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Prescription dependence of renormalization-group functions

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The dependence on the renormalization scheme of the important QCD renormalization-group parameters β, γ, and δ is discussed. We compute the second coefficient for all three parameters for several renormalization schemes which include all the standard ones.

In any realistic quantum field theory one has to choose a renormalization prescription for handling the ultraviolet divergences which appear in the perturbative calculation of any physical quantity. In QED one has a natural choice of the renormalization scheme (RS), i.e., on-mass-shell renormalization. This is so because in electrodynamics one has a classical limit which allows one to define the parameters which appear in QED, or equivalently, QED has no essential infrared singularities. The problem of finding an appropriate choice is closely related to the one of the definition of a suitable expansion parameter which optimizes the convergence of the perturbative expansion.[1]

This problem shows up clearly in QCD. There is no classical limit, so one has an unbiased freedom to choose the RS. Of course, the price one might have to pay for this freedom is too large higher-order contributions. This is known to happen with 't Hooft's minimal-subtraction (MS) scheme[2] which in, e.g., heavy-quarkonia decay leads to much larger QCD corrections than other RS's such as the modified minimal-subtraction ($\overline{\text{MS}}$) scheme[3] or Weinberg's (W) momentum-subtraction scheme.[4] This seems to be a general feature and both $\overline{\text{MS}}$ and W schemes are preferred in a comparison with experiment because of their smaller higher-order corrections. Most theorists prefer the MS scheme because of its computational simplicity, others however take the W scheme because of its clearer physical interpretation. Thus two-loop calculations in the $\overline{\text{MS}}$ and W schemes are and will be done. In particular, the choice of a RS involves the renormalization group and the aim of this paper is to study the RS dependence of the renormalization-group functions in four different flavor-independent RS's at the two-loop level. The schemes are the MS, $\overline{\text{MS}}$, W, and a modification of Weinberg's scheme ($\overline{\text{W}}$) which we will suggest later on. As a matter of fact there has been some confusion on their properties as, e.g., that the two-loop β function is RS independent as long as the schemes are flavor independent; this however needs qualification.

The RS independence of the first two coefficients of the Callan-Symanzik β function[5] was first stated, although in a different form because of the presence of a mass term, in a later paper by Symanzik.[6] However, there the field theory which Symanzik considers is $\lambda\phi^4$ and not a gauge theory. His results can only be taken over to a gauge theory if the gauge parameter is held fixed, that is, if it does not itself satisfy a renormalization-group equation which governs its dependence on the renormalization point.[7] However, when the renormalization-group equations are obtained holding all bare parameters of the theory fixed, so that all the renormalized parameters vary, there is a difference between field theories with and without a gauge symmetry. This is because quantization introduces a new parameter in gauge theories, the gauge parameter, which gets renormalized and thus depends on the renormalization point μ. When the renormalization-group equations are obtained holding the bare gauge parameter fixed, a new dependence in μ comes in through the renormalized gauge parameter. This requires a revision of the properties of RS independence of the coefficients of the renormalization-group functions. This is what we would like to study here. Certainly this revision is not necessary if the renormalization-group functions are defined holding the renormalized gauge parameter fixed.

In any RS there is a massive parameter μ which enters and which specifies the renormalization point. In dimensional RS's as the MS and $\overline{\text{MS}}$ schemes, it enters when the bare coupling constant α_{0n}, which in $n=4+2\epsilon$ dimensions is not dimensionless, is written in a way that exhibits the dimension explictly, $\alpha_0=\alpha_{0n}(\mu^2)^\epsilon$. In momentum subtraction RS's such as the W and $\overline{\text{W}}$ schemes it enters as the point in Euclidean momentum space where the subtraction is performed, $p^2=-\mu^2$. In these latter schemes we will, for the sake of simplicity, use the same μ for the dimensions of the bare coupling constant and the subtraction point.

