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Classical solutions of the leading-logarithm approximation with nontrivial topology

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Exact solutions of the classical equations corresponding to the leading-logarithm approximation are obtained. They are classified by an (integer) topological number.

The leading-logarithm approximation to the Euclidean functional integral of the Yang-Mills (YM) theory

$$Z[J]=\frac{1}{N}\int[dA]\exp\left\{-S_{\rm cl}[A]+\int J_\mu(x)A_\mu(x)\,d^4x\right\}\ ,\quad S_{\rm cl}[A]=\frac{1}{2}\int\frac{1}{g^2}{\rm Tr}[F_{\mu\nu}(x)F_{\mu\nu}(x)]\,d^4x \tag{1}$$

consists in replacing the classical action $S_{\rm cl}[A]$ by the effective action[1,2]

$$\begin{aligned}&S_{\rm eff}[A]=\int(\mathcal{L}_{\rm eff}[F^2]-\mathcal{L}_{\rm eff}^{\rm min})\,d^4x\ ,\quad \mathcal{L}_{\rm eff}^{\rm min}=\mathcal{L}_{\rm eff}[\kappa^2]=\frac{-\beta_0\kappa^2}{4}\ ,\\ &\mathcal{L}_{\rm eff}[F^2]=\frac{\beta_0}{4}F^2\ln\left[\frac{F^2}{e\kappa^2}\right]\ ,\quad F^2\equiv{\rm Tr}(F_{\mu\nu}F_{\mu\nu})\ ,\end{aligned} \tag{2}$$

where β_0 is the first coefficient of the Callan-Symanzik β function, and κ is the square of the renormalization mass. The integral (1) is then evaluated over the classical paths only:

$$Z[J]\simeq\frac{1}{N}\int[dA]\exp\left\{-S_{\rm eff}[A]+\int J_\mu(x)A_\mu(x)\,d^4x\right\}\delta(A-\bar{A})\ ,\qquad \left.\frac{\delta S_{\rm eff}[A]}{\delta A}\right|_{A=\bar{A}}=J\ . \tag{3}$$

This approximation to the exact Euclidean functional incorporates the renormalization-group features to one-loop order. Moreover, $S_{\rm eff}$ has other attractive properties, as (explicit) local gauge invariance, asymptotic freedom, and dimensional transmutation.

In Refs. 1 and 2 this leading-log model has been studied in connection with the problem of quark confinement. Here we shall investigate the topological structure of the solutions of its equations of motion

$$[1+\ln(F^2|e\kappa^2)]\{\partial_\mu F_{\mu\nu}+[A_\mu,F_{\mu\nu}]\}=0\ , \tag{4}$$

with the boundary condition

$$\lim_{x\to\infty}{\rm Tr}[F_{\mu\nu}(x)F_{\mu\nu}(x)]=\kappa^2\ , \tag{5}$$

which has to be satisfied in order to obtain a finite action.

We shall now construct a family of (vacuum) solutions of Eqs. (4) and (5) by simply imposing that

$${\rm Tr}[F_{\mu\nu}(x)F_{\mu\nu}(x)]=\kappa^2,\quad x\in\mathcal{R}^4\ . \tag{6}$$

We prove below that the following field-strength tensor is a solution of Eq. (6):

$$\begin{aligned}&F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu,A_\nu]\ ,\\ &A_\mu(x)=A_\mu^{(i)}(x)+if(x^2)x_\mu\sigma_3\ ,\end{aligned} \tag{7}$$

where $A_\mu^{(i)}$ is the instanton potential[3]

$$\begin{aligned}&A_\mu^{(i)}(x)=\frac{x^2}{x^2+a^2}g^{-1}(x)\,\partial_\mu g(x)\ ,\\ &g(x)=\frac{s_\mu x_\mu}{\sqrt{x^2}},\quad s_\mu=(1,i\vec{\sigma})\ ,\end{aligned} \tag{8}$$

and $f(x^2)$ is the function

$$f(x^2)=\frac{1}{x^2}\left[\frac{\kappa^2}{32}(x^2+a^2)^2-\frac{2a^4}{(x^2+a^2)^2}\right]^{1/2}\ . \tag{9}$$

It is not difficult to see that $F_{\mu\nu}$ given by Eq. (7) satisfies Eqs. (4) and (5). In fact, one explicitly gets

$$F_{\mu\nu}(x)=F_{\mu\nu}^{(i)}(x)+f(x^2)\frac{x^2}{x^2+a^2}F_{\mu\nu}^{(0)}(x)\ , \tag{10}$$

where

$$F_{\mu\nu}^{(i)}(x)=\frac{4a^2}{(x^2+a^2)^2}\sigma_{\mu\nu},\quad \sigma_{\mu\nu}=\tfrac{1}{4}(s_\mu\bar{s}_\nu-s_\nu\bar{s}_\mu),\quad \bar{s}_\mu=(1,-i\vec{\sigma})\ ,$$
$$F_{\mu\nu}^{(0)}(x)=i[\sigma_3,x_\mu A_\nu^{(0)}(x)-x_\nu A_\mu^{(0)}(x)],\quad A_\mu^{(0)}(x)=g^{-1}(x)\partial_\mu g(x)\ . \tag{11}$$

