Exact solutions of the massive Klein-Gordon-Schwarzschild equation

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A continuous, one-complex-parameter family pair of solutions of the Klein-Gordon equation for a massive particle in Schwarzschild space-time, given in terms of elementary functions, is derived. The domain of analyticity of the solutions is $0 \le r \le +\infty$ for extensive regions of the complex plane of the parameter.

Attaining solutions of the field equations in a curved background is interesting in different contexts: (i) for the computation of the quantum propagator in a curved background; (ii) in classical black-hole scattering; (iii) for questions related to a possible proof of nontriviality of the $\lambda\phi^4$ theory in a curved background; (iv) in the study of gravitational collapse; etc. We shall here deal only with the case of a spin-zero particle of mass m in a Schwarzschild background, as created by a noncharged, nonrotating black hole of mass M. The metric is

$$ds^2 = \left[1-\frac{2GM}{r}\right]dt^2 - \left[1-\frac{2GM}{r}\right]^{-1}dr^2 - r^2(d\theta^2+\sin^2\theta\, d\phi^2) , \qquad (1)$$

and the corresponding field equation is

$$(\nabla_\mu \nabla^\mu - m^2)\Phi = 0 . \qquad (2)$$

The solutions of (2) can be written as

$$r^{-1} f_{\omega l}(r) Y_{lm}(\theta,\phi)\exp(-i\omega t) , \qquad (3)$$

the Y_{lm} being spherical harmonics. The functions $f_{\omega l}(r)$ satisfy the radial equation[1,2]

$$\frac{d^2 f_{\omega l}(r)}{dr_*^2} + \left[\omega^2 - \left[m^2 + \frac{l(l+1)}{r^2} + \frac{2GM}{r^3}\right]\times\left[1-\frac{2GM}{r}\right]\right] f_{\omega l}(r) = 0 , \qquad (4)$$

with

$$r_* = r + 2GM \ln\left|\frac{r}{2GM}-1\right| . \qquad (5)$$

Notice that adopting this definition with absolute value, we will be able to use the coordinate r_* also when $r < 2GM = r_S$ (r_S is the Schwarzschild radius).

Our aim is to find exact solutions of Eq. (4) valid in extensive regions (as large as possible) of space-time. We start with an ansatz of the form

$$f_{\omega l}(r) = \alpha \exp\{+i[\omega r_* + g(\rho)]\} , \qquad (6)$$

where

$$\rho = 1 - \frac{2GM}{r} , \qquad (7)$$

and where for the function $g(\rho)$ we write

$$g(\rho) = b\ln(1-\rho) + \sum_{n=-1}^{\infty}(a_n + b_n\rho)z^n , \quad z = 4\rho(1-\rho) . \qquad (8)$$

For the moment we shall only consider the plus sign in front of the i in (6) but an ansatz with $-i$ would be equally good. Notice that the region exterior to the event horizon, $r > r_S$, is here $0<\rho<1$. We always have $0 \le z \le 1$, and $z \to 0$ both when $r \to r_S$ and when $r \to \infty$. Substituting (6) and (8) into (4), we see that the radial Klein-Gordon-Schwarzschild equation is exactly satisfied, provided that

$$b = -\frac{(\bar\omega - \bar k)^2}{2\bar k}, \quad b_0 = a_{-1} = 0, \quad b_{-1} = 4(\bar k - \bar\omega) \equiv 4a ,$$

$$a_1 = \frac{2\bar\omega + i}{4(4\bar\omega^2+1)}[-2ab - (\bar l + 1) + i(a-b)] , \qquad (9)$$

$$b_1 = \frac{1}{2\bar k}\left[2aa_1 + \frac{b^2}{4} - \frac{1}{4} + i\left[a_1 - \frac{b}{2}\right]\right] - 2a_1 ,$$

where we have set

$$\bar\omega = 2GM\omega, \quad \bar k = 2GMk , \quad k = (\omega^2 - m^2)^{1/2}, \quad \bar l = l(l+1) , \qquad (10)$$