The renormalization-group functions β, γ, and δ are defined in terms of the renormalized coupling constant α, renormalized mass m_i, and renormalized gauge parameter a by

$$\mu\frac{d\alpha}{d\mu}=\alpha\beta(\alpha,x_i,a)\ , \tag{1a}$$

$$\frac{\mu}{m_j}\frac{dm_j}{d\mu}=-\gamma_j(\alpha,x_i,a)\ , \tag{1b}$$

$$\mu\frac{da}{d\mu}=\delta(\alpha,x_i,a)\ , \tag{1c}$$

where $i,j=1,\ldots,N_f$, N_f being the number of flavors, and all derivatives are performed with the bare quantities α_{0n}, x_{0i}, and a_0 held fixed. Furthermore $x_i\equiv m_i/\mu$. Of course α,x_i, and a are functions of μ which are obtained by integration of Eqs. (1a)–(1c). Since we will, without further recalling, refer only to flavor-independent schemes, the right-hand sides of Eqs. (1a)–(1c) will be independent of x_i, and γ_j will be flavor independent and we will thus drop its flavor index.

The expansions of the three functions β, γ, and δ in α are of the type

$$\begin{aligned}\beta&=\beta_0+\frac{\alpha}{\pi}\beta_1+\left[\frac{\alpha}{\pi}\right]^2\beta_2+\left[\frac{\alpha}{\pi}\right]^3\beta_3+\cdots\ ,\\ \gamma&=\frac{\alpha}{\pi}\gamma_1+\left[\frac{\alpha}{\pi}\right]^2\gamma_2+\cdots\ ,\\ \delta&=\frac{\alpha}{\pi}\delta_1+\left[\frac{\alpha}{\pi}\right]^2\delta_2+\cdots\ ,\end{aligned} \tag{2}$$

where $\beta_0=2\epsilon$ appears because of the fact that α_0, the dimensionless bare coupling constant, depends on μ for α_{0n} fixed. The coefficients β_j, γ_j, and δ_j are functions of a only. For $j=1$ all three are essentially (we will qualify later) RS independent and their values are

$$\begin{aligned}\beta_1^{\text{MS}}&=-\tfrac{11}{6}C_2(G)+\frac{N_f}{3}\ ,\\ \gamma_1^{\text{MS}}&=\tfrac{3}{2}C_2(R)\ ,\\ \delta_1^{\text{MS}}&=a\left[C_2(G)\left[\tfrac{13}{12}-\frac{a}{4}\right]-\frac{N_f}{3}\right]\ ,\end{aligned} \tag{3}$$

where $C_2(G)=N, C_2(R)=(N^2-1)/2N$ for color SU(N). For $j=2$ all three are known in the MS scheme,

$$\begin{aligned}\beta_2^{\text{MS}}&=-\tfrac{17}{12}C_2{}^2(G)+\tfrac{5}{12}C_2(G)N_f+\tfrac{1}{4}C_2(R)N_f\quad\text{(Jones,[8] Caswell[9])}\ ,\\ \gamma_2^{\text{MS}}&=\tfrac{3}{16}C_2{}^2(R)+\tfrac{97}{48}C_2(R)C_2(G)-\tfrac{5}{24}C_2(R)N_f\quad\text{[Nanopoulos and Ross (misprint),[10] Tarrach[11]]}\ ,\\ \delta_2^{\text{MS}}&=-\frac{a}{8}[\tfrac{1}{8}C_2{}^2(G)(2a^2+11a-59)+2N_fC_2(R)+\tfrac{5}{2}N_fC_2(G)]\quad\text{(Egoryan and Tarasov[12])}\end{aligned} \tag{4}$$

Recently, β_3^{MS} has also been computed.[13] The aim of this work is to compute β_2, γ_2, and δ_2 in the other three RS's: $\overline{\text{MS}}$, W, and $\overline{\text{W}}$. This will clarify some misunderstandings and will be necessary for renormalization-group calculations in these RS's.

The coupling-constant, mass, and gauge-parameter renormalization constants Z_α, Z_m, and Z_a are defined in a given RS as

$$\begin{aligned}\alpha_0&=\alpha_{0n}(\mu^2)^\epsilon=Z_\alpha^R\alpha^R(\mu)\ ,\\ m_0&=Z_m^Rm^R(\mu)\ ,\\ a_0&=Z_a^Ra^R(\mu)\ ,\end{aligned} \tag{5}$$

where R stands for MS, $\overline{\text{MS}}$, W, or $\overline{\text{W}}$. From this and Eqs. (1), one has

$$\begin{aligned}\beta^R-\beta_0&=-(Z_\alpha^R)^{-1}\mu\frac{dZ_\alpha^R}{d\mu}\ ,\\ \gamma^R&=(Z_m^R)^{-1}\mu\frac{dZ_m^R}{d\mu}\ ,\\ \delta^R&=-a^R(\mu)(Z_a^R)^{-1}\mu\frac{dZ_a^R}{d\mu}\end{aligned} \tag{6}$$

