Analyticity properties and unitarity constraints of heavy meson form factors

Eduardo de Rafael
Centre de Physique Théorique, Centre National de la Recherche Scientifique Luminy, Case 907, F-13288 Marseille Cedex 9, France

Josep Taron
Department of Physics, The Ohio State University, 174 West 18th Avenue, Columbus, Ohio 43210-1106
(Received 11 January 1994)

We derive new bounds on the b-number form factor $F(q^2)$ of the B meson for q^2 values relevant to the kinematics of the decays $\bar{B} \to Dl\bar{\nu}_l$ and $\bar{B} \to D^* l\bar{\nu}_l$. The new bounds take into account the experimentally known properties of the Υ states below the onset of the physical $\bar{B}B$ threshold.

PACS number(s): 13.20.He, 12.39.Hg, 14.40.Gx, 14.40.Nd

I. INTRODUCTION

The possibility of obtaining model-independent bounds [1] on the b-number form factor $F(q^2)$ of the B meson,

$$\langle B(p')|\bar{b}\gamma^\mu b|B(p)\rangle = (p+p')^\mu \, F[q^2 = (p-p')^2], \quad (1)$$

has recently attracted some attention (Refs. [2–8]). The interest of these bounds for phenomenology lies in their relevance to the semileptonic B decays

$$\bar{B} \to Dl\bar{\nu}_l \quad \text{and } \bar{B} \to D^* l\bar{\nu}_l. \quad (2)$$

It has been shown [9] that in the limit of very large b and c quark masses, there are new approximate symmeries of QCD which allow one to express the six form factors which govern these decays in terms of the b-number form factor $F(q^2)$ in (1) alone. The conservation of b number by the strong interactions implies

$$F(q^2 = 0) = 1. \quad (3)$$

Further model-independent information on this form factor, e.g., about its slope at the origin, would be very useful to extract the value of the mixing matrix element $|V_{cb}|$ from the data [10,11].

The bounds proposed in Ref. [1] are based on very general QCD properties of the two-point function $\Pi(q^2)$ defined as

$$(q^\mu q^\nu - q^2 g^{\mu\nu})\Pi(q^2) = i\int d^4x \, e^{iq\cdot x}\langle 0|T(V^\mu(x)V^\nu(0))|0\rangle, \quad (4a)$$

with

$$V^\mu = \bar{b}(x)\gamma^\mu b(x) \, , \quad (4b)$$

as well as on analyticity properties which the b-number form factor $F(q^2)$ in Eq. (1) is *assumed* to satisfy in the large b-quark mass limit. In QCD, the function $\Pi(q^2)$ obeys a once subtracted dispersion relation. It is therefore convenient to consider the first derivative of $\Pi(q^2)$ ($Q^2 = -q^2$, our metric is $+---$, $Q^2 > 0$ corresponds to the spacelike region),

$$\chi(Q^2) = -\frac{\partial \Pi(Q^2)}{\partial Q^2} = \int_0^\infty dt \frac{1}{(t+Q^2)^2}\frac{1}{\pi}\mathrm{Im}\Pi(t), \quad (5)$$

with Im $\Pi(t)$ the b-number spectral function defined by the relation

$$(q^\mu q^\nu - g^{\mu\nu}q^2)\mathrm{Im}\Pi(q^2) = \frac{1}{2}\sum_\Gamma \int d\mu(\Gamma)(2\pi)^4\delta^{(4)}\left(q - \sum_\Gamma p\right) \langle 0|V^\mu(0)|\Gamma\rangle\langle\Gamma|V^\nu(0)|0\rangle, \quad (6)$$

where the summation is extended to all possible hadronic states Γ with the quantum numbers of the V^μ current, and with the integral over phase space extended to each intermediate state.

Two bounds were derived in Ref. [1]. The first bound follows from the saturation of the right-hand side (RHS) in Eq. (6) with the lowest $\bar{B}B$ state. Since each hadronic state contributes positively to the spectral function, we have, in this case,

$$\frac{1}{\pi}\mathrm{Im}\Pi(t) \geq \frac{1}{16\pi^2}\frac{1}{3}\left(1 - \frac{4M_B^2}{t}\right)^{3/2}|F(t)|^2\theta(t - 4M_B^2), \quad (7)$$

with $F(t)$ the same b-number form factor as in (1). The second bound takes also into account the other two-meson states $\bar{B}B^*$, $\bar{B}^*B$, and $\bar{B}^*B^*$, *assuming further* that, in the large b-quark mass limit, the four states

$\bar{B}B$, $\bar{B}B^*$, $\bar{B}^*B$, and $\bar{B}^*B^*$ are related to each other by the resulting new spin-flavor symmetry. (The assumption here is in fact similar to the one previously made in Refs. [12,13] to predict the ratios of $e^+e^- \to \bar{B}B, \bar{B}B^*, \bar{B}^*B, \bar{B}^*B^*$ cross sections.) Needless to say, the second bound is stronger than the first, and, when compared to the existing model-dependent determination of the $F(q^2)$ form factor [14] and the fits to the present experimental data, turns out to be surprisingly restrictive [15]. This has prompted several authors to reconsider critically some of the assumptions which were made in [1].