and, in general, for $n \ge 2$, that

$$a_n=\frac{1}{4n(2\bar{\omega}-in)}\{-2\bar{\omega}(2n-1)b_{n-1}+2(n-1)(b-2a)a_{n-1}-2(n-1)ab_{n-1}+\tfrac{1}{2}(2n-3)bb_{n-2}$$
$$+S_3(n-1)+i[n^2b_{n-1}-(4n-1)(n-1)a_{n-1}-\tfrac{1}{4}(2n-1)(2n-3)b_{n-2}]\} \tag{11}$$

and

$$b_n=\frac{1}{2\bar{k}n}\{-2(\bar{\omega}+\bar{k})na_n+\tfrac{1}{2}[(2n-1)a+(n-1)b]b_{n-1}+b(n-1)a_{n-1}$$
$$+S_1(n)-S_2(n)+S_3(n)+in[na_n+\tfrac{1}{4}b_{n-1}-(n-1)a_{n-1}]\}\ , \tag{12}$$

where

$$\begin{aligned}
S_1(n)&=\tfrac{1}{4}\sum_{k=1}^{n}[kb_{k-1}-2(k-1)a_{k-1}+4ka_k][(n-k+1)b_{n-k}-2(n-k)a_{n-k}+4(n-k+1)a_{n-k+1}]\ ,\\
S_2(n)&=\tfrac{1}{16}\sum_{k=1}^{n-1}[(2k-1)b_{k-1}+4ka_k][(2n-2k-1)b_{n-k-1}+4(n-k)a_{n-k}]\ ,\\
S_3(n)&=\tfrac{1}{4}\sum_{k=1}^{n}\{[(2k-1)b_{k-1}+4ka_k](n-k)(b_{n-k}+2a_{n-k})\\
&\qquad-[kb_{k-1}-2(k-1)a_{k-1}+4ka_k][(2n-2k+1)b_{n-k}+(n-k+1)a_{n-k+1}]\}\ .
\end{aligned} \tag{13}$$

We are interested in well-behaved solutions of (4). To this end we make the following assumption: a_n and b_n are analytical in $1/n$ for n large enough. One gets

$$\begin{aligned}
\frac{a_n}{a_{n-1}}&=\alpha_0+\frac{\alpha_1}{n}+\frac{\alpha_2}{n^2}+\cdots,\\
\frac{b_n}{b_{n-1}}&=\beta_0+\frac{\beta_1}{n}+\frac{\beta_2}{n^2}+\cdots\ ,\\
\frac{a_n}{b_n}&=\gamma_0+\frac{\gamma_1}{n}+\frac{\gamma_2}{n^2}+\cdots\ .
\end{aligned} \tag{14}$$

Substituting these expansions into (11) and (12), we come out with the (*unique*) solution

$$\begin{aligned}
\frac{a_n}{a_{n-1}}&=1-\frac{1}{2n}+O(n^{-2})\ ,\\
\frac{b_n}{b_{n-1}}&=1-\frac{1}{2n}+O(n^{-2})\ ,\\
\frac{a_n}{b_n}&=-\tfrac{1}{2}+O(n^{-2})\ .
\end{aligned} \tag{15}$$

Even more, by restricting us to the two first terms in each case, i.e., setting the terms of $O(n^{-2})$ strictly equal to zero,

$$\frac{a_n}{a_{n-1}}=1-\frac{1}{2n},\quad \frac{b_n}{b_{n-1}}=1-\frac{1}{2n},\quad \frac{a_n}{b_n}=-\frac{1}{2}\ , \tag{16}$$

we obtain an *exact* solution of the recurrence equations (11) and (12), given by

$$a_n=c\frac{(2n-1)!!}{(2n)!!},\quad b_n=-2c\frac{(2n-1)!!}{(2n)!!}\ . \tag{17}$$

For the function $g(\rho)$ of (8), we get

$$g(\rho)=\frac{a}{1-\rho}+b\ln(1-\rho)+c(1-2\rho)\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}z^n\ . \tag{18}$$