Before going on, notice that β_1^{MS}, β_2^{MS}, γ_1^{MS}, and γ_2^{MS} are gauge-parameter independent. For the coefficients of the β function this is a consequence of a result due to Caswell and Wilczek.[14] They state that in a RS where the renormalized coupling constant is gauge-parameter independent also the β function is gauge-parameter independent. This happens in the MS or $\overline{\text{MS}}$ schemes, but not in the W scheme. Then also the proof that β_2 is independent of the RS fails. Indeed let the relation between $\alpha^{\text{MS}}(\mu)$ and $\alpha^R(\mu)$ be written as

$$\alpha^{\text{MS}}(\mu)=\alpha^R(\mu)\left[1+A^R(a^R(\mu))\frac{\alpha^R(\mu)}{\pi}+B^R(a^R(\mu))\left[\frac{\alpha^R(\mu)}{\pi}\right]^2+\cdots\right], \tag{7}$$

where the whole μ dependence of A^R, B^R, etc., comes through the renormalized gauge parameter $a^R(\mu)$. Differentiating both sides with respect to μ we find immediately, using Eqs. (1) and (2),

$$\begin{aligned}\beta_1^R&=\beta_1^{\text{MS}}-\beta_0A^R\,,\\ \beta_2^R&=\beta_2^{\text{MS}}-\delta_1^R\frac{dA^R}{da^R}+2(A^{R^2}-B^R)\beta_0\,,\end{aligned} \tag{8}$$

so that β_1 is RS independent in the limit $\epsilon\to0$ and β_2 in the limit $\epsilon\to0$, and when A^R is gauge-parameter independent. Let us compute A^R in the different RS's. Recall that the MS scheme is defined such that the renormalization constants only have poles in ϵ. Thus at the one-loop level

$$Z_\alpha^{\text{MS}(2)}=1+\frac{\alpha^{\text{MS}}(\mu)}{\pi}\frac{1}{\epsilon}[\tfrac{11}{12}C_2(G)-\tfrac{1}{6}N_f]\,. \tag{9}$$

For the $\overline{\text{MS}}$ scheme the numerical factor $\gamma_E-\ln 4\pi$, γ_E being Euler's constant which always accompanies $1/\epsilon$, is included in the renormalization constant, i.e.,

$$Z_\alpha^{\overline{\text{MS}}(2)}=1+\frac{\alpha^{\overline{\text{MS}}}(\mu)}{\pi}\left[\frac{1}{\epsilon}+\gamma_E-\ln 4\pi\right]\times[\tfrac{11}{12}C_2(G)-\tfrac{1}{6}N_f]\,. \tag{10}$$

The W scheme is defined by the requirement that the renormalized Green's functions do not have radiative corrections at a certain Euclidean point in the limit of vanishing quark masses. For the W scheme the result can be taken from Ref. 15, and choosing, e.g., the three-gluon vertex subtracted at the symmetric point one has

$$\begin{aligned}Z_\alpha^{W(2)}=1+\frac{\alpha^W(\mu)}{\pi}\Big\{&[\tfrac{11}{12}C_2(G)-\tfrac{1}{6}N_f]\left[\frac{1}{\epsilon}+\gamma_E-\ln 4\pi\right]\\ &+C_2(G)[-\tfrac{23}{12}-\tfrac{1}{72}R-(1-a^W(\mu))\tfrac{5}{48}R+\tfrac{1}{16}(1-a^W(\mu))^2(1-\tfrac{2}{3}R)\\ &+\tfrac{1}{48}(1-a^W(\mu))^3]+N_f(\tfrac{1}{3}+\tfrac{2}{9}R)\Big\}\,,\end{aligned} \tag{11}$$

where $R=2.343\,907\ldots$. From the coupling-constant renormalization constants and Eqs. (5) and (7), one deduces

$$\begin{aligned}A^{\overline{\text{MS}}}&=(\gamma_E-\ln 4\pi)[\tfrac{11}{12}C_2(G)-\tfrac{1}{6}N_f]\,,\\ A^W&=(\gamma_E-\ln 4\pi)[\tfrac{11}{12}C_2(G)-\tfrac{1}{6}N_f]\\ &\quad+C_2(G)[-\tfrac{23}{12}-\tfrac{1}{72}R-(1-a^W(\mu))\tfrac{5}{48}R+\tfrac{1}{16}(1-a^W(\mu))^2(1-\tfrac{2}{3}R)\\ &\quad+\tfrac{1}{48}(1-a^W(\mu))^3]+N_f(\tfrac{1}{3}+\tfrac{2}{9}R)\,.\end{aligned} \tag{12}$$