The basic criticism of Refs. [2–5] focuses on the analyticity properties which in Ref. [1] are implicitly attributed to the b-number form factor $F(q^2)$ in the large b-quark mass limit, and in particular the neglect of the effect of heavy-heavy "quarkonium" states below the $4M_B^2$ threshold. None of these references, however, offers the derivation of new useful bounds compatible with the *rectified* analyticity properties. The main purpose of this article is to show how to derive new bounds on $F(q^2)$, for q^2 values relevant to the kinematics of the decays in (2), with inclusion of the experimentally known properties of the Υ states below the onset of the physical $\bar{B}B$ threshold.

The paper is organized as follows. Section II reviews general positivity properties of the b-number spectral function, as well as the analyticity properties of the b-number form factor of the B meson. The new bounds on this form factor, in the presence of the Υ states below the onset of the physical $\bar{B}B$ threshold, are derived in Sec. III. Section IV discusses a number of observations relevant to the heavy quark effective theory (HQET) which stem from this work. The technical details to derive the new bounds are explained in the Appendix.

II. UNITARITY CONSTRAINTS, ANALYTICITY PROPERTIES, AND QCD

The starting point is the dispersion relation in Eq. (5) and the positivity property of the b-number spectral function defined in Eq. (6). The contribution to this spectral function from each of the $\bar{B}B$ intermediate states B^-B^+, $\bar{B}^0B^0$, and $\bar{B}_s^0B_s^0$ is the same in the limit where the light quark mass differences are neglected. As pointed out to us by Broadhurst (and emphasized in Refs. [6,7]), this brings a factor n_f which counts the number of light flavors in the RHS of Eq. (7): i.e.,

$$\frac{1}{\pi}\mathrm{Im}\Pi(t) \geq \frac{n_f}{16\pi^2}\frac{1}{3}\left(1-\frac{4M_B^2}{t}\right)^{3/2}|F(t)|^2\theta(t-4M_B^2). \tag{8}$$

If necessary, one can improve this inequality by explicitly taking into account the contribution to the b-number spectral function from the Υ states below the two-meson threshold. (We refrain from including the contribution of the Υ states above threshold as well, because of the danger of possible double counting with the $B\bar{B}$ continuum.) These contributions can be extracted from the $e^+e^- \to \Upsilon$ experimental cross section, due to the fact that the hadronic electromagnetic current brings in the b-number current via the term $-1/3\ \bar{b}\gamma^\mu b$. Using the simple parametrization

$$\sigma(e^+e^- \to \Upsilon) = 12\pi^2\delta(t-M_\Upsilon^2)\frac{\Gamma_\Upsilon^{ee}}{M_\Upsilon} \tag{9}$$

and the relation ($-e/3$ is the b-quark electric charge)

$$\sigma(e^+e^- \to \Upsilon) = \frac{4\pi^2\alpha}{t}\frac{e^2}{9}\frac{1}{\pi}\mathrm{Im}\Pi_\Upsilon(t), \tag{10}$$

we can improve Eq. (8) to

$$\frac{1}{\pi}\mathrm{Im}\Pi(t) \geq \frac{27}{4\pi\alpha^2}\sum_i M_{\Upsilon_i}\Gamma_{\Upsilon_i}^{ee}\delta(t-M_{\Upsilon_i}^2) + \frac{n_f}{16\pi^2}\frac{1}{3}\left(1-\frac{4M_B^2}{t}\right)^{3/2}|F(t)|^2\theta(t-4M_B^2). \tag{11}$$

Inclusion of the contribution from other two-meson intermediate states to the b-number spectral function necessarily requires additional dynamical assumptions at the present stage. As pointed out in Ref. [16], the HQET cannot be reliably applied to relate the various physical amplitudes: $\langle\bar{B}B|V^\mu|0\rangle$, $\langle\bar{B}^*B|V^\mu|0\rangle$, and $\langle\bar{B}^*B^*|V^\mu|0\rangle$ in the timelike region. We shall therefore limit ourselves to the derivation of a new bound based on the inequality (11) only.

The function $\chi(Q^2)$, for a heavy quark mass m_b and spacelike values $Q^2 \geq 0$, can be reliably computed using QCD perturbation theory. At the one-loop level, asymptotic freedom results in (N_c = number of colors)

$$\chi(Q^2) = \frac{N_c}{4\pi^2}\int_0^1 dx\frac{2x^2(1-x)^2}{m_b^2+x(1-x)Q^2}\ . \tag{12}$$

The knowledge of this function and the lower bound for the spectral function in (11) inserted into the dispersion relation in (5) lead to the unitary inequality

$$16\pi^2M_B^2\chi(Q^2) \geq \frac{27\pi}{4\alpha^2}\sum_i\frac{M_{\Upsilon_i}\Gamma_{\Upsilon_i}^{ee}}{M_B^2}\left(\frac{M_{\Upsilon_i}^2+Q^2}{4M_B^2}\right)^{-2} + \frac{n_f}{12}\int_1^\infty dy\left(y+\frac{Q^2}{4M_B^2}\right)^{-2}y^{-3/2}(y-1)^{3/2}|F(4M_B^2y)|^2, \tag{13}$$

where we have set $y = t/4M_B^2$.