This series can be easily summed up:

$$\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}z^n=(1-z)^{-1/2}-1\ , \tag{19}$$

and we arrive at

$$g(\rho)=\frac{a}{1-\rho}+b\ln(1-\rho)+2c(1-\rho)\ . \tag{20}$$

Actually, the coefficients a, b, and c are not independent. In order to obtain a solution of (4), the initial conditions (9) have to be satisfied [as well as the recurrences (11) and (12)]. We must impose (16) for $n=2$,

$$4a_2=3a_1,\quad 4b_2=3b_1,\quad b_1=-2a_1\ . \tag{21}$$

Equivalently, we could have substituted (20) and (6) into (4). In any case we obtain the following values for the coefficients a,b,c:

$$a=i\bar{l},\quad b=i,\quad c=0\ , \tag{22}$$

and the following relations between the parameters $\bar{\omega}$, $\bar{l}$, and $\bar{m}\equiv 2GMm$:

$$\begin{aligned}
\bar{l}&=-1\pm(1+2i\bar{\omega})^{1/2},\\
\bar{m}^2&=-2+2(1+i\bar{\omega})[2\mp(1+2i\bar{\omega})^{1/2}]\ .
\end{aligned} \tag{23}$$

This yields the one-complex-parameter continuous family of solutions of Eq. (4):

$$f_{\omega l}(r)=\alpha\exp\left[i\omega r_*+[1\mp(1+2i\bar{\omega})^{1/2}]\frac{r}{2GM}+\ln\frac{r}{2GM}\right]\ , \tag{24}$$

with $\varpi \in C$ arbitrary. A second family of solutions, independent of this one, is obtained by changing the sign of the i's in (24).

In principle, these solutions are valid only for $r > r_S$ (region I of the Kruskal diagram). This is so in spite of the fact that the function $f_{\omega l}(r)$ of (24) is well defined for $0 < r < r_S$ also. The reason is the following. In the coordinates (t,r) the Schwarzschild metric is singular at $r=0$ and at $r=r_S$. Even though Eq. (4) holds in both domains $0 < r < r_S$ and $r_S < r < +\infty$ and (24) is a solution in both these domains, how can we make sure that the solution at both sides of the singularity $r=r_S$ is the *same*? In other words, as it stands, we do not know if (24) for $r < r_S$ is the continuation of (24) for $r > r_S$ when we cross the singularity $r=r_S$ (transit from region I to region II of the Kruskal diagram).

To proceed correctly we have to use a nonsingular coordinate system, such as the Kruskal coordinates (t',r') (Ref. 3)

$$t'=\frac{U+V}{2}, \quad r'=\frac{V-U}{2}, \tag{25}$$

where

$$U=-\exp\left[-\frac{u}{4GM}\right], \quad u=t-r_* ,$$
$$V=\exp\left[\frac{v}{4GM}\right], \quad v=t+r_* , \tag{26}$$

or, equivalently, the coordinates (U,V). The Schwarzschild metric (1) becomes

$$ds^2=\frac{4(2GM)^3}{r}\exp\left[-\frac{r}{2GM}\right]dU\,dV - r^2(d\theta^2+\sin^2\theta\, d\phi^2) \tag{27}$$

and is no more singular at $r=r_S$ (i.e., at $U=0$ or $V=0$). The coordinates (U,V) extend over the whole regions I, II, III, and IV of the Kruskal diagram. Equation (26) gives the relation between (U,V) and (t,r) at region I. Using it we can write the corresponding Klein-Gordon-Schwarzschild radial equation in terms of (U,V), which will be valid over the whole extended Kruskal space, but for the only singular "point" $r=0$. Let a family of solutions of this equation be $g(U,V)$ which in terms of (t,r) is given in region I by Eq. (24). Let us rename this part of the whole solution $f_{\rm I}(t,r)$, i.e.,

$$g(U,V)=\left[1-\frac{2GM}{r}\right]f_{\rm I}(t,r), \quad U<0, \quad V>0 ,$$
$$f_{\rm I}(t,r)=\exp(-i\omega t)f_{\omega l}(r), \quad r>r_S , \tag{28}$$