This shows that even for the finite part ($\epsilon \to 0$), $\beta_2^{\rm W} \neq \beta_2^{\rm MS}$, and $\beta_2^{\rm W}$ is furthermore gauge-parameter dependent. Indeed, and as $\delta_1^{\rm W} = \delta_1^{\rm MS}$ we find for $\epsilon \to 0$

$$\beta_2^{\rm W} = \beta_2^{\rm MS} - a^{\rm W}(\mu) \left[C_2(G) \left[\frac{13}{12} - \frac{a^{\rm W}(\mu)}{4} \right] - \frac{N_f}{3} \right] C_2(G)$$

$$\times [\tfrac{5}{48} R - \tfrac{1}{8}(1 - a^{\rm W}(\mu))(1 - \tfrac{2}{3} R) - \tfrac{1}{16}(1 - a^{\rm W}(\mu))^2] \,. \tag{13}$$

Of course $\beta_2^{\rm W} = \beta_2^{\rm MS}$ had we held the renormalized gauge parameter fixed in the definition of the β function.

There exists one modification of the standard W scheme, $\overline{\rm W}$, in which $\beta_2^{\overline{\rm W}} = \beta_2^{\rm MS}$ for $\epsilon \to 0$. It corresponds to making the subtraction for the three-gluon vertex at a configuration where one of the momenta is zero, the other two Euclidean. Although this is an exceptional configuration, the renormalized charge in this scheme $\alpha^{\overline{\rm W}}(\mu)$ is infrared finite. More interestingly, it is gauge-parameter independent. Indeed, the coupling-constant renormalization constant is[16]

$$Z_\alpha^{\overline{\rm W}(2)} = 1 + \frac{\alpha^{\overline{\rm W}}(\mu)}{\pi} \left\{ [\tfrac{11}{12} C_2(G) - \tfrac{1}{6} N_f] \left[\frac{1}{\epsilon} + \gamma_E - \ln 4\pi \right] + C_2(G)(-\tfrac{35}{18}) + \tfrac{11}{18} N_f \right\} \tag{14}$$

and

$$A^{\overline{\rm W}} = (\gamma_E - \ln 4\pi)[\tfrac{11}{12} C_2(G) - \tfrac{1}{6} N_f] - \tfrac{35}{18} C_2(G) + \tfrac{11}{18} N_f \,, \tag{15}$$

which does not depend on $a^{\rm W}(\mu)$. There is a complete cancellation of the gauge-parameter-dependent parts of the gluon self-energy and the three-gluon vertex in this exceptional subtraction point, so that the renormalized coupling constant does not depend on it. This is a modification of Weinberg's scheme which is also very convenient from the computational point of view.

The dependence on $a^{\rm W}(\mu)$ of $\beta_2^{\rm W}$ brings up a new problem. Recall that when β_2^R does not depend on $a^R(\mu)$ the integration of Eq. (1a) leads to

$$\frac{\alpha^R(\mu)}{\pi} = \frac{1}{-\beta_1^R \frac{1}{2} \ln(\mu^2/\Lambda_R{}^2)} - \frac{1}{[-\beta_1^R \frac{1}{2} \ln(\mu^2/\Lambda_R{}^2)]^2} \frac{\beta_2^R}{\beta_1^R} \ln \ln \frac{\mu^2}{\Lambda_R{}^2} \,, \tag{16}$$

where Λ_R is the RS-dependent, but μ-independent, integration constant, which is only correctly defined by the two-loop expression given in Eq. (16). In the W scheme, however, $\beta_2^{\rm W}$ depends on $a^{\rm W}(\mu)$ and Eq. (1a) is coupled to Eq. (1c) at the two-loop level, and no definition of $\Lambda_{\rm W}$ analogous to $\Lambda_{\rm MS}$ exists *a priori*. We will show now that fortunately this is not so at the two-loop level.

