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Series solutions for the Klein-Gordon equation in Schwarzschild space-time

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Three pairs of independent solutions of the radial equation corresponding to a massive Klein-Gordon field in Schwarzschild space-time are given. They are defined through different series expansions for the cases $r \to 0$, $r \to \infty$, and $r \sim 2GM$.

I. INTRODUCTION

It is well known[1] that the mode solutions of the Klein-Gordon equation

$$(\Box^2 + m^2)\phi = 0 \tag{1.1}$$

in Schwarzschild space-time, described by the line element

$$ds^2 = \left[1 - \frac{2GM}{r}\right] dt^2 - \left[1 - \frac{2GM}{r}\right]^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) , \tag{1.2}$$

can be written in the form[1]

$$r^{-1} R_{\omega l}(r) Y_{lm}(\theta,\varphi) \exp(-i\omega t) , \tag{1.3}$$

where the Y_{lm} are spherical harmonics and the radial functions $R_{\omega l}$ satisfy

$$\frac{d^2 R_{\omega l}}{dr^{*2}} + \left[\omega^2 - \left[m^2 + \frac{l(l+1)}{r^2} + \frac{2GM}{r^3}\right]\left[1 - \frac{2GM}{r}\right]\right] R_{\omega l} = 0 \tag{1.4}$$

with r^* (the Regge-Wheeler coordinate) defined by

$$r^* = r + 2GM \ln \left| \frac{r}{2GM} - r \right| . \tag{1.5}$$

For most purposes, it is not necessary to solve Eq. (1.4) in detail, in particular, when attention is restricted to observations made in the asymptotic region $r \to \infty$. In this region the solutions of (1.4) are

$$\exp(\pm ikr^*), \quad k = (\omega^2 - m^2)^{1/2} , \tag{1.6}$$

and (1.3) reduces to

$$r^{-1} Y_{lm}(\theta,\varphi) \exp[-i(\omega t \mp kr^*)] . \tag{1.7}$$

Particle creation by a collapsing spherical body—the celebrated Hawking effect[2]—is a beautiful example showing how one can proceed without knowing the solutions of the wave equation (1.1) in terms of simple functions.

However, a more complete investigation of field theories in curved space-time certainly require a knowledge of the solutions of (1.4), at least of approximate solutions. In a recent paper with Tarrach[3] on the existence of a nontrivial $\lambda\phi^4$ theory on a Schwarzschild background—in which the present study has originated—we became aware of this necessity. Here we shall construct three different pairs of independent solutions of Eq. (1.4) in the form of series expansions for $r \to 0$, $r \to \infty$, and near the Schwarzschild radius $r = 2GM$. We shall show how these series can be thoroughly obtained by a recurrent procedure. For related work dealing with solutions of Eq. (1.4) see Ref. 4.

II. SOLUTIONS IN THE REGION INTERIOR TO THE SCHWARZSCHILD HORIZON

In the region $0 < r < 2GM$, an appropriate ansatz for solving Eq. (1.4) turns out to be

$$R(r^*) = \alpha \exp\left\{ \pm i \left[kr^* + a_0 \ln\left[\frac{r}{2GM}\right] + a_1 \frac{r}{2GM} + a_2 \left[\frac{r}{2GM}\right]^2 + a_3 \left[\frac{r}{2GM}\right]^3 + \cdots \right]\right\} , \tag{2.1}$$

where the a_s, $s = 0,1,2,\ldots$ are dimensionless (complex) constants which, in general, depend on ω and l. They are not difficult to obtain by direct substitution of an ansatz (2.1) into Eq. (1.5) and comparison of the different powers of r^{s-4}, $s = 0,1,2,3$. We obtain

$$\begin{aligned} &a_0 = \mp i, \quad a_1 = \pm i l(l+1) , \\ &a_2 = kGM \pm \frac{i}{4} l(l+1)[l(l+1)+2] , \\ &a_3 = \tfrac{2}{3} kGM \pm i \left[\tfrac{4}{9} m^2 G^2 M^2 + \frac{l(l+1)}{9}[3 + \tfrac{5}{2} l(l+1) + l^2(l+1)^2] \right] . \end{aligned} \tag{2.2}$$

After some further calculations, we obtain the following recurrent expression for the general coefficient a_s, $s \geq 4$, in terms of the preceding ones:

$$a_{s+4}=\frac{1}{(s+4)^2}\left[(s+3)(2s+7)a_{s+3}+[-(s+3)\pm 4ikGM](s+2)a_{s+2}\right.$$
$$\pm i\left[-4kGM(s+1)a_{s+1}+\sum_{j=1}^{s+1} j(s-j+2)a_j a_{s-j+2}\right.$$
$$\left.\left.-2\sum_{j=1}^{s+2} j(s-j+3)a_j a_{s-j+3}+\sum_{j=1}^{s+3} j(s-j+4)a_j a_{s-j+4}\right]\right],\quad s\geq 0\ . \tag{2.3}$$

