Gamma_n, \vec{P}'_n = m_{ab,n}\vec{v}\,' \rangle^{k<\mu}
= 2 m_{ab,n} v^0 (2\pi)^3 \delta^{(3)}(m_{ab,n}(\vec{v}-\vec{v}\,'))
\times \frac{f_H^2}{2N_c} |\tilde{\psi}_{ab,n}(\vec{0})|^2. \tag{5.19}
$$

Then, putting together high and low momentum contributions, we have

$$
\langle \Gamma_n, \vec{P}_n = m_{ab,n}\vec{v} | \Gamma_n, \vec{P}'_n = m_{ab,n}\vec{v}\,' \rangle
= 2 m_{ab,n} v^0 (2\pi)^3 \delta^{(3)}(m_{ab,n}(\vec{v}-\vec{v}\,'))
\times \left[\int_{k>\mu} \frac{d^3\vec{k}}{(2\pi)^3} |\tilde{\psi}_{ab,n}(\vec{k})|^2 + \frac{f_H^2}{2N_c} |\tilde{\psi}_{ab,n}(\vec{0})|^2 \right]
= 2 m_{ab,n} v^0 (2\pi)^3 \delta^{(3)}(m_{ab,n}(\vec{v}-\vec{v}\,'))
\times \left[1 + \frac{\bar{f}_H^2}{2N_c} |\tilde{\psi}_{ab,n}(\vec{0})|^2 \right], \tag{5.20}
$$

where $\bar{f}_H^2$ is defined in (5.14). Notice that the result is cutoff independent.

Finally, the normalization factor reads

$$
N_n(\mu, f_H) = \frac{1}{1 + \frac{\tilde{f}_H^2}{2N_c} |\tilde{\psi}_{ab,n}(\vec{0})|^2}. \tag{5.21}
$$

$N_n(\mu, f_H)$ can also be obtained from requiring that

$$
\left\langle \Gamma_n, \vec{P}_n = m_{ab,n}\vec{v} \left| \int d^3\vec{x} \bar{Q}^b \gamma^0 Q^b(\vec{x}) \right| \Gamma_n, \vec{P}'_n = m_{ab,n}\vec{v}\,' \right\rangle
= 2 m_{ab,n} v^0 (2\pi)^3 \delta^{(3)}(m_{ab,n}(\vec{v}-\vec{v}\,')) \tag{5.22}
$$

as we shall see later on. Once we have taken into account the correct normalization, (5.9) reads

$$
|f_{\psi_Q(n)}|^2 = 4 m_{ab,n} \left[N_c |\psi_{ab,n}(0)|^2 + \frac{1}{2} [\tilde{\psi}^*_{ab,n}(0)\psi_{ab,n}(0)
+ \psi^*_{ab,n}(0)\tilde{\psi}_{ab,n}(0)] \bar{f}_H^2
- |\psi_{ab,n}(0)|^2 |\tilde{\psi}_{ab,n}(0)|^2 \frac{\bar{f}_H^2}{2} \right]. \tag{5.23}
$$

We shall relegate to Sec. VII the discussion on the applicability of the limit (5.13) and formula (5.23) to physical situations.

VI. MATRIX ELEMENTS AT ZERO RECOIL

We are interested in Green functions of the kind

$$
G_{\Gamma\Gamma'\Gamma''}(p_1, p_2)
= \int d^4x_1 d^4x_2 e^{ip_1x_1 + ip_2x_2} \langle 0 | T\{ \bar{Q}^a \Gamma'' Q^b(x_1)
\times \bar{Q}^b \Gamma Q^c(0) \bar{Q}^c \Gamma' Q^a(x_2) \} | 0 \rangle. \tag{6.1}
$$

For the momentum range

$$
p_1 = m_{ab,n} v + k'_1, \quad p_2 = -m_{ac,m} v - k'_2,
\quad k'_1, k'_2 \to 0. \tag{6.2}
$$

We separate each current in almost on-shell momenta and off-shell momenta according to (5.2). The leading contribution is given by the term

$$
G_{\Gamma\Gamma'\Gamma''}(p_1, p_2)
= \int d^4x_1 d^4x_2 e^{ip_1x_1 + ip_2x_2} \langle 0 | T\{ [\bar{Q}^a \Gamma'' Q^b(x_1)]_{\rm off}
\times [\bar{Q}^b \Gamma Q^c(0)]_{\rm off} [\bar{Q}^c \Gamma' Q^a(x_2)]_{\rm off} \} | 0 \rangle
= N_c {\rm tr}(p_- \Gamma'' p_+ \Gamma p_+ \Gamma') \psi^*_{ac,m}(0) \psi_{ab,n}(0)
\times \int \frac{d^3\vec{k}}{(2\pi)^3} \tilde{\psi}^*_{ab,n}(\vec{k}) \tilde{\psi}_{ac,m}(\vec{k}) \frac{1}{v\cdot k'_1 + i\epsilon} \frac{1}{v\cdot k'_2 + i\epsilon} \tag{6.3}
$$