Let us integrate Eq. (1c) to one-loop order. The result is[17]

$$a^R(\mu) = \frac{\hat{a}^R}{\dfrac{\hat{a}^R}{\frac{13}{3} - \frac{4}{3} N_f / C_2(G)} + \left[\frac{1}{2} \ln \dfrac{\mu^2}{\Lambda_R{}^2} \right]^{(1/\beta_1^R)[(13/12)C_2(G) - (1/3)N_f]}} \,, \tag{17}$$

where $\hat{a}^R$ is the RS-dependent, μ-independent constant of integration. The limit for large μ is

$$\begin{aligned} a^R(\mu) \underset{\mu^2/\Lambda_R{}^2 \to \infty}{=} & \frac{13}{3} - \frac{4}{3} \frac{N_f}{C_2(G)} \quad \text{if } N_f < \tfrac{13}{4} C_2(G) \,, \\ a^R(\mu) \underset{\mu^2/\Lambda_R{}^2 \to \infty}{=} & 0 \quad \text{if } \tfrac{13}{4} C_2(G) < N_f < \tfrac{11}{2} C_2(G) \,. \end{aligned} \tag{18}$$

Corrections to these limits are of the order of negative rational powers of $\ln(\mu^2/\Lambda_R{}^2)$. Thus to leading two-loop order one can substitute the gauge parameter $a^{\rm W}(\mu)$ in $\beta_2^{\rm W}$ by its limits. But at precisely these two values the gauge-dependent part is zero [see Eq. (13)]. Thus to leading two-loop order $\beta_2^{\rm W} = \beta_2^{\rm MS}$ for $\epsilon \to 0$, and the two-loop integration of Eq. (1a) can be done for the W scheme as for the MS scheme. Because of this one can introduce a $\Lambda_{\rm W}$ in analogy to $\Lambda_{\rm MS}$.

Let us now study γ_2^R. As for the coupling constant, the renormalized masses in two different RS's are also related by a power series in the coupling constant

$$m^{\mathrm{MS}}(\mu)=m^{R}(\mu)\left[1+C^{R}(a^{R}(\mu))\frac{\alpha^{R}(\mu)}{\pi}+D^{R}(a^{R}(\mu))\left[\frac{\alpha^{R}(\mu)}{\pi}\right]^{2}+\cdots\right]. \tag{19}$$

Taking derivatives on both sides one finds

$$\begin{aligned}\gamma_1^R&=\gamma_1^{\mathrm{MS}}+\beta_0C^R\,,\\ \gamma_2^R&=\gamma_2^{\mathrm{MS}}+A^R\gamma_1^{\mathrm{MS}}+C^R\beta_1^{\mathrm{MS}}+\frac{dC^R}{da^R}\delta_1^R+(2D^R-C^{R^2}-C^RA^R)\beta_0\,,\end{aligned} \tag{20}$$

so that γ_1 is RS independent for $\epsilon\to 0$. The situation for γ_2^R is more complicated. The mass renormalization constants for the different schemes can be obtained from Ref. 15:

$$\begin{aligned}Z_m^{\mathrm{MS}(2)}&=1+\frac{\alpha^{\mathrm{MS}}(\mu)}{\pi}\frac{1}{\epsilon}\frac{3}{4}C_2(R)\,,\\ Z_m^{\overline{\mathrm{MS}}(2)}&=1+\frac{\alpha^{\overline{\mathrm{MS}}}(\mu)}{\pi}\frac{1}{\epsilon}\frac{3}{4}C_2(R)[1+\epsilon(\gamma_E-\ln 4\pi)]\,,\\ Z_m^{\overset{(-)}{\mathrm{W}}(2)}&=1+\frac{\alpha^{\overset{(-)}{\mathrm{W}}}(\mu)}{\pi}\left\{\tfrac{3}{4}C_2(R)\left[\frac{1}{\epsilon}+\gamma_E-\ln 4\pi\right]-\tfrac{1}{4}C_2(R)[4+a^{\overset{(-)}{\mathrm{W}}}(\mu)]\right\},\end{aligned} \tag{21}$$

so that

$$\begin{aligned}C^{\overline{\mathrm{MS}}}&=(\gamma_E-\ln 4\pi)\tfrac{3}{4}C_2(R)\,,\\ C^{\overset{(-)}{\mathrm{W}}}&=(\gamma_E-\ln 4\pi)\tfrac{3}{4}C_2(R)-\tfrac{1}{4}C_2(R)[4+a^{\overset{(-)}{\mathrm{W}}}(\mu)]\,.\end{aligned} \tag{22}$$