with $f_{\omega l}(r)$ given by (24). We then deduce that the corresponding expression of $g(U,V)$ in the other three regions of the Kruskal diagram, in terms of (t,r), is consistently given by

$$f_{\rm II}(t,r(r_*))=\frac{r(r_*-\sigma)}{r}\exp\left[\frac{r(r_*-\sigma)-r}{2GM}\right]f_{\rm I}(t+\sigma,r(r_*-\sigma)) ,$$
$$f_{\rm III}(t,r(r_*))=\frac{r(r_*-\sigma}{r}\exp\left[\frac{r(r_*-\sigma)-r}{2GM}\right]f_{\rm I}(t-\sigma,r(r_*-\sigma)) , \tag{29}$$
$$f_{\rm IV}(t,r(r_*))=\frac{r(r_*-2\sigma)}{r}\exp\left[\frac{r(r_*-2\sigma)-r}{2GM}\right]f_{\rm I}(t,r(r_*-2\sigma)) ,$$

where $\sigma \equiv 2GM\ln(-1)$ [of course, the same determination of $\ln(-1)$ is to be used everywhere]. In this way, our exact solution of Eq. (4) is extended to the whole Kruskal space. Equation (29) provides its explicit expressions in each of the regions in terms of the coordinates (t,r). Notice, however, that this expression changes from one domain to another. In fact, in the black-hole region II (i.e., $0<r<r_S$), we have

$$f_{\rm II}(t,r(r_*))=\alpha\frac{r(r_*-\sigma)}{r}\exp\left[2\pi\varpi+\frac{r(r_*-\sigma)-r}{2GM}\right]$$
$$\times\exp\left[-i\omega u+[1\mp(1+2i\varpi)^{1/2}]\frac{\psi(r_*-\sigma)}{2GM}\ln\left[\frac{\psi(r_*-\sigma)}{2GM}\right]\right], \quad 0<r<r_S , \tag{30}$$

where $\psi(r_*)=r$ is the inverse function of (5). Combining adequately Eqs. (29) we can follow the variation of the solution of each transit from one region to another along a geodesic $u=$const or $v=$const. For instance, Eq. (30) gives us the variation of the solution in the crossover from region I to the black-hole region II along a geodesic $v=$const.

We observe that the solution (24) is best suited for studying the transit from region I to region II. However, it is immediate from Eqs. (29) that this solution is not adequate for dealing with the transition from regions III to I. In this case the other family of solutions (24) with i replaced by $-i$ has to be employed. If we call it $\bar{f}_{\omega l}(r)$, with the same notations above the corresponding functions $\bar{f}_{\rm III}$ and $\bar{f}_{\rm I}$ account for the transition from the white-hole region III to region I along a geodesic $u=$const.

Let us now investigate the asymptotic behavior of the solution (24). For $r \to r_S$, it behaves as

$$f_{\omega l}(r) \sim \exp\left[i\varpi \ln \left| \frac{r}{2GM} - 1 \right| \right], \quad r \sim r_S \ , \tag{31}$$

and, for $r \to +\infty$, as

$$f_{\omega l}(r) \sim \frac{r}{2GM} \exp\left[i\omega r + [1 \mp (1+2i\varpi)^{1/2}] \frac{r}{2GM} \right], \quad r \to +\infty \ . \tag{32}$$

Setting $\varpi = \alpha + i\beta$, and taking the minus sign, it is easy to see that for

$$\beta < 0, \quad \alpha^2 > \beta(5\beta+3) \ , \tag{33}$$

all the solutions of the family (24) are simultaneously convergent at $r=r_S$ and at $r=+\infty$. Only for $\beta>0$ do we not obtain any solution convergent at $r=r_S$, in accordance with the general results about the analytical properties of the solutions of (4) which have been already established in the literature.[4] But it turns out that in order to have $m^2>0$ it must be $\beta>0$ [when we take the minus sign in (30)]. However, the following finding is very remarkable: for α, β, and β/α small and such that the preceding condition (33) is satisfied (this is immediate to fulfill), our solution (24) does actually converge in the compact region $r_S \le r \le +\infty$ and has $m^2>0$. Moreover, when we take the plus sign in (30) and $\varpi = i\beta$, $\beta<0$ arbitrary, we find a solution convergent at $r=r_S$ and with any desired value of $m^2>0$. On the other hand, the solution (30), corresponding to region II, also converges as $r \to 0$. Thus, we have obtained a family of solutions which is analytic in the whole range $0 \le r \le +\infty$.