and the next-to-leading contribution by the term

$$
G^{\rm on}_{\Gamma\Gamma'\Gamma''}(p_1, p_2) = G^{\rm on,1}_{\Gamma\Gamma'\Gamma''}(p_1, p_2) + G^{\rm on,2}_{\Gamma\Gamma'\Gamma''}(p_1, p_2)
+ G^{\rm on,3}_{\Gamma\Gamma'\Gamma''}(p_1, p_2), \tag{6.4}
$$

$$
G^{\rm on,1}_{\Gamma\Gamma'\Gamma''}(p_1, p_2)
= \int d^4x_1 d^4x_2 e^{ip_1x_1 + ip_2x_2} \langle 0 | T\{ [\bar{Q}^a \Gamma'' Q^b(x_1)]_{\rm on}
\times [\bar{Q}^b \Gamma Q^c(0)]_{\rm off} [\bar{Q}^c \Gamma' Q^a(x_2)]_{\rm off} \} | 0 \rangle, \tag{6.5}
$$

$$
G^{\rm on,2}_{\Gamma\Gamma'\Gamma''}(p_1, p_2)
= \int d^4x_1 d^4x_2 e^{ip_1x_1 + ip_2x_2} \langle 0 | T\{ [\bar{Q}^a \Gamma'' Q^b(x_1)]_{\rm off}
\times [\bar{Q}^b \Gamma Q^c(0)]_{\rm on} [\bar{Q}^c \Gamma' Q^a(x_2)]_{\rm off} \} | 0 \rangle, \tag{6.6}
$$

$$
G^{\rm on,3}_{\Gamma\Gamma'\Gamma''}(p_1, p_2)
= \int d^4x_1 d^4x_2 e^{ip_1x_1 + ip_2x_2} \langle 0 | T\{ [\bar{Q}^a \Gamma'' Q^b(x_1)]_{\rm off}
\times [\bar{Q}^b \Gamma Q^c(0)]_{\rm off} [\bar{Q}^c \Gamma' Q^a(x_2)]_{\rm on} \} | 0 \rangle. \tag{6.7}
$$

The calculation of (6.5) and (6.7) is analogous to the ones carried out in Sec. II. We obtain

$$
G^{\rm on,1}_{\Gamma\Gamma'\Gamma''}(p_1, p_2)
= \int d^4x_1 d^4x_2 e^{ik'_1x_1 - ik'_2x_2} i C_1 \langle 0 | T\{ \bar{h}^a_- \Gamma'' h^b_+(x_1)
\times \bar{h}^b_+ \Gamma p_+ \Gamma' h^a_-(0) \} | 0 \rangle \int d^4q \frac{e^{iqx_2}}{v\cdot q + i\epsilon}, \tag{6.8}
$$

$$
C_1 = \psi^*_{ac,m}(0) \tilde{\psi}_{ab,n}(0) \int \frac{d^3\vec{k}}{(2\pi)^3} \tilde{\psi}^*_{ab,n}(\vec{k}) \tilde{\psi}_{ac,m}(\vec{k}),
$$

$$G^{\mathrm{on},3}_{\Gamma\Gamma'\Gamma''}(p_1,p_2)$$
$$=\int d^4x_1 d^4x_2 e^{ik_1'x_1-ik_2'x_2} i C_3 \left\langle 0\left| T\left\{ \bar{h}^a_- \Gamma'' p_+ \Gamma h^c_+(0) \right.\right.\right.$$
$$\times \bar{h}^c_+ \Gamma' h^a_-(x_2)\Big| 0 \int d^4q \, \frac{e^{-iqx_1}}{v\cdot q + i\epsilon}, \tag{6.9}$$

$$C_3 = \tilde{\psi}^*_{ac,m}(0)\psi_{ab,n}(0)\int \frac{d^3\vec{k}}{(2\pi)^3}\,\tilde{\psi}^*_{ab,n}(\vec{k})\psi_{ac,m}(\vec{k}).$$

Notice that (6.5) and (6.7) cannot be written in terms of local Green functions in the HQET. One propagator must be kept explicit.

The calculation of (6.6) is more subtle. We describe it in some detail in the Appendix B. We obtain

$$G^{\mathrm{on},2}_{\Gamma\Gamma'\Gamma''}(p_1,p_2)$$
$$=\int d^4x_1 d^4x_2 e^{ik_1'x_1-ik_2'x_2} C_2 \langle 0| T\{\bar{h}^a_- \Gamma'' h^b_+(x_1)\bar{h}^b_+ \Gamma$$
$$\times h^c_+(0)\bar{h}^c_+ \Gamma' h^a_-(x_2)\}|0\rangle,$$
$$C_2 = \psi^*_{ac,m}(0)\psi_{ab,n}(0)\tilde{\psi}^*_{ac,m}(0)\tilde{\psi}_{ab,n}(0). \tag{6.10}$$

This term is the only one in (6.4) which remains in the matrix elements [see (6.14) below].