We next turn our attention to the b-number form factor $F(t)$ of the B meson. On general quantum field theory grounds $F(t)$ obeys a dispersion relation; and it follows from the QCD inequality above that the dispersion relation for $F(t)$ has at most one subtraction. Since the value of $F(t)$ at $t=0$ is known [see Eq. (3)], it is convenient to use $t=0$ as the subtraction point: i.e.,

$$F(t) = F(0) + \frac{t}{\pi}\int_0^\infty \frac{dt'}{t'}\frac{\mathrm{Im}F(t')}{t'-t-i\epsilon} \ . \tag{14}$$

In full generality

$$(p+p')^\mu \mathrm{Im}F(t) = \frac{1}{2}\sum_\Gamma \int d\mu(\Gamma)(2\pi)^4\delta^{(4)}\left(q-\sum_\Gamma p\right)\langle \bar{B}B|\Gamma\rangle\langle\Gamma|V^\mu(0)|0\rangle, \tag{15}$$

with the summation extended to all possible hadronic states Γ with the quantum numbers of the V^μ current. It appears then that the b-number form factor of the B meson has a succession of branch cuts starting at the $\pi\pi$ threshold, the $K\bar{K}$ threshold, the $D\bar{D}$ threshold, the $B\bar{B}$ threshold, etc. Since the V^μ current only involves b quarks and b quarks are heavy, their coupling to hadronic states of lighter flavors, which can only proceed through annihilation via gluonic interactions, are suppressed. Other possible hadronic states below the $B\bar{B}$ threshold are the three Υ states $\Upsilon(1S)$, $\Upsilon(2S)$, and $\Upsilon(3S)$ which to a good approximation appear as poles in the positive real axis of the complex t plane. Their contribution to the b-number form factor can be parametrized as

$$F(t) = F(0) + t\sum_i 3\frac{g_{\Upsilon_i B\bar{B}} f_{\Upsilon_i}}{M^2_{\Upsilon_i} - t - i\epsilon} + F_{\mathrm{reg}}(t), \tag{16}$$

where f_{Υ_i} denote the coupling constants which govern the electronic width of the Υ_i resonances,

$$\Gamma[\Upsilon_i \to (\gamma) \to e^+e^-] = f^2_{\Upsilon_i} M_{\Upsilon_i}\frac{4\pi}{3}\alpha^2, \tag{17}$$

and $g_{\Upsilon_i B\bar{B}}$ the coupling constants of the Υ_i resonances to the $B\bar{B}$ system. More precisely, we are considering an effective Lagrangian interaction

$$\mathcal{L}_{\Upsilon\gamma} = \frac{|e|}{2} f_\Upsilon(\partial_\mu\Upsilon_\nu - \partial_\nu\Upsilon_\mu)\, F^{\mu\nu}, \tag{18}$$

to implement the coupling of a massive spin-1 field which describes a generic Υ resonance, with the electromagnetic strength tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, and an effective interaction Lagrangian

$$\mathcal{L}_{\Upsilon B\bar{B}} = i g_{\Upsilon B\bar{B}}\Upsilon_\mu(B^\dagger\partial^\mu B - \partial^\mu B^\dagger B), \tag{19}$$

to implement the coupling of the Υ field to the B pseudoscalars. The coupling constants $g_{\Upsilon B\bar{B}}$ and f_Υ are dimensionless and real.

The naive scaling of the coupling constants $g_{\Upsilon B\bar{B}}$ and f_Υ in the large b-quark mass limit implies [3]

$$g_{\Upsilon B\bar{B}} \to (m_b)^{1/2} \quad \text{and } f_\Upsilon \to (m_b)^{-1/2}. \tag{20}$$

In this limit, the contribution from the Υ states to the RHS in Eq. (13) decouples, and, therefore, as was done in Ref. [1], this contribution in this limit can be ignored. However, in the same limit, the residues at the Υ poles in the B-number form factor in Eq. (16) scale as $(m_b)^2$. If naive scaling holds, then the Υ poles of the b-number form factor of the B meson below the $B\bar{B}$ threshold cannot be neglected, contrary to what was done in the derivation of the bounds in Ref. [1]. New bounds, if possible, have to be derived.

III. THE NEW BOUNDS

The derivation of the new bounds is possible with an appropriate generalization of the method we already used in [1]. The technical details are explained in the Appendix. To adapt our problem to the framework of the Appendix we shall map the entire complex y plane ($y = \frac{t}{4M_B^2}$) onto the unit disc $|z| \le 1$ via the transformation

$$i\frac{1+z}{1-z} = \sqrt{y-1} = i\sqrt{\frac{1+v\cdot v'}{2}}. \tag{21}$$

Here $v\cdot v'$ is the Isgur-Wise variable which denotes the product of the four velocities of the incoming and outgoing B mesons in the vertex in Eq. (1): $q^2 = 2M_B^2(1-v\cdot v')$. Eventually, we are interested in bounds of the b-number form factor F in the physical region relevant to the decays in (2), i.e.,

$$1 \le v\cdot v' \le \frac{1}{2}(M/M' + M'/M) \simeq 1.6. \tag{22}$$

By the transformation in (21), the physical cut $1 \le y \le \infty$ is mapped into the unit circle $z = e^{i\theta}$, the $\bar{B}B$ threshold at $y=1$ into $z=-1$, and the position of the Υ poles below the $\bar{B}B$ threshold at $y_i = \frac{M^2_{\Upsilon_i}}{4M_B^2}$ into the real points z_i: $-1 < z_i < 0, i = 1,2,3$. The integral on the RHS of Eq. (13) can then be written as an integral on the unit circle.