Notice that by making use of this expression *any* term of the series expansion (2.1) is obtained in a straightforward way. In particular, up to terms of $O((r/2GM)^3)$, we have

$$R_{\omega l}(r^*)=\alpha\frac{r}{2GM}\exp\left[-\frac{l(l+1)}{2GM}r\left[1+\frac{l(l+1)+2}{8GM}r\right]+O\left[\left[\frac{r}{2GM}\right]^3\right]\right]$$
$$\times\exp\left\{\pm ik\left[\left[r^*+\frac{r^2}{4GM}\right]+2GM\,O\left[\left[\frac{r}{2GM}\right]^3\right]\right]\right\},\quad 0<r<2GM\ . \tag{2.4}$$

III. SOLUTIONS IN THE ASYMPTOTIC REGION $r\to\infty$

In the asymptotic region $r\to\infty$, that is $r/(2GM)\gg 1$, an ansatz which proves to be very convenient in order to solve Eq. (1.4) is

$$R(r^*)=\alpha\exp\left\{\pm i\left[kr^*+b_0\ln\left[\frac{r}{2GM}\right]+b_1\frac{2GM}{r}+b_2\left[\frac{2GM}{r}\right]^2+b_3\left[\frac{2GM}{r}\right]^3+\cdots\right]\right\},\quad \frac{r}{2GM}>1, \tag{3.1}$$

where the b_s, $s=0,1,2,3,\ldots$, are dimensionless constants. Substituting (3.1) into (1.4) and equating the successive powers of r^{-s}, $s=1,2,3,4,\ldots$, we obtain

$$b_0=\frac{GMm^2}{k},\quad b_1=\frac{m^2}{4k}\left[GM\left[\frac{m^2}{k^2}-4\right]+\frac{l(l+1)}{GMm^2}\pm\frac{i}{k}\right],$$
$$b_2=\frac{m^2}{16k}\left[-2GM\left[\frac{m^4}{2k^4}-\frac{m^2}{k^2}+4\right]-\frac{l^2+l-1}{k^2GM}+\frac{2}{GMm^2}\mp i\left[\frac{2m^2}{k^3}+\frac{l(l+1)}{kG^2M^2m^2}\right]\right], \tag{3.2}$$
$$b_3=\frac{1}{3}\left[2\left[1-\frac{b_0}{2kGM}\right]b_2+\frac{b_1}{4kGM}(b_1+2b_0)-\frac{1}{4kGM}\pm\frac{i}{2kGM}(-3b_2+5b_1+b_0)\right].$$

The recurrence for an arbitrary coefficient b_s, $s\geq 4$, in terms of the preceding ones, is given by

$$b_{s-1}=\frac{1}{s-1}\left[\left[1-\frac{m^2}{2k^2}\right](s-2)b_{s-2}+\frac{m^2}{k^2}(s-3)b_{s-3}-\frac{m^2}{2k^2}(s-4)b_{s-4}\right.$$
$$+\frac{1}{4kGM}\left[\sum_{j=1}^{s-3} j(s-j-2)b_j b_{s-j-2}-2\sum_{j=1}^{s-4} j(s-j-3)b_j b_{s-j-3}+\sum_{j=1}^{s-5} j(s-j-4)b_j b_{s-j-4}\right]$$
$$\left.\pm\frac{i}{4kGM}[-(s-1)(s-2)b_{s-2}+(2s-3)(s-3)b_{s-3}-(s-2)(s-4)b_{s-4}]\right],\quad s\geq 5\ . \tag{3.3}$$

As before, this expression provides a direct way to obtain any of the coefficients b_s of the series expansion (3.1) where the first of them are given by (3.2). For the radial wave component $R_{\omega l}(r^*)$, we get

$$R_{\omega l}(r^*)=\alpha\exp\left[-\frac{GMm^2}{2k^2r}+\frac{G^2M^2m^4}{2k^4r^2}+\frac{l(l+1)}{4k^2r^2}+O\left[\left[\frac{2GM}{r}\right]^3\right]\right]$$
$$\times\exp\left\{\pm i\left[kr^*+\frac{GMm^2}{k}\ln\left[\frac{r}{2GM}\right]+\frac{G^2M^2m^2}{2kr}\left[\frac{m^2}{k^2}-4\right]+\frac{l(l+1)}{2kr}\right.\right.$$
$$\left.\left.-\frac{G^3M^3m^2}{2kr^2}\left[\frac{m^4}{2k^4}-\frac{m^2}{k^2}+4\right]-\frac{GMm^2(l^2+l-1)}{4k^3r^2}+\frac{GM}{2kr^2}\right]+O\left[\left[\frac{2GM}{r}\right]^3\right]\right\}. \tag{3.4}$$