We calculate (6.8)–(6.10) using the hadronic effective Lagrangian [see formulas (4.18) and (4.19)]. We obtain

$$G^{\mathrm{on},1}_{\Gamma\Gamma'\Gamma''}(p_1,p_2) = C_1 \frac{f_H^2}{2}\,\mathrm{tr}(p_-\Gamma'' p_+ \Gamma p_+ \Gamma')\,\frac{1}{v\cdot k_1' + i\epsilon}$$
$$\times \frac{1}{v.k_2' + i\epsilon}, \tag{6.11}$$

$$G^{\mathrm{on},2}_{\Gamma\Gamma'\Gamma''}(p_1,p_2) = C_2 \frac{f_H^2}{2}\,\mathrm{tr}(p_-\Gamma'' p_+ \Gamma p_+ \Gamma')$$
$$\times \frac{1}{v\cdot k_1' + i\epsilon}\,\frac{1}{v\cdot k_2' + i\epsilon}, \tag{6.12}$$

$$G^{\mathrm{on},3}_{\Gamma'\Gamma''}(p_1,p_2) = C_3 \frac{f_H^2}{2}\,\mathrm{tr}(p_-\Gamma'' p_+ \Gamma p_+ \Gamma')$$
$$\times \frac{1}{v\cdot k_1' + i\epsilon}\,\frac{1}{v\cdot k_2' + i\epsilon}. \tag{6.13}$$

The matrix element at zero recoil then reads

$$\langle \Gamma_n, \vec{P}_n = m_{ab,n}\vec{v}|\bar{Q}^b \Gamma Q^c(0)|\Gamma_m, \vec{P}_m = m_{ac,n}\vec{v}\rangle$$
$$= -\sqrt{m_{ab,n} m_{ac,m}}\ \mathrm{tr}(\vec{\Gamma}_n \Gamma \Gamma_m)$$
$$\times\Bigg(\int \frac{d^3\vec{k}}{(2\pi)^3}\,\tilde{\psi}^*_{ab,n}(\vec{k})\tilde{\psi}_{ac,m}(\vec{k})$$
$$+ \frac{f_H^2}{2N_c}\,\tilde{\psi}_{ac,m}(0)\tilde{\psi}^*_{ab,n}(0)\Bigg), \tag{6.14}$$

$\Gamma_n = i\gamma_5 p_-$, $i\not{\epsilon}^i p$ for the pseudoscalar and vector particle respectively. The integral in (6.14) must be understood with an infrared cutoff μ. From (6.14) it is apparent that our physical states are not properly normalized. Indeed, for $b=c$ and $\Gamma=\gamma^0$ one should obtain (5.22) but one does not. The reason for this has been discussed at the end of Sec. V. The solution consists of introducing the normalization factor $N_n(\mu, f_H)$ defined in (5.21). The properly normalized result reads

$$\langle \Gamma_n, \vec{P}_n = m_{ab,n}\vec{v}|\bar{Q}^b \Gamma Q^c(0)|\Gamma_m, \vec{P}_m = m_{ac,m}\vec{v}\rangle$$
$$= -\sqrt{m_{ab,n} m_{ac,m}}\ \mathrm{tr}(\bar{\Gamma}_n \Gamma \Gamma_m)$$
$$\times\Bigg[\int \frac{d^3\vec{k}}{(2\pi)^3}\,\tilde{\psi}^*_{ab,n}(\vec{k})\tilde{\psi}_{ac,m}(\vec{k})$$
$$\times\left(1 - \frac{\bar{f}_H^2}{4N_c}|\tilde{\psi}_{ab,n}(0)|^2 - \frac{\bar{f}_H^2}{4N_c}|\tilde{\psi}_{ac,m}(0)|^2\right)$$
$$+ \frac{\bar{f}_H^2}{2N_c}\,\tilde{\psi}_{ac,m}(0)\tilde{\psi}^*_{ab,n}(0)\Bigg]. \tag{6.15}$$

Notice that the nonperturbative correction depends only on a single parameter $\bar{f}_H^2$ which may be extracted from the decay constants calculated in Sec. V. This is a nontrivial prediction which turns out to be a direct consequence of the U($4N_{hf}$) symmetry being spontaneously broken down to U($2N_{hf}$)$\otimes$U($2N_{hf}$).

VII. APPLICATIONS

If the charm and bottom mass were large enough we could apply the results above to the physics of Υ, η_b, B_c, B_c^*, J/ψ, and η_c. (The top is believed to be too heavy to form hadronic bound states and will be ignored.) We analyze in this section whether this is so or not. In the systems where the formalism actually applies, we are mainly interested in estimating the importance of the new nonperturbative contribution rather than in obtaining accurate results. The latter is a much harder task which is definitely beyond the scope of the present work.

Let us first focus on bottom. The fact that the almost on-shell momentum excitations in heavy quarkonium are Goldstone modes [11] implies that the Υ and η_b spectrum does not receive additional nonperturbative contributions. We may then extract the bottom mass from the Υ mass by means of the formulas given in [8,9], which take into account the leading order in the multipole expansion. Since we have established a link between quarkonium and the HQET we can next use m_b to extract Λ, the non perturbative parameter relating the mass of the B meson to m_b. Moreover, taking into account that Λ is flavor independent, we may next extract the charm mass m_c. We summarize the results in Table I.