In order to use the results derived in the Appendix, Eq. (13) should be cast into the form

$$1 \ge \frac{1}{2\pi}\int_0^{2\pi} d\theta |f(e^{i\theta})|^2. \tag{23}$$

That this is always possible is guaranteed by the fact that the integrand in (13) is positive, and a theorem. Let $\phi(e^{i\theta})$ be real and positive, then

$$h(z) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} d\theta \frac{e^{i\theta}+z}{e^{i\theta}-z}\ln\phi(e^{i\theta})\right) , \tag{24}$$

verifies

$$|h(e^{i\theta})| = \phi(e^{i\theta}), \tag{25}$$

is analytic and has no zeros in the unit disc. The function $h(z)$ is unique up to a global phase. This is actually the solution of the Dirichlet problem of finding an analytic function $h(z)$ with no zeros in the unit disc with a boundary condition on the unit circle as given by (25). The problem can be immediately solved with the help of the Poisson kernels, applied to the real and imaginary parts of $\ln h(z)$ [17]. The solution is given by (24).

It is easy to find directly the function h which corresponds to the two factors which multiply $|F|^2$ in the integrand in Eq. (13) with the help of the relations

$$\begin{aligned} y - y_i &= -4\frac{(z-z_i)(1-zz_i)}{(1-z_i)^2(1-z)^2} \\ &= \frac{z_i - z}{1-z_i z}\left(\frac{1+z_i}{1-z_i}+\frac{1+z}{1-z}\right)^2 , \end{aligned} \tag{26}$$

where z, z_i are the images by (21) of y, y_i, respectively. Factors of the type $(z-a)/(1-za^*)$ $(a \in C)$ are ubiquitous in the analysis; they are unimodular on the unit circle and, therefore, drop from the integrand in Eq. (23).

For the sake of simplicity we shall choose $Q^2 = 0$ in the unitary inequality in (13) and ignore here the question of optimizing the choice of Q^2. We shall also adopt the lowest-order result in Eq. (12) as a good estimate of the QCD evaluation of $\chi(0)$. Perturbative α_s corrections to this result are known to be small at the m_b^2 scale. Since we shall be ignoring α_s corrections, which are positive, it seems prudent to neglect as well the contribution from the Υ states on the RHS of the unitarity inequality. As already mentioned they decouple in the large b-quark mass limit in any case. We shall also take M_B and m_b to be equal. The unitarity inequality in (12) then reads

$$1 \geq \frac{5n_f}{16N_c}\int_1^\infty dy\, y^{-7/2}(y-1)^{3/2}|F|^2. \tag{27}$$

But for the n_f factor, this coincides with Eq. (14) in Ref. [1].

Equation (27) can now be written in the canonical form of Eq. (23), by setting

$$f(z) = \varphi(z)F[q^2(z)], \tag{28}$$

with

$$\varphi(z) = \varphi(0)\sqrt{1-z}(1+z)^2 \tag{29a}$$

and

$$\varphi(0) = \frac{1}{16}\sqrt{\frac{5\pi}{2}\frac{n_f}{N_c}} . \tag{29b}$$

As a function of the variable z, $F[q^2(z)]$ is an analytic function in the unit disc, except for the three poles at $-1 < z_i < 0$, corresponding to the locations of the three Υ states below the $\bar{B}B$ threshold. These poles of $F(q^2)$ give rise to poles of the function $f(z)$ in (28) at the same location $-1 < z_i < 0$, with residues

$$R_i = \varphi(z_i)\frac{1-z_i}{1+z_i}z_i\eta_i = \varphi(0)z_i(1+z_i)(1-z_i)^{3/2}\eta_i , \quad i = 1,2,3, \tag{30a}$$

where η_i denotes the product of coupling constants:

$$\eta_i \equiv 3g_{\Upsilon_i B\bar{B}}f_{\Upsilon_i} . \tag{30b}$$

The modulus of the couplings $|f_{\Upsilon_i}|$ can be determined from the experimental electronic widths [see Eq. (17)]. Unfortunately, the couplings $g_{\Upsilon_i B\bar{B}}$ for the three Υ's below the $\bar{B}B$ threshold are unknown, and therefore the sizes of the residues are also unkown. As discussed in the Appendix, it is nervertheless possible to obtain upper and lower bounds on the form factor $F(q^2)$ using the fact that $F(0) = 1$ [see Eq. (3)], if the locations of the poles z_i of the function $f(z)$ are known. These are determined, via Eq. (21), by the masses of the three Υ states below the $\bar{B}B$ threshold; i.e., the parameters

$$a_i \equiv \frac{M_{\Upsilon_i}^2}{4M_B^2}. \tag{31}$$

We find, in this case [see Eq. (A17) in the Appendix]

$$F_-(z) \leq F(z) \leq F_+(z), \tag{32}$$

where

$$F_\pm = -\frac{F(0)}{\sqrt{1-z}(1+z)^2}\prod_i z_i\left(\frac{1-z_i z}{z-z_i}\right)\left[1 \pm \sqrt{\frac{z^2}{1-z^2}}\sqrt{\frac{512}{5\pi}\frac{N_c}{n_f}\frac{1}{F^2(0)}\prod_i\frac{1}{z_i^2}-1}\,\right]. \tag{33}$$

In the case where $a_i \to 1$ $(z_i \to -1)$, and correcting for the n_f factor, the formula coincides with Eq. (16) of [1]. Using the experimental values for the Υ masses below threshold, we obtain in particular upper and lower bounds on the slope of the b-number form factor of the B meson at the origin:

$$-6.0 \leq F'(v\cdot v' = 1) \leq 4.5. \tag{34}$$

The lower bound, although rather generous, is not trivial.