IV. SOLUTIONS NEAR THE SCHWARZSCHILD RADIUS $r_S=2GM$

Let us start once more from Eq. (1.4). We shall now find a couple of independent solutions of it in the neighborhood of the Schwarzschild radius $r_S=2GM$. To this end, let us set

$$\rho=1-\frac{2GM}{r} . \tag{4.1}$$

It is easy to see that (1.4) can be exactly transformed into

$$\frac{d^2R_{\omega l}}{dr^{*2}}+\left[\omega^2-\left[m^2+\frac{l^2+l+1}{4G^2M^2}\right]\rho+\frac{2l^2+2l+3}{4G^2M^2}\rho^2-\frac{l^2+l+3}{4G^2M^2}\rho^3+\frac{1}{4G^2M^2}\rho^4\right]R_{\omega l}=0 . \tag{4.2}$$

The following ansatz turns out to be a good one for solving Eq. (4.2):

$$R_{\omega l}(r^*)=\alpha\exp\{\pm i\omega[r^*+g_{\omega l}(r^*)]\},\quad g(r^*)=c_1(1-\rho)+c_2\rho^2+c_3\rho^3+\cdots , \tag{4.3}$$

where the c_s are constants with dimensions of length, which in general depend on ω and l. Substituting the ansatz (4.3) into Eq. (4.2), we obtain

$$\begin{aligned}\Bigg\{&-\omega^2(1-\rho)^2\left[4GM\left[-c_1\rho+\sum_{j=2}^{\infty}jc_j\rho^j\right]+(1-\rho)^2\left[-c_1\rho+\sum_{j=2}^{\infty}jc_j\rho^j\right]^2\right]\\&\pm i\omega\rho(1-\rho)^3\left[-c_1+(3c_1+4c_2)\rho-\sum_{j=2}^{\infty}[j(j+2)c_j-(j+1)^2c_{j+1}]\rho^j\right]\\&\qquad-(4G^2M^2m^2+l^2+l+1)\rho+(2l^2+2l+3)\rho^2-(l^2+l+3)\rho^3+\rho^4\Bigg\}R=0 .\end{aligned} \tag{4.4}$$

Setting the coefficients of the terms in ρ and ρ^2 equal to zero we obtain, respectively,

$$\begin{aligned}c_1&=2GM\frac{\overline{m}^2+l^2+l+1}{1+4\varpi^2}\left[2\pm\frac{i}{\varpi}\right] ,\\ c_2&=-\frac{GM}{1+\varpi^2}\left[\frac{(\overline{m}^2+l^2+l+1)^2(4\varpi^2-5)}{2(1+4\varpi^2)^2}+\frac{(\overline{m}^2+l^2+l+1)(4\varpi^2+7)}{1+4\varpi^2}\right.\\&\qquad\left.\pm\frac{i}{\varpi}\left[\frac{(\overline{m}^2+l^2+l+1)(8\varpi^2-1)}{2(1+\varpi^2)(1+4\varpi^2)^2}+\frac{3(\overline{m}^2+l^2+l+1)}{(1+\varpi^2)(1+4\varpi^2)}-(l^2+l+\tfrac{3}{2})\right]\right] ,\end{aligned} \tag{4.5}$$

where we have introduced the dimensionless quantites

$$\overline{m}\equiv 2GMm,\quad \varpi\equiv 2GM\omega . \tag{4.6}$$

Moreover, from the coefficients of ρ^3 and ρ^4 one obtains the relations

$$\begin{aligned}c_3=\frac{2}{9+4\varpi^2}&\left[\tfrac{2}{3}\varpi^2\left[c_1+4c_2+\frac{c_1{}^2+c_1c_2}{GM}\right]+2(3c_1+4c_2)-\tfrac{2}{3}GM(l^2+l+3)\right.\\&\left.\pm i\varpi\left[-3c_1-\tfrac{4}{3}c_2+\frac{c_1{}^2+c_1c_2}{GM}-\frac{GM(l^2+l+3)}{\varpi^2}\right]\right]\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}c_4=\frac{1}{2(4+\varpi^2)}&\left[\frac{GM}{2}-\varpi^2\left[c_2-3c_3+\frac{1}{2GM}(\tfrac{3}{2}c_1{}^2+4c_1c_2+c_2{}^2)\right]-5c_1-18c_2+21c_3\right.\\&\left.\pm i\varpi\left[\frac{GM}{\varpi^2}+\tfrac{5}{2}c_1+7c_2-\tfrac{9}{2}c_3+\frac{1}{GM}(\tfrac{3}{2}c_1{}^2+4c_1c_2+c_2{}^2)\right]\right] ,\end{aligned} \tag{4.8}$$

respectively.