In Table I the values we obtain for m_b are about 3% lower than those obtained in QCD sum rules [22] but compatible with a recent QCD-based evaluation [23] and with the lattice calculation [24]. The values we obtain for Λ are a bit lower but otherwise compatible with those extracted from QCD sum rules [6]. Our values for m_c are again about 6% lower

TABLE I. We use $\Lambda_{\rm QCD}$ as an input and take the one-loop running coupling constant α at the scale of the inverse Bohr radius, i.e., $\alpha=\alpha(1/a_{bb,0})$. For the gluon condensate we take the fixed value $\langle B^2\rangle=(585\ \text{MeV})^4$. The error in m_b has been taken from estimations of the hyperfine splittings $O(\alpha^2)$, which are also the main source of error in $\bar{\Lambda}$. For m_c the error comes both from $\bar{\Lambda}$ and the $1/m_c$ corrections. The last column gives our model-independent determination of m_{B_c}.

$\Lambda_{\rm QCD}$ (MeV)	m_b (MeV)	$\bar{\Lambda}$ (MeV)	m_c (MeV)	m_{B_c} (MeV)
200	4877±35	436±35	1539±70	6212±110
150	4843±35	470±35	1505±70	6242±110
100	4802±35	511±35	1464±70	6312±110

than the typical values in QCD sum rules [22]. We should emphasize that our numbers in Table I are model independent.

We can next extract the nonperturbative parameter $\bar{f}_H^2$ from f_Υ (this is done in Table II). We use

$$f_\Upsilon=2\sqrt{3m_\Upsilon}\psi_{bb,0}(0)\left[1+\frac{\tilde{\psi}_{bb,0}(0)\bar{f}_H^2}{6\psi_{bb,0}(0)}-\frac{|\tilde{\psi}_{bb,0}(0)|^2\bar{f}_H^2}{12}-\frac{8\alpha(m_b)}{3\pi}+8.77m_b^2\langle B^2\rangle\left(\frac{a_{bb,0}}{2}\right)^6\right., \tag{7.1}$$

where the one-loop QCD corrections and the leading correction from the multipole expansion[2] [9] are taken into account.

The numbers in Table II are very sensitive to the scale at which α is taken. Notice that we choose $\alpha=\alpha(1/a_{bb,0})$ in the Bohr radius and binding energy but $\alpha=\alpha(m_b)$ in the one-loop perturbative correction included in (7.1). From Table II we see that for the actual values of m_b and $\Lambda_{\rm QCD}=100$. 150 MeV the on-shell contribution ($\bar{f}_H$) does not dominate over the condensate but it is certainly sizeable. For $\Lambda_{\rm QCD}=200$ MeV all corrections are about the same order and for any value of m the "on-shell" contribution dominates over the condensate.

Observe that the conditions (5.13), in particular $a_{bb,0}^{-1}\gg\mu\gg\Lambda_{\rm QCD}$, may be considered as reasonably well fulfilled if we take the cutoff $\mu\sim700$ MeV (see Table III below).

Let us next turn our attention to charm. The charm mass is known not to be heavy enough for the multipole expansion to work [8]. This means that the nonperturbative contribution overwhelms the perturbative one. Therefore any approximation whose leading order is a perturbative contribution, like our approach, will not be able to say much about charmonium. In particular, for the on-shell contributions the difficulty lies on the second-to-last condition in (5.13) being fulfilled. There is little room to accommodate the cutoff μ between the inverse Bohr radius and $\Lambda_{\rm QCD}$, as should be clear from Table III. We refrain from giving any numbers for charmonium.

[2]We use the formula given in Ref. [9] which differs from the ones in Ref. [8].

TABLE II. We display the relative weight, with its sign, of the one-loop $[\alpha(m_b)]$, the condensate ($\langle B^2\rangle$) and the "on-shell" ($\bar{f}_H$) contribution with respect to the Coulomb-type contribution (normalized to 1). The last columns display the mass $m_{\rm cr}$ from which the "on-shell" contribution dominates over the condensate and the value of $(\bar{f}_H^2)^{1/3}$.

$\Lambda_{\rm QCD}$ (MeV)	$\alpha(m_b)$	$\langle B^2\rangle$	$\bar{f}_H$	$m_{\rm cr}$ (GeV)	$(\bar{f}_H^2)^{1/3}$ (MeV)
200	−0.19	0.10	−0.11	-	260
150	−0.17	0.19	−0.08	90	210
100	−0.15	0.41	−0.12	160	210

Unfortunately, the situation is not much better for the B_c, which has received considerable attention lately [25–27]. Nonetheless, once we have $\bar{f}_H^2$, we shall give some numbers in this case in Table IV.

From Table IV we see that for $\Lambda_{\rm QCD}=100$, 150 MeV the contribution of the condensate is too large for the approach to be reliable. For $\Lambda_{\rm QCD}=200$ MeV we are at the boundary of its validity since the on-shell correction is large. We may thus give a rough estimate for f_{B_c} only for $\Lambda_{\rm QCD}\sim200$ MeV, which turns out to be compatible with the estimate obtained by QCD sum rules [26], but about 30% lower than potential model estimates [27].

From Table V it follows that the new nonperturbative contribution is not very important in the matrix elements between Υ-B_c states.