It is easy to see that

$$\mathrm{Tr}[F_{\mu\nu}^{(i)}(x)F_{\mu\nu}^{(0)}(x)]=0\ . \tag{12}$$

Therefore,

$$\mathrm{Tr}[F_{\mu\nu}(x)F_{\mu\nu}(x)]=\mathrm{Tr}[F_{\mu\nu}^{(i)}(x)F_{\mu\nu}^{(i)}(x)]+f^2(x^2)\frac{x^4}{(x^2+a^2)^2}\ \mathrm{Tr}[F_{\mu\nu}^{(0)}(x)F_{\mu\nu}^{(0)}(x)] \tag{13}$$

and, substituting

$$\mathrm{Tr}[F_{\mu\nu}^{(i)}(x)F_{\mu\nu}^{(i)}(x)]=\frac{64a^4}{(x^2+a^2)^4}\ ,\quad \mathrm{Tr}[F_{\mu\nu}^{(0)}(x)F_{\mu\nu}^{(0)}(x)]=32\ , \tag{14}$$

we finally obtain

$$\mathrm{Tr}[F_{\mu\nu}(x)F_{\mu\nu}(x)]=\frac{32}{(x^2+a^2)^2}\left[x^4f^2(x^2)+\frac{2a^4}{(x^2+a^2)^2}\right]=\kappa^2\ . \tag{15}$$

Thus, $F_{\mu\nu}$ given by (7) is indeed a solution of (4) and (5).

We shall now demonstrate that the topological index of each of these solutions is equal to the one corresponding to the associated instanton potential. In fact, in the particular example we are considering,

$$\mathrm{Tr}[\tilde{F}_{\mu\nu}(x)F_{\mu\nu}(x)]=\mathrm{Tr}[\tilde{F}_{\mu\nu}^{(i)}(x)F_{\mu\nu}^{(i)}(x)]+f^2(x^2)\frac{x^4}{(x^2+a^2)^2}\mathrm{Tr}[\tilde{F}_{\mu\nu}^{(0)}(x)F_{\mu\nu}^{(0)}(x)]$$
$$+2f(x^2)\frac{x^2}{x^2+a^2}\mathrm{Tr}[\tilde{F}_{\mu\nu}^{(i)}(x)F_{\mu\nu}^{(0)}(x)]\ . \tag{16}$$

By $\tilde{F}_{\mu\nu}$ we denote the dual tensor $\tilde{F}_{\mu\nu}=\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}/2$. A direct calculation gives

$$\mathrm{Tr}[\tilde{F}_{\mu\nu}^{(0)}F_{\mu\nu}^{(0)}]=0\ , \tag{17}$$

while using (12) and the self-duality of the instanton solution, one gets

$$\mathrm{Tr}[\tilde{F}_{\mu\nu}^{(i)}F_{\mu\nu}^{(0)}]=0\ . \tag{18}$$

Therefore,

$$\mathrm{Tr}[\tilde{F}_{\mu\nu}(x)F_{\mu\nu}(x)]=\mathrm{Tr}[\tilde{F}_{\mu\nu}^{(i)}(x)F_{\mu\nu}^{(i)}(x)]\ , \tag{19}$$

which substituted into the expression for the topological index

$$q=\frac{1}{16\pi^2}\int_{\mathbb{R}^4}\mathrm{Tr}[\tilde{F}_{\mu\nu}(x)F_{\mu\nu}(x)]d^4x \tag{20}$$

yields the value

$$q=1\ . \tag{21}$$

This proves the assertion we have made above.

[1]H. Pagels and E. Tomboulis, Nucl. Phys. B143, 485 (1978).
[2]S. L. Adler, Phys. Rev. D 23, 2905 (1981); in Current Problems in Particle Theory: Unified Theory and Beyond, proceedings of the Johns Hopkins workshop, 1981 (unpublished), p. 5.
[3]A. A. Belavin, A. M. Polyakov, A. S. Schwartz, and Yu. S. Tyupkin, Phys. Lett. 59B, 85 (1975).
[4]G. 't Hooft, Phys. Rev. Lett. 37, 8 (1976); R. Jackiw, C. Nohl, and C. Rebbi, Phys. Rev. D 15, 1642 (1977); M. F. Atiyah, N. J. Hitchin, V. G. Drinfeld, and Yu. I. Manin, Phys. Lett. 65A, 185 (1978).