These results together with Eqs. (8), (12), and (15) lead in the limit $\epsilon\to 0$ to

$$\begin{aligned}\gamma_2^{\overline{\mathrm{MS}}}&=\gamma_2^{\mathrm{MS}}\,,\\ \gamma_2^{\mathrm{W}}&=\gamma_2^{\mathrm{MS}}+\tfrac{1}{4}C_2(R)\{\,C_2(G)[\,-\tfrac{19}{6}-\tfrac{1}{12}R-(1-a^{\mathrm{W}}(\mu))(\tfrac{5}{4}+\tfrac{5}{8}R)+(1-a^{\mathrm{W}}(\mu))^2(\tfrac{5}{8}-\tfrac{1}{4}R)\\ &\qquad+\tfrac{1}{8}(1-a^{\mathrm{W}}(\mu))^3]+N_f(\tfrac{2}{3}+\tfrac{4}{3}R)\,\}\,,\\ \gamma_2^{\overline{\mathrm{W}}}&=\gamma_2^{\mathrm{MS}}+\tfrac{1}{4}C_2(R)\{\,C_2(G)[-\tfrac{10}{3}-(1-a^{\overline{\mathrm{W}}}(\mu))\tfrac{5}{4}+(1-a^{\overline{\mathrm{W}}}(\mu))^2\tfrac{1}{4}]+\tfrac{7}{3}N_f\,\}\,.\end{aligned} \tag{23}$$

Notice that γ_2 does not change when going from the MS to the $\overline{\mathrm{MS}}$ scheme. We have here the same problem as for the integration of Eq. (1a) but now instead for the integration of (1b). For those RS's for which γ_2 is gauge-parameter independent the integration to two-loop order of Eq. (1b) gives[11]

$$m^R(\mu)=\frac{\hat{m}^R}{[\frac{1}{2}\ln(\mu^2/\Lambda_R{}^2)]^{\gamma_1^R/\beta_1^R}}\left[1+\frac{\gamma_2^R-\gamma_1^R\beta_2^R/\beta_1^R}{(\beta_1^R)^2\frac{1}{2}\ln(\mu^2/\Lambda_R{}^2)}-\frac{\gamma_1^R\beta_2^R\ln\ln(\mu^2/\Lambda_R{}^2)}{(\beta_1^R)^3\frac{1}{2}\ln(\mu^2/\Lambda_R{}^2)}\right], \tag{24}$$

where $\hat{m}^R$ is a RS-dependent, μ-independent constant of integration.

This is then valid for the MS and $\overline{\mathrm{MS}}$ schemes. However, if one wants to use this expression for the W or $\overline{\mathrm{W}}$ schemes one has to use γ_2^{W} and $\gamma_2^{\overline{\mathrm{W}}}$ as given in Eq. (23), substituting the gauge parameter in these expressions by its limits as given in Eq. (18). In these limits neither of them coincides with γ_2^{MS}.

Although certainly Eq. (1c) is of less physical relevance than the other two, we will for the sake of completeness quickly consider δ_2^R. As before

$$a^{\mathrm{MS}}(\mu)=a^{R}(\mu)\left[1+E^{R}(a^{R}(\mu))\frac{\alpha^{R}(\mu)}{\pi}+F^{R}(a^{R}(\mu))\left[\frac{\alpha^{R}(\mu)}{\pi}\right]^{2}+\cdots\right] \tag{25}$$

and differentiating both sides

$$\delta_1^R=\delta_1^{\mathrm{MS}}-a^R E^R\beta_0\,,$$

$$\bar{\delta}_2^R=\delta_2^{\mathrm{MS}}-\delta_1^{\mathrm{MS}}(E^R-A^R)-a^R E^R\beta_1^{\mathrm{MS}}+\frac{d\delta_1^{\mathrm{MS}}}{da^{\mathrm{MS}}}E^R a^R-a^R\delta_1^R\frac{dE^R}{da^R}+a^R(E^{R^2}+A^R E^R-2F^R)\beta_0\,, \tag{26}$$

so that again δ_1 is RS independent for $\epsilon\to 0$. For δ_2 one needs the gauge-parameter renormalization constants which are[15]