The decay constants and matrix elements above receive contributions from corrections of several types: (i) QCD perturbative corrections to the Coulomb potential $O(\alpha(1/a_n))$. These have been evaluated at one loop level in [28] (see also [23]); (ii) relativistic corrections to the Coulomb potential $O(\alpha(1/a_n))$ (see also [28,23]); (iii) QCD perturbative corrections to the Green functions $O(\alpha(m))$. These corrections have been taken into account in (7.1). They correspond to the only QCD corrections in heavy-light systems. In our case they are important for the calculation of matrix elements at nonzero recoil. (iv) Nonperturbative corrections arising from the multipole expansion in the off-shell momentum region $O(\Lambda_{\rm QCD}^4 a_n^4/\alpha^2(1/a_n))$ [8,9]. These corrections have also been taken into account in (7.1). (v)Finite mass corrections $O(\Lambda_{\rm QCD}^2/m)$ in the hadronic HQET Lagrangian.

VIII. CONCLUSIONS

We have demonstrated that, contrary to the common belief, HQET techniques are also useful for the study of systems composed of two heavy quarks. In particular, we have

TABLE III. We give the $\bar{c}c,\bar{b}c,\bar{b}b$ inverse Bohr radius as a function of $\Lambda_{\rm QCD}$.

$\Lambda_{\rm QCD}$ (MeV)	$1/a_{cc,0}$ (MeV)	$1/a_{bc,0}$ (MeV)	$1/a_{bb,0}$ (MeV)
200	630	790	1240
150	540	700	1120
100	450	590	980

TABLE IV. We display the analogy to Table II for B_c. We have also given our predictions for f_{B_c} in the last column.

Λ_{QCD} (MeV)	$\alpha(2\mu_{bc})$	$\langle B^2\rangle$	$\bar{f}_H$	f_{B_c} (MeV)
200	−0.24	0.35	−0.44	370
150	−0.22	0.74	−0.34	540
100	−0.19	1.93	−0.54	780

identified new nonperturbative contributions to the decay constants and to certain matrix elements which are described by a hadronic Lagrangian based on the HQET. All these new contributions are parametrized at leading order by a single constant f_H. This is nontrivial and can be traced back to the fact that a $U(4N_{hf})$ symmetry is spontaneously broken down to $U(2N_{hf})\otimes U(2N_{hf})$.

It is remarkable that strong cutoff dependencies coming from a totally different origin match perfectly. Indeed, at the off-shell end the cutoff arises from an integral over a Coulomb type wave function, whereas at the on-shell end it arises from a Feynman integral.

We should also stress that we have been able to put in the same context (i.e., the HQET) both heavy-heavy and heavy-light systems. This allows for a model independent determination of heavy quark masses from quarkonium, which may then be used to extract the parameter $\bar{\Lambda}$ relating the mass of the heavy-light systems to the mass of the heavy quark.

As far as practical applications are concerned, our formalism is suitable for the ground state of the Υ and η_b family. Unfortunately the charm mass is too small for the formalism to become applicable in general to J/Ψ and B_c systems. Nevertheless one may stretch it in some cases to obtain information on the mass and decay constant of the latter.

Let us finally mention that the hadronic HQET Lagrangian can easily incorporate heavy-light mesons. The formalism can then be extended to the calculation of matrix elements between quarkonium and heavy-light systems. The leading nonperturbative contributions to those are also given by f_H and another nonperturbative parameter which is related to heavy-light decay constants. Nonrecoil contributions can also be evaluated within the formalism.

Note added in proof. We have presented a technique which allows one to disentangle the on-shell contributions from the rest and match them to the HQET. The matching has been carried out at the tree level. We have already shown in [29] that the matching also goes through at the one loop level. Nevertheless, a word of caution is needed. It would be desirable to have a more direct and systematic derivation of these results from QCD. Progress in this direction is being made [30].

TABLE V. We give the relative weight, with its sign, of the "on-shell" contribution with respect to the Coulomb-type contribution (normalized to 1) in the matrix elements (6.15) between Υ-B_c states.

Λ_{QCD} (MeV)	200	150	100
$B_c-\Upsilon$	−0.10	−0.08	−0.14

ACKNOWLEDGMENTS

This work has been supported in part by CICYT grant AEN93-0695. A.P. acknowledges financial support from CIRIT. J.S. has benefited from the Fermilab Summer Visitors Program. He thanks E. Eichten for illuminating conversations and J. M. Sanchis for sending his papers [7] prior to publication. Thanks are also given to J. L. Goity and P. Pascual for the critical reading of the manuscript.

APPENDIX A: A TOY MODEL

Because of the similarity, both in physics and techniques, to the chiral perturbation theory it is interesting to consider a toy model which contains the on-shell contributions only. At quark level the model is described by the HQET with quarks and antiquarks with the same velocity as in Sec. III. At hadronic level it is described by the effective hadronic Lagrangian of Sec. IV.