As indicated in the Appendix, it is also possible, with the same input, to obtain bounds on the η_i residues:

TABLE I. Data and coupling constants for the Υ states. The coupling constants f_Υ and $g_{\Upsilon B\bar{B}}$ are defined by the effective Lagrangians in Eqs. (18) and (19).

State	Mass (MeV)	$\Gamma(i \to e^+e^-)$ (keV)	$\lvert f_{\Upsilon_i}\rvert \times 10^2$	$\Gamma(i \to B\bar{B})$ (MeV)	$\lvert g_{\Upsilon_i B\bar{B}}\rvert$	$\lvert\eta_i\rvert = 3\lvert g_{\Upsilon_i D\bar{D}} f_{\Upsilon_i}\rvert$
$\Upsilon(1S)$	9460.32 ± 0.22	1.34 ± 0.04	2.5		?	
$\Upsilon(2S)$	10023.30 ± 0.31	0.56	1.6		?	
$\Upsilon(3S)$	10355.3 ± 0.5	0.44	1.4		?	
$\Upsilon(4S)$	10580.1 ± 3.5	0.24 ± 0.05	1.0	$\leq 23.8 \pm 2.2$	≤ 25	≤ 0.75

$$
\begin{aligned}
-3.3 \times 10^3 \leq \eta_1 \leq 3.3 \times 10^3 \ , \\
-5.7 \times 10^3 \leq \eta_2 \leq 5.7 \times 10^3 \ , \\
-2.7 \times 10^3 \leq \eta_3 \leq 2.7 \times 10^3 \ .
\end{aligned} \tag{35}
$$

As we shall next discuss, these bounds allow for huge values of the unknown couplings $g_{\Upsilon_i B\bar{B}}$.

In order to get a feeling for what is a reasonable expected size for the η_i residues, we propose to extract the value of the η_4 residue corresponding to the $\Upsilon(4S)$ state which is already above the $4M_B^2$ threshold from experiment. The experimental data, as well as the corresponding couplings, are shown in Table I. The decay rate of the $\Upsilon(4S)$ into $\bar{B}B$ calculated with the effective Lagrangian in Eq. (19) is

$$
\Gamma(\Upsilon(4S) \to B\bar{B}) = \frac{1}{48\pi} M_{\Upsilon_4} \left(1 - \frac{4M_B^2}{M_{\Upsilon_4}^2}\right)^{3/2} g^2_{\Upsilon_4 B\bar{B}}. \tag{36}
$$

From Eqs. (17) and (36) and the knowledge of the experimental total width, we obtain for η_4 in (30b),

$$
|\eta_4(\text{expt})| \leq 0.75 \pm 0.15, \tag{37}
$$

i.e., a value about 3 orders of magnitude smaller than the limits allowed by the bounds for the other $\eta_i, i = 1, 2, 3$.

It is also instructive to extract from experiment the corresponding residues for the charmonium states $\psi(3S)$ and $\psi(4S)$ which are above the $D\bar{D}$ threshold. The experimental data as well as the corresponding couplings are shown in Table II. It is noteworthy that the experimental upper values of the η residues for the $\psi(3S)$ and $\psi(4S)$ states, and the η_4 residue of the $\Upsilon(4S)$ state are of the same size. We propose to use this phenomenological observation as a guiding ansatz for possible input values of the η_i residues, $i = 1, 2, 3$, and to derive the corresponding bounds[1] for $F(q^2)$.

The relevant analytic form of the upper and the lower bounds for $F(q^2)$ when both the positions and the modulus of the residues of the Υ poles are known is given by Eqs. (A16) and (A19) of the Appendix. In the limit where $|\eta_i| \to 0$, $i = 1, 2, 3$, we reproduce the first bound given in [1] (corrected by the famous n_f factor): i.e.,

$$
-0.89 \leq F'(1) \leq 0.52 \ . \tag{38}
$$

The corresponding upper and lower bounds of the slope $F'(v \cdot v' = 1)$ for a specific set of input values of the modulus of the reduced residues $|\eta_i|$ of about the same size as the upper bounds known from experiment are given in Table III. We observe from the results in this table that the bounds are rather insensitive to small variations of the $|\eta_i|$'s. (Increasing all $|\eta_i|$ by a factor 4 diminishes the lower bound by 50%.) The upper bounds in Table III are not interesting since they all have a positive slope and we expect from Bjorken's bound [19] that $F(v \cdot v')$ is a decreasing function for $v \cdot v' \geq 1$. The lower bounds however are certainly nontrivial and may be useful for phenomenology and model building.

We conclude from our analysis above that the only rigorous lower bound we have at present on the slope of the b-number form factor of the B meson at the origin is the one in Eq. (34). Nevertheless, on phenomenological grounds, we consider that a lower bound

$$
F'(1) \geq -1.7 \tag{39}
$$

is a conservative estimate.