In general, for the coefficient c_s, $s\geq 5$, we derive the equation

$$c_s=\frac{2}{s(s^2+4\varpi^2)}\left[\frac{\varpi^2}{GM}\Sigma_s+2\varpi^2[2(s-1)c_{s-1}-(s-2)c_{s-2}]\right.$$
$$+s[(s-1)(2s-1)c_{s-1}-3(s-1)(s-2)c_{s-2}+(s-3)(2s-3)c_{s-3}-\tfrac{1}{2}(s-2)(s-4)c_{s-4}]$$
$$\pm i\varpi\left[\frac{s}{2GM}\Sigma_s-2(s-1)^2c_{s-1}+(s-2)(5s-6)c_{s-2}-2(s-3)(2s-3)c_{s-3}\right.$$
$$\left.\left.+(s-2)(s-4)c_{s-4}\right]\right],\quad s\geq 5\ , \tag{4.9}$$

where

$$\Sigma_s\equiv c_1[(s-1)c_{s-1}-4(s-2)c_{s-2}+6(s-3)c_{s-3}-4(s-4)c_{s-4}+(s-5)c_{s-5}]$$
$$-\tfrac{1}{2}\sum_{j=2}^{s-2}j(s-j)c_jc_{s-j}+2\sum_{j=3}^{s-3}j(s-j-1)c_jc_{s-j-1}-3\sum_{j=4}^{s-4}j(s-j-2)c_jc_{s-j-2}$$
$$+2\sum_{j=5}^{s-5}j(s-j-3)c_jc_{s-j-3}-\tfrac{1}{2}\sum_{j=6}^{s-6}j(s-j-4)c_jc_{s-j-4}\ . \tag{4.10}$$

As in the preceding cases, the recurrence (4.9) permits us to obtain c_s in a straightforward way for any value of s.

Notice that Eqs. (4.9) and (4.10) and also the former ones (2.3) and (3.3) are very well suited for computer manipulations. This is very convenient because of the lack of closed algebraic expressions for the exact solutions of Eq. (1.4) in any of the ranges considered.

Substituting (4.6) into (1.4), we obtain

$$R_{\omega l}(r^*)=\alpha\exp\left\{-\frac{\overline{m}^2+l^2+l+1}{1+4\varpi^2}\frac{2GM}{r}+\left[\frac{\overline{m}^2+l^2+l+1}{2(1+\varpi^2)(1+4\varpi^2)}\left[\frac{(8\varpi^2-1)(\overline{m}^2+l^2+l+1)}{2(1+4\varpi^2)}+3\right]\right.\right.$$
$$\left.\left.-\frac{2l^2+2l+3}{4(1+\varpi^2)}\right]\left[1-\frac{2GM}{r}\right]+O\left[\left[1-\frac{2GM}{r}\right]^2\right]\right\}$$
$$\times\exp\left[\pm i\varpi\left\{\frac{r^*}{2GM}+\frac{2(\overline{m}^2+l^2+l+1)}{1+4\varpi^2}\frac{2GM}{r}\right.\right.$$
$$-\left[\frac{\overline{m}^2+l^2+l+1}{2(1+\varpi^2)(1+4\varpi^2)}\left[\frac{(4\varpi^2-5)(\overline{m}^2+l^2+l+1)}{2(1+4\varpi^2)}+4\varpi^2+7\right]\right.$$
$$\left.\left.\left.-\frac{2l^2+2l+3}{4(1+\varpi^2)}\right]\left[1-\frac{2GM}{r}\right]\right\}+O\left[\left[1-\frac{2GM}{r}\right]^2\right]\right],\quad r\sim 2GM\ . \tag{4.11}$$

These solutions (4.11) are apparently best suited for performing the calculations involved in the study of quantum scalar fields in a Schwarzschild space-time background.[1,3] The reason is that they extend to $r\to\infty$ without any problem and, moreover, viewed as functions of ω, they do not develop singularities at $k=0$ or at any other value of ω—as happens to be the case with the asymptotic solutions (3.4). In fact, for $r\to\infty$ the solutions (4.11) reduce to

$$R_{\omega l}(r^*)=\beta\exp(\pm i\omega r^*)\ , \tag{4.12}$$

as it should be [cf. Eq. (1.6)], where β is some constant. Moreover, from (4.5)–(4.8) and the general recurrence (4.9) we see that these coefficients will never become singular for any real value of ω. This is a necessary condition when one has to integrate over the full range of k (massive case) or ω (massless case).

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[1]See, for instance, N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982).

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