Within this model, the interactions between $(\eta_Q,\eta_{Q'})$, $(\eta_Q,\psi_{Q'})$, and $(\psi_Q,\psi_{Q'})$, when the two particles move roughly at the same velocity, are described by a single unknown constant. This is analogous to the fact that at lowest order in 3-flavor chiral perturbation theory the elastic scattering of (π,π), (K,K), and (π,K) is also described by a single constant. When heavy–light mesons are included in the effective Lagrangian the same constant describes the elastic scattering of heavy–light mesons with quarkonium. This is also analogous to the fact that the local vertex $\pi-\pi-N-N$ at leading order in the chiral Lagrangian is described by the same constant as the (π,π) elastic scattering. Let us mention at this point that when one actually calculates the scattering amplitudes, one obtains zero. This has to do with the universality of the leading-order effective Lagrangians for Goldstone modes [18–20]. Any theory undergoing a $U(4N_{hf})$ spontaneous symmetry breaking down to $U(2N_{hf})\otimes U(2N_{hf})$ has the same low energy effective Lagrangian (4.12) provided the rest of the symmetries in the theory are also the same. It was shown in [11], that even when the gluons are switched off, spontaneous symmetry breaking occurs in the HQET. In that case there is no interaction in the fundamental theory and hence it is not surprising that the scattering amplitudes in the effective Lagrangian vanish. Universality implies that there will be vanishing scattering amplitudes when the gluons are switched on as well.

Within this model one can also treat $1/m$ corrections in a way similar to the one in which quarks masses are dealt with in chiral perturbation theory. At the quark level the leading $1/m$ corrections to the HQET are given by a kinetic term

$$-\sum_{a=1}^{N_{hf}}\frac{1}{2m_a}D_i\bar{h}^aD_ih^a \tag{A1}$$

and a spin-breaking term

$$\sum_{a=1}^{N_{hf}}\frac{1}{4m_a}\bar{h}^aS^lG^lh^a,$$

$$G^l=-\frac{1}{2}\epsilon^{jkl}e_j^{\mu}e_k^{\nu}G_{\mu\nu}. \tag{A2}$$

The kinetic term (A1) does not break the global $U(2N_{hf})\otimes U(2N_{hf})$ symmetry but it breaks its local version. In order to construct at the hadronic level terms which break the $U(4N_{hf})$ symmetry in the same fashion as (A1) does, we introduce the $u(4N_{hf})$-valued sources ϕ and a_i transforming as

$$\phi \to g\phi g^{-1},$$
$$a_i \to g a_i g^{-1} + g\partial_i g^{-1}. \quad \text{(A3)}$$

Then the term

$$-d_i \bar{h} \not{v} \phi d_i h,$$
$$d_i h := (D_i + a_i)h, \quad d_i \bar{h}\not{v} := D_i \bar{h}\not{v} - \bar{h}\not{v} a_i \quad \text{(A4)}$$

is on one hand invariant under $U(4N_{hf})$ and on the other reduces to (A1) upon setting

$$a_i = 0, \quad \phi = \begin{pmatrix} \frac{1}{2m_a} & & \\ & \frac{1}{2m_b} & \\ & & \ddots \end{pmatrix} \not{v}. \quad \text{(A5)}$$

At the hadronic level, we must then construct invariant terms linear in ϕ, which may also contain a_i. Up to two space derivatives we have

$$\mathrm{tr}(S\phi), \quad \text{(A6)}$$
$$\mathrm{tr}(S\phi d_i S d_i S), \quad \text{(A7)}$$
$$\mathrm{tr}(S\phi)\mathrm{tr}(d_i S d_i S), \quad \text{(A8)}$$
$$d_i S := \partial_i S + a_i S - S a_i .$$

We have not written down terms which coincide or vanish upon using (A5).

For the spin breaking term (A2) we may introduce a $u(4N_{hf})$-valued source R^l transforming as

$$R^l \to g R^l g^{-1} \quad \text{(A9)}$$

so that (A2) is substituted by

$$\bar{h}\not{v} R^l G^l h. \quad \text{(A10)}$$

We recover (A2) upon setting

$$R^l = \begin{pmatrix} \frac{1}{4m_a} & & \\ & \frac{1}{4m_b} & \\ & & \ddots \end{pmatrix} \not{v} S^l. \quad \text{(A11)}$$

There are no terms at the hadronic level with the same symmetry transformation properties at lower orders in derivatives. The first possible term appears at third order.

Therefore the leading $1/m$ corrections introduce three new parameters. Equation (A6) provides a mass term $O(\Lambda^2_{\rm QCD}/m)$ and (A7) and (A8) give rise to the usual nonrelativistic kinetic term. The procedure above can easily be extended to any finite order in $1/m$.

APPENDIX B: DETAILS OF SEC. VI

We present in this appendix some technical details on the evaluation of the off-shell short distance effects carried out in Sec. VI.

Consider the following matrix element

$$\langle \Gamma_n, m_{ab,n}\vec{v} | \bar{Q}^b \Gamma Q^c(x) | \Gamma_m, m_{ac,n}\vec{v}\rangle. \quad \text{(B1)}$$

Since two different bound states are involved, it is not clear *a priori* which c.m. dependence one should subtract before using (2.5). Nevertheless, translation invariance implies that the result of the calculation must satisfy

$$\langle \Gamma_n, m_{ab,n}\vec{v} | \bar{Q}^b \Gamma Q^c(x+a) | \Gamma_m, m_{ac,n}\vec{v}\rangle$$
$$= e^{i m_{ab,n} v\cdot a - i m_{ac,m} v\cdot a} \langle \Gamma_n, m_{ab,n}\vec{v} | \bar{Q}^b \Gamma Q^c(x) | \Gamma_m, m_{ac,n}\vec{v}\rangle. \quad \text{(B2)}$$