For the sake of comparison, it is interesting to point out that in the case $m=0$ a discrete family of solutions (for $\omega = in$, $n \in N$) was already known.[4,5] For $n=1$ one has $l=(-1+i\sqrt{3})/2$. Actually, this solution is contained in our continuous family, as the particular case $\varpi = i/2$. In contrast, for $m \neq 0$, we have obtained a continuous family involving real and complex ω's and with particular solutions in which m^2 and l are positive, for imaginary ω. For a general discussion of the mathematical and physical implications of imaginary frequency solutions of Eq. (4) see Ref. 6.

Finally, we shall now give some approximate solutions of (4) which, in spite of not being exact, can be interesting on their own. From Eq. (23) we observe that, even if we take ω to be real—which is perfectly allowed in any case [also by (33)]—the exact solutions will involve l or m complex. Notice, however, that for $\omega = i\beta$, $\beta<0$, we obtain a real and positive m. On the other hand, it is no novelty that we cannot obtain solutions with real l coverging both at $r=r_S$ and at $r \to +\infty$. That this is not possible was first proved by Zerilli.[7] Anyway, it may be useful to give some approximate solutions involving real values of *all* the parameters: ω, l, and m. This is achieved by using the same expressions (9), (11), and (12) for given values of these parameters. The way to proceed is the following. We calculate a certain number of coefficients a_k and b_k, say $k=1,2,\ldots,p$, until we reach stability (15). This has to be checked numerically. In general, the number p will have to be larger than the smaller ϖ. Then we make use of Eq. (20), previously modified as follows:

$$g(\rho) = \frac{a}{1-\rho} + b\ln(1-\rho) + c(1-2\rho)\left[\frac{2\rho}{1-2\rho} - \sum_{k=1}^{p} \frac{(2k-1)!!}{(2k)!!}[4\rho(1-\rho)]^k \right] + \sum_{k=1}^{p} (a_k + b_k\rho)[4\rho(1-\rho)]^k \ , \tag{34}$$

with the a and b given in (9), and

$$c = \frac{(2p)!!}{(2p-1)!!} a_p \ . \tag{35}$$

We have carried out a numerical analysis for different values of ϖ. For ϖ large the stability of the series is very remarkable. A standard FORTRAN program with double precision complex variables easily allows for one thousand iterations without accumulating errors. Formula (34) can be used with $p=2$ or 3 with a very great approximation. For instance, for $\varpi=20$, $l=0$, $\bar{m}=2.5$, we obtain

$$\begin{aligned} &a=-0.16, \quad b=-6.5\times10^{-4} \ , \\ &a_1=-(6.3+1.2i)\times10^{-3}, \quad b_1=(6.3+2.2i)\times10^{-3} \ , \\ &a_2=-(2.4+0.7i)\times10^{-3}, \quad b_2=(4.8+1.5i)\times10^{-3} \ , \\ &a_3=-(2.0+0.6i)\times10^{-3}, \quad b_3=(4.0+1.3i)\times10^{-3} \ . \end{aligned} \tag{36}$$

In this case, for $n \ge 3$ the relations (15) are fulfilled with an accuracy of 10^{-5}. The same is true for $l=1$. When $\varpi \simeq 2$ one must take $p \simeq 10$. The series does not stabilize for $\varpi \le 1$. This has of course to be so, because for these values of the parameters ϖ and $\bar{m}$ no solution of Eq. (4) *valid in the whole region* $r_S < r < +\infty$ can exist. In fact, the range of ϖ for spinless particle emission from a noncharged, nonrotating black hole begins about this value.[8]

The possible existence of more solutions of Eq. (4) similar to the ones derived here is under investigation.[9]

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