$$Z_a^{\mathrm{MS}(2)}=1+\frac{\alpha^{\mathrm{MS}}(\mu)}{\pi}\frac{1}{\epsilon}[-\tfrac{1}{8}C_2(G)(\tfrac{13}{3}-a^{\mathrm{MS}}(\mu))+\tfrac{1}{6}N_f]\,,$$

$$Z_a^{\overline{\mathrm{MS}}(2)}=1+\frac{\alpha^{\overline{\mathrm{MS}}}(\mu)}{\pi}\left[\frac{1}{\epsilon}+\gamma_E-\ln 4\pi\right][-\tfrac{1}{8}C_2(G)(\tfrac{13}{3}-a^{\overline{\mathrm{MS}}}(\mu))+\tfrac{1}{6}N_f]\,, \tag{27}$$

$$Z_a^{\overset{(-)}{\mathrm{W}}(2)}=1+\frac{\alpha^{\overset{(-)}{\mathrm{W}}}(\mu)}{\pi}\left\{\left[\frac{1}{\epsilon}+\gamma_E-\ln 4\pi\right][-\tfrac{1}{8}C_2(G)(\tfrac{13}{3}-a^{\overset{(-)}{\mathrm{W}}}(\mu))+\tfrac{1}{6}N_f]\right.$$
$$\left.+\tfrac{1}{8}C_2(G)[\tfrac{62}{9}-2(1-a^{\overset{(-)}{\mathrm{W}}}(\mu))+\tfrac{1}{2}(1-a^{\overset{(-)}{\mathrm{W}}}(\mu))^2]-\tfrac{5}{18}N_f\right\}\,,$$

so that

$$E^{\overline{\mathrm{MS}}}=(\gamma_E-\ln 4\pi)[-\tfrac{1}{8}C_2(G)(\tfrac{13}{3}-a^{\overline{\mathrm{MS}}}(\mu))+\tfrac{1}{6}N_f]\,,$$

$$E^{\overset{(-)}{\mathrm{W}}}=(\gamma_E-\ln 4\pi)[-\tfrac{1}{8}C_2(G)(\tfrac{13}{3}-a^{\overset{(-)}{\mathrm{W}}}(\mu))+\tfrac{1}{6}N_f] \tag{28}$$
$$+\tfrac{1}{8}C_2(G)[\tfrac{62}{9}-2(1-a^{\overset{(-)}{\mathrm{W}}}(\mu))+\tfrac{1}{2}(1-a^{\overset{(-)}{\mathrm{W}}}(\mu))^2]-\tfrac{5}{18}N_f\,,$$

and one finds for $\epsilon\to 0$

$$\delta_2^{\overline{\mathrm{MS}}}=\delta_2^{\mathrm{MS}} \tag{29}$$

and more complicated expressions for δ_2^{W} and $\delta_2^{\overline{\mathrm{W}}}$. Notice that again the two-loop coefficient is the same for the MS and $\overline{\mathrm{MS}}$ schemes. This does not happen for δ_2^{W} and $\delta_2^{\overline{\mathrm{W}}}$ which are different from δ_2^{MS} even in the asymptotic limit of Eq. (18) for $N_f<\frac{13}{4}C_2(G)$. For $\frac{13}{4}C_2(G)<N_f<\frac{11}{2}C_2(G)$ the asymptotic limit is 0 and then of course all the δ_2's are equal as they are all zero.

Let us summarize our results for the RS's we have studied. Recall, however, that these statements refer to renormalization-group functions defined holding fixed the bare parameters of the theory. The second coefficient of the β function is RS independent among schemes for which the one-loop renormalized charge is gauge-parameter independent. When this is not so there is a new gauge-parameter-dependent contribution which is however asymptotically vanishing. The second coefficient of the γ function is RS independent among the MS and $\overline{\mathrm{MS}}$ schemes for which both the one-loop renormalized charge and the one-loop renormalized mass are gauge-parameter independent. In the other schemes there is a new contribution, even asymptotically. The same situation happens for the second coefficient of the function. Notice that all the second coefficients do not change in going from the MS to the $\overline{\mathrm{MS}}$ scheme. This is probably due to the fact that the modification of the MS scheme can be introduced already at the level of regularization by weighting the n-dimensional Feynman-integral measure with the appropriate Euler function. Then the renormalization proceeds as before, just subtracting the poles.

Note added. While we revised the manuscript a paper by Stevenson[18] has appeared where similar problems are touched upon.

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