We also have

$$\langle \Gamma_n, m_{ab,n}\vec{v} | \bar{Q}^b \Gamma Q^c(x) | \Gamma_m, m_{ac,n}\vec{v}\rangle$$
$$= e^{i m_{ab,n} v\cdot \xi - i m_{ac,m} v\cdot \xi}$$
$$\times \langle \Gamma_n, m_{ab,n}\vec{v} | \bar{Q}^b \Gamma Q^c(x-\xi) | \Gamma_m, m_{ac,n}\vec{v}\rangle. \quad \text{(B3)}$$

If we assign $\xi \to \xi + a$ under translations (B3) fulfills (B2). If we also require ξ to be a linear function of x, then necessarily $\xi = x$ and the result is well defined:

$$\langle \Gamma_n, m_{ab,n}\vec{v} | \bar{Q}^b \Gamma Q^c(x) | \Gamma_m, m_{ac,n}\vec{v}\rangle$$
$$= e^{i m_{ab,n} v\cdot x - i m_{ac,m} v\cdot x} \langle \Gamma_n, m_{ab,n}\vec{v} | \bar{Q}^b \Gamma Q^c(0) | \Gamma_m, m_{ac,n}\vec{v}\rangle$$
$$= e^{i m_{ab,n} v\cdot x - i m_{ac,m} v\cdot x}$$
$$\times \left(-\mathrm{tr}(\bar{\Gamma}_n \Gamma \Gamma_m) \int \frac{d^3\vec{k}}{(2\pi)^3} \tilde{\psi}^*_{ab,n}(\vec{k}) \tilde{\psi}_{ac,m}(\vec{k}) \right). \quad \text{(B4)}$$

Consider next

$$\langle \Gamma_n, m_{ab,n}\vec{v} | \bar{Q}^{bi_3}_{\alpha_3}(x_3) Q^{ci_4}_{\alpha_4}(x_4) | \Gamma_m, m_{ac,n}\vec{v}\rangle. \quad \text{(B5)}$$

We are in a similar situation as above. However now translation invariance does not completely fix the result. Under the same assumptions we obtain

$$\langle \Gamma_n, m_{ab,n}\vec{v} | \bar{Q}^{bi_3}_{\alpha_3}(x_3) Q^{ci_4}_{\alpha_4}(x_4) | \Gamma_m, m_{ac,m}\vec{v}\rangle$$
$$= e^{i[\alpha x_3 + (1-\alpha)x_4][E_{ac,m} - E_{ab,n}] + i m_b v\cdot x_3 - i m_c v\cdot x_4}$$
$$\times \left\{ -\frac{1}{N_c} (\Gamma_m \bar{\Gamma}_n)_{\alpha_4\alpha_3} \delta_{i_3 i_4} \int \frac{d^3\vec{k}}{(2\pi)^3} \tilde{\psi}^*_{ab,n}(\vec{k}) \right.$$
$$\times \tilde{\psi}_{ac,m}(\vec{k}) \exp i \left[\vec{k}\cdot\vec{v}(x_3^0 - x_4^0) - (\vec{x}_3 - \vec{x}_4) \right.$$
$$\left.\left. \times \left(\vec{k} + \frac{\vec{k}\cdot\vec{v}}{1+v^0}\vec{v} \right) \right] \right\}, \quad \text{(B6)}$$

where α is arbitrary and parametrizes the ambiguity. Usually one never runs into calculations of the kind (B5) but rather of matrix elements of currents as in (B1), which are not ambiguous. We find expressions like (B5) in our calculation because we insist on enforcing on shellness in certain currents. In our concrete case we have a current with a momentum insertion

$$\bar{Q}^b \Gamma Q^c(x) e^{ip\cdot x}, \quad p=(-m_b+m_c+E_{ab,n}-E_{ac,m})v. \tag{B7}$$

In order to enforce on shellness we substitute it by

$$\bar{Q}^{bi_3}_{\alpha_3}(x_3) Q^{ci_4}_{\alpha_4}(x_4) e^{ip_3x_3+ip_4x_4}(\Gamma)_{\alpha_3\alpha_4}\delta_{i_3i_4},$$

$$p_3=-\left(m_b-\frac{m_b}{m_{ab}}E_{ab,n}\right)v-k'_3,$$

$$p_4=\left(m_c-\frac{m_c}{m_{ac}}E_{ac,m}\right)v+k'_4, \quad k'_3,k'_4\to 0, \tag{B8}$$

as mentioned in (5.3). However in doing so there is a momentum mismatch

$$\left(\frac{m_a}{m_{ab}}E_{ab,n}-\frac{m_a}{m_{ac}}E_{ac,m}\right)v$$

which should be fixed somehow in order to get (B7) back in the $x_3=x_4=x$ limit. The most general way of distributing this momentum mismatch between x_3 and x_4 is by inserting, in (B8),

$$\exp\left[i[\beta x_3+(1-\beta)x_4]\left(\frac{m_a}{m_{ab}}E_{ab,n}-\frac{m_a}{m_{ac}}E_{ac,m}\right)v\right]. \tag{B9}$$

Any β is equally good since we are eventually interested in the limit $x_3=x_4=x$. Notice that the ambiguity in α in (B6) is proportional to the ambiguity in β in (B9). Since we can choose β at will, we do it in such a way that the dependence in both α and β cancels. This is how we are able to obtain a representation of (6.6) in terms of the HQET (6.10).