TABLE II. Data and coupling constants for the charmonium states. The coupling constants f_ψ and $g_{\psi\bar{D}D}$ are those of effective Lagrangians analogous to Eqs. (18) and (19).

State	Mass (MeV)	$\Gamma(i \to e^+e^-)$ (keV)	$\lvert f_{\psi_i}\rvert \times 10^2$	$\Gamma(i \to D\bar{D})$ (MeV)	$\lvert g_{\psi_i D\bar{D}}\rvert$	$\lvert\eta_i\rvert = 1.5\lvert g_{\psi_i D\bar{D}} f_{\psi_i}\rvert$
$J/\psi(1S)$	3096.93 ± 0.09	5.35 ± 0.29	8.8		?	
$\psi(2S)$	3686.00 ± 0.10	2.14 ± 0.21	5.1		?	
$\psi(3S)$	3769.9 ± 2.5	0.26 ± 0.4	1.8	$\leq 23.6 \pm 2.7$	≤ 16.8	≤ 0.47
$\psi(4S)$	4040 ± 10	0.75 ± 0.15	2.9	$\leq 52 \pm 10$	≤ 4.0	≤ 0.17

[1] We have assumed flavor SU(3) symmetry and used the same constants $g_{B\bar{B}\Upsilon_i}$, $g_{D\bar{D}\psi_i}$ for any of the three light flavor species u, d, s of the mesons B and D, respectively. This allows us to improve the bounds on the residues by including all possible channels. Notice that not all the channels are always allowed by phase space.

TABLE III. Upper and lower bounds for the slope of the b-number form factor of the B meson for various phenomenological input values of the residues η_i [see Eqs. (30b) and (16) in the text].

$\|\eta_1\|$	$\|\eta_2\|$	$\|\eta_3\|$	$F'(1)_{\text{lower}}$	$F'(1)_{\text{upper}}$
0.5	0.5	0.5	−1.23	0.79
1.0	1.0	1.0	−1.51	1.00
1.0	0.5	0.3	−1.36	0.92
1.5	1.5	1.5	−1.73	1.04

IV. COMMENTS ON THE HEAVY QUARK EFFECTIVE THEORY

Related to the work described in the previous sections, there are a number of observations we wish to make which are relevant to the heavy quark effective theory formulation [20] of the Isgur-Wise symmetries.

First we shall comment on the limit $m_b \to \infty$ of our bounds. As m_b grows bigger and bigger, there appear an increasing number of Υ resonances below the $\bar{B}B$ threshold (a semiclassical estimate based on nonrelativistic potential models gives that this growth goes like $(m_b)^{1/2}$ [21]). In order to be able to take the $m_b \to \infty$ limit on our expressions for the bounds, more information should be known about the m_b dependence of the location of the poles $z_i(m_b)$, as well as of the residues $\eta_i(m_b)$. However, if the behavior $\eta_i(m_b) \to$ const and a finite number of poles below threshold are assumed in the $m_b \to \infty$ limit, as done in [4], then, since $z_i(m_b) \to -1$, the results in Ref. [1] are again recovered. The authors of Ref. [4] found that the effect of such poles is to broaden the bounds. This is just an artifact of the approximation they use. As shown in the Appendix, stronger bounds can be derived leading to the same results as in [1].

The second observation is of a phenomenological nature. Based on naive scaling [3] of the coupling constants f_Υ and f_ψ in the large b-quark mass limit and the large c-quark mass limit, one expects the ratios of these couplings to scale as

$$2\,\frac{f_{\Upsilon_i}}{f_{\psi_i}} \to \left(\frac{m_c}{m_b}\right)^{1/2}, \tag{40}$$

where the factor 2 takes care of the different quark charges of the b and c quarks. Except for the $i = 3$ states [and in fact the electronic width of the $\Upsilon(3S)$ is poorly known], the experimental ratios

$$2\,\frac{f_{\Upsilon_1}}{f_{\psi_1}} \simeq 0.57, \quad 2\,\frac{f_{\Upsilon_2}}{f_{\psi_2}} \simeq 0.63, \quad 2\,\frac{f_{\Upsilon_4}}{f_{\psi_4}} \simeq 0.69, \tag{41}$$

are not incompatible with the empirically allowed range of quark mass values [18]:

$$0.5 \le (m_c/m_b)^{1/2} \le 0.6. \tag{42}$$

Our last comment has to do with the compatibility of the bounds with models of the Isgur-Wise function. In Ref. [1] we proposed as a simple-minded model of this function the one provided by the triangle graph vertex with two heavy quark lines and one light quark across with a constituent mass which acts as a regulator and no gluons across. The resulting Isgur-Wise function has the form

$$\xi(v\cdot v' = \omega) = \frac{1}{\sqrt{\omega^2-1}}\,\ln(\omega + \sqrt{\omega^2-1}). \tag{43}$$

The slope at zero recoil is $\xi'(1) = -1/3$, in confortable agreement with the lower bound in Eq. (38). The function in (43) has been recently found again in a toy field theory model which tries to implement both heavy and light quark symmetries [8]. Bardeen and Hill dismiss however this solution on the grounds of "residual mass invariance" of the heavy quark effective theory [22,23] and conclude that the Isgur-Wise function in their model is given by

$$\xi(v\cdot v' = \omega) = \frac{2}{1+v\cdot v'}. \tag{44}$$

The slope at zero recoil of this function is $\xi'(1) = -1/2$, also compatible with the lower bound in (38).

ACKNOWLEDGMENTS

It is a pleasure to thank G. Anderson, J. Bijnens, D. Broadhurst, H. Dosch, S. Narison, M. Neubert, M. Peskin, A. Pich, S. Raby, J. Sloan, J. Soto, and B. Stech for enlightening discussions. This work was supported in part by USA Grant No. DE-FG02-91-ER40690.

APPENDIX

The mathematical tools needed to derive the bounds on the form factor $F(q^2)$ in Eq. (1) follow from analyticity properties and positivity. Let $f(z)$ be an analytic function on the unit disc, and let

$$I[f] \equiv \frac{1}{2\pi}\int_0^{2\pi} d\theta |f(e^{i\theta})|^2 = \frac{1}{2\pi i}\oint_{|\omega|=1}\frac{d\omega}{\omega}|f(\omega)|^2, \qquad \omega = e^{i\theta}. \tag{A1}$$

The basic inequality follows from

$$0 \le \frac{1}{2\pi}\int_0^{2\pi} d\theta |f(e^{i\theta}) - f(0)|^2 = I[f] - |f(0)|^2, \tag{A2}$$

i.e.,

$$|f(0)|^2 \le I[f]. \tag{A3}$$

A similar inequality may be derived at any point z in the unit disc. With the help of the Möbius transformation

$$\omega = \frac{z-x}{1-z^*x}, \quad |z|^2 < 1, \tag{A4}$$

at fixed z, the problem of finding a bound on $|f(z)|$ is reduced to finding a bound at $x = 0$. Indeed, (A4) maps the unit circle $|\omega| = 1$ onto the unit circle $|x| = 1$, and the point $\omega = z$ is mapped into $x = 0$. In terms of the variable x,

$$I[f] = (1-|z|^2)\frac{1}{2\pi i}\oint_{|x|=1}\frac{dx}{x}\left|\frac{f(\frac{z-x}{1-z^*x})}{1-z^*x}\right|^2, \qquad (A5)$$

for which inequality (A3) gives, at $x = 0$,

$$|f(z)|^2 \le \frac{I[f]}{1-|z|^2}. \qquad (A6)$$

This is the generalization of (A3).

Suppose f has a simple zero in the disc, at $z = a$. One may build

$$\psi(z) = f(z)\frac{1-a^*z}{z-a}, \qquad (A7)$$

which is analytic on the disc and is such that $|f(z)| = |\psi(z)|$ at $|z| = 1$; therefore $I[\psi] = I[f]$ and applying (A6) to $\psi(z)$ yields

$$|f(z)|^2 \le \frac{I[f]}{1-|z|^2}\left|\frac{z-a}{1-a^*z}\right|^2. \qquad (A8)$$

Notice that

$$\left|\frac{z-a}{1-a^*z}\right|^2 = 1 - \frac{(1-|a|^2)(1-|z|^2)}{|1-a^*z|^2} < 1, \qquad (A9)$$

since $|z|$, $|a| < 1$. Knowledge of the location of a simple zero inside the disc leads to an inequality (A8), which is more constraining. The generalization to higher-order zeros is immediate. One can proceed similarly when $f(z)$ has a simple pole at $z = p$ in the unit disc. In that case one may build

$$\psi(z) = f(z)\frac{z-p}{1-p^*z}, \qquad (A10)$$

which is analytic on the disc. Equation (A6) then reads

$$|f(z)|^2 \le \frac{I[f]}{1-|z|^2}\left|\frac{1-p^*z}{z-p}\right|^2, \qquad (A11)$$

which for a given $I[f]$ is less constraining than[2] (A6). Letting $z \to p$, a bound on the residue R is also obtained:

$$|R|^2 \le I[f](1-|p|^2). \qquad (A12)$$

Notice that in each case the bounds only depend on the information that is provided about the function.

[2]Equation (A11) is however more constraining than the bounds obtained in [4], which would give in this case

$$|f(z)|^2 \le \frac{I[f]}{1-|z|^2}\left|\frac{1-p^*z}{z-p}\right|^2 + I[f]\left|\frac{p}{z-p}\right|^2,$$

and for the residue

$$|R|^2 \le I[f].$$

If the residue R is also known, $\psi(z) = f(z) - \frac{R}{z-p}$ is analytic on the disc, Eq. (A6) applies and yields

$$\left|f(z) - \frac{R}{z-p}\right|^2 \le \frac{I[\psi]}{1-|z|^2}, \qquad (A13)$$

with

$$I[\psi] = I[f] - \frac{|R|^2}{1-|p|^2}. \qquad (A14)$$

After these examples, let us consider the cases of interest to us discussed in the text: (i) $f(0)$ is known; (ii) $f(0)$ is known as well as the location of the poles $-1 < z_1 < z_2 < z_3 < 0$; (iii) the residues R_1, R_2, R_3 of these poles are also known.

(i) Build $\psi(z) = f(z) - f(0)$, for which $\psi(0) = 0$. Equation (A8) applies with $a = 0$ and yields

$$|f(z) - f(0)|^2 \le \frac{I[f] - |f(0)|^2}{1-|z|^2}|z|^2. \qquad (A15)$$

This is the inequality used in [1].