[1] M. B. Voloshin and M. A. Shifman, Yad. Fiz. **45**, 463 (1987) [Sov. J. Nucl. Phys. **45**, 292 (1987)]; H. D. Politzer and M. B. Wise, Phys. Lett. B **206**, 681 (1988); **208**, 504 (1988).

[2] N. Isgur and M. B. Wise, Phys. Lett. B **232**, 113 (1989); **237**, 527 (1990).

[3] E. Eichten and B. Hill, Phys. Lett. B **234**, 511 (1990).

[4] H. Georgi, Phys. Lett. B **240**, 447 (1990).

[5] B. Grinstein, Nucl. Phys. **B339**, 253 (1990).

[6] B. Grinstein, in *High Energy Phenomenology,* Proceedings of the Workshop, Mexico City, Mexico, 1991, edited by M. A. Perez and R. Huerta (World Scientific, Singapore, 1992), pp. 161–216; in *Intersections Between Particle and Nuclear Physics*, Proceedings of the Conference, Tucson, Arizona, 1991, edited by N. van Oers, AIP Conf. Proc. No. 243 (AIP, New York, 1992), pp. 112–126; T. Mannel, Chin. J. Phys. **31**, 1 (1993); M. Neubert, Phys. Rep. **245**, 259 (1994).

[7] R. Casalbuoni, A. Deandrea, N. Di Bartolomeo, R. Gatto, F. Ferruglio, and G. Nardulli, Phys. Lett. B **309**, 163 (1993); **302**, 95 (1993); M. A. Sanchis, *ibid.* **312**, 333 (1993); Z. Phys. C **62**, 271 (1994).

[8] H. Leutwyler, Phys. Lett. **98B**, 447 (1981).

[9] M. B. Voloshin, Nucl. Phys. **B154**, 365 (1979); Yad. Fiz. **36**, 247 (1982) [Sov. J. Nucl. Phys. **36**, 143 (1982)].

[10] J. Soto and R. Tzani, Phys. Lett. B **297**, 358 (1992).

[11] J. Soto and R. Tzani, Int. J. Mod. Phys. A **9**, 4949 (1994).

[12] A. Das and V. S. Mathur, Phys. Rev. D **49**, 2508 (1994); A. Das and M. Hott, Mod. Phys. Lett. A **9**, 2217 (1995).

[13] T. Mannel, W. Roberts, and Z. Ryzak, Nucl. Phys. **B368**, 204 (1992).

[14] A. F. Falk, H. Georgi, B. Grinstein, and M. B. Wise, Nucl. Phys. **B343**, 1 (1990).

[15] B. Grinstein, W. Kilian, T. Mannel, and M. Wise, Nucl. Phys. **B363**, 19 (1991); W. Kilian, T. Mannel, and T. Ohl, Phys. Lett. B **304**, 311 (1993).

[16] W. Lucha, F. F. Schoberl, and D. Gromes, Phys. Rep. **200**, 127 (1991).

[17] F. E. Close and Z. Li, Phys. Lett. B **289**, 143 (1992).

[18] S. Coleman, J. Wess, and B. Zumino, Phys. Rev. **177**, 2239 (1969); C. Callan, S. Coleman, J. Wess, and B. Zumino, *ibid.* **177**, 2247 (1969).

[19] H. Leutwyler, Ann. Phys. (N.Y.) **235**, 165 (1994).

[20] H. Leutwyler, Phys. Rev. D **49**, 3033 (1994).

[21] J. Gasser and H. Leutwyler, Ann. Phys. (N.Y.) **158**, 142 (1984); Nucl. Phys. **B250**, 465 (1985).

[22] S. Narison, *QCD Spectral Sum Rules*, Lecture Notes in Physics Vol. 26 (World Scientific, Singapore, 1989).

[23] S. Titard and F. J. Yndurаín, Phys. Rev. D **49**, 6007 (1994).

[24] C. T. H. Davies, K. Hornbostel, A. Langnau, G. P. Lepage, A. Lidsey, C. J. Morningstar, J. Shigemitsu, and J. Sloan, Phys. Rev. Lett. **73**, 2654 (1994).

[25] E. Jenkins, M. Luke, A. V. Manohar, and M. Savage, Nucl. Phys. **B390**, 463 (1993).

[26] E. Bagan, H. G. Dosch, P. Gosdzinsky, S. Narison, and J. M. Richard, Z. Phys. C **64**, 57 (1994).

[27] E. J. Eichten and C. Quigg, Phys. Rev. D **49**, 5845 (1994); G. S. Gershtein, V. V. Kiselev, A. K. Likhoded, and A. V. Tkabladze, *ibid*. **51**, 3613 (1995).

[28] J. Pantaleone, S. H. Henry Tye, and Y. J. Ng, Phys. Rev. D **33**, 777 (1986).

[29] A. Pineda and J. Soto, Phys. Lett. B **361**, 95 (1995).

[30] A. Pineda and J. Soto (in progress).