(ii) Build

$$\psi(z) = \frac{z-z_1}{1-z_1^*z}\frac{z-z_2}{1-z_2^*z}\frac{z-z_3}{1-z_3^*z}f(z) + z_1z_2z_3f(0),$$

$\psi(0) = 0$. The bounds are given by (A8):

$$|\psi(z)|^2 \le \frac{I[\psi]}{1-|z|^2}|z|^2, \qquad (A16)$$

with

$$I[\psi] = I[f] - |z_1z_2z_3f(0)|^2. \qquad (A17)$$

Bounds on the residues follow from (A16),

$$\left|\frac{z_1-z_2}{1-z_2^*z_1}\frac{z_1-z_3}{1-z_3^*z_1}\frac{R_1}{1-|z_1|^2} + z_1z_2z_3f(0)\right|^2 \le \frac{I[\psi]}{1-|z_1|^2}|z_1|^2, \qquad (A18)$$

and similarly, *mutatis mutandis*, for R_2 and R_3.

(iii) Build

$$\psi(z) = f(z) - \frac{R_1}{z-z_1} - \frac{R_2}{z-z_2} - \frac{R_3}{z-z_3} - \left(\frac{R_1}{z_1} + \frac{R_2}{z_2} + \frac{R_3}{z_3} + f(0)\right),$$

$\psi(0) = 0$. The bounds are given by Eq. (A16) with

$$I[\psi] = I[f] - \left|\frac{R_1}{z_1} + \frac{R_2}{z_2} + \frac{R_3}{z_3} + f(0)\right|^2 - \frac{|R_1|^2}{1-|z_1|^2} - \frac{|R_2|^2}{1-|z_2|^2} - \frac{|R_3|^2}{1-|z_3|^2} - 2\,\mathrm{Re}\left(\frac{R_1R_2^*}{1-z_1z_2^*} + \frac{R_1R_3^*}{1-z_1z_3^*} + \frac{R_2R_3^*}{1-z_2z_3^*}\right). \qquad (A19)$$

[1] E. de Rafael and J. Taron, Phys. Lett. B **282**, 215 (1992).
[2] A.F. Falk, M. Luke, and M.B. Wise, Phys. Lett. B **299**, 123 (1993).
[3] B. Grinstein and P.F. Mende, Phys. Lett. B **299**, 127 (1993).
[4] C.A. Dominguez, J.G. Korner, and D. Pirjol, Phys. Lett. B **301**, 257 (1993).
[5] C.E. Carlson, J. Milana, N. Isgur, T. Mannel, and W. Roberts, Phys. Lett. B **299**, 133 (1993).
[6] D.J. Broadhurst (unpublished).
[7] A.G. Grozin, *Introduction to the Heavy Quark Effective Theory - Part I* (Novosibirsk, 1992).
[8] W.A. Bardeen and Ch.T. Hill, Phys. Rev. D **49**, 409 (1994).
[9] N. Isgur and M.B. Wise, Phys. Lett. B **232**, 113 (1989); **237**, 527 (1990).
[10] ARGUS Collaboration, H. Albrecht *et al.*, Phys. Lett. B **275**, 195 (1992); and data on the decay $\bar{B}^0 \to D^{*+} l^- \bar{\nu}$, in *Proceedings of the XXVIth International Conference on High Energy Physics*, Dallas, Texas, 1992, edited by J. Sanford, AIP Conf. Proc. No. 272 (AIP, New York, 1993).
[11] D. Bortoletto and S. Stone, Phys. Rev. Lett. **65**, 2951 (1990).
[12] T. Mannel and Z. Ryzak, Phys. Lett. B **247**, 412 (1990).
[13] A.F. Falk and B. Grinstein, Phys. Lett. B **249**, 314 (1990).
[14] J.L. Rosner, Phys. Rev. D **42**, 3732 (1990); T. Mannel, W. Roberts, and Z. Ryzak, Phys. Lett. B **254**, 274 (1991); M. Neubert, *ibid.* **264**, 455 (1991).
[15] J. Taron, in *Proceedings of the XXVIth International Conference on High Energy Physics* [10].
[16] P. Ball, H.G. Dosch, and M.A. Shifman, Phys. Rev. D **47**, 4077 (1993).
[17] E.C. Titchmarsh, *The Theory of Functions* (Oxford University Press, Oxford, 1939).
[18] Particle Data Group, K. Hikasa *et al.*, Phys. Rev. D **45**, S1 (1992).
[19] J.D. Bjorken, in *Results and Perspectives in Particle Physics*, Proceedings of the 4th Rencontres de Physique de la Vallée D'Aoste, La Thuile, Italy, 1990, edited by M. Greco (Editions Frontières, Gif-sur-Yvette, France, 1990).
[20] H. Georgi, Phys. Lett. B **240**, 447 (1990); Mark Wise, in *Particle Physics—The Factory Era*, Proceedings of the Lake Louise Winter Institute, Lake Louise, Canada, 1991, edited by B.A. Campbell *et al.* (World Scientific, Singapore, 1991).
[21] C. Quigg and J.L. Rosner, Phys. Lett. **72B**, 462 (1978).
[22] M. Luke and A. Manohar, Phys. Lett. B **286**, 348 (1992).
[23] A. Falk, M. Neubert, and M. Luke, Nucl. Phys. **B388**, 363 (1992).