Derivation of the gauge-invariant action for open and closed free bosonic string field theories

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The gauge-invariant actions for open and closed free bosonic string field theories are obtained from the string field equations in the conformal gauge using the cohomology operations of Banks and Peskin. For the closed-string theory no restrictions are imposed on the gauge parameters.

I. INTRODUCTION

The construction of a gauge-invariant field theory is a subject of recent interest. The first attempt was done by Siegel[1] who constructed a gauge-fixed covariant action based on the Becchi-Rouet-Stora-Tyutin (BRST) formalism. Later on, Kaku[2] and Banks and Peskin[3] presented a nonlocal action which becomes local with the introduction of Stuckelberg dynamical fields.[4–7] Independently, Neveu *et al.* constructed actions with a finite[8] and infinite[9,10] number of Stuckelberg and Lagrangian multiplier fields. These authors realize the existence of additional symmetries[11] in their actions. The analysis of these symmetries leads one to consider a larger gauge-invariant action that can be written as $\langle \varphi | Q | \varphi \rangle$ where Q is the Becchi-Rouet-Stora operator of the first quantized string.[12] This action has been proposed independently by Witten.[13] When we integrate the Lagrange multiplier fields we recover the action of Refs. 4–7. The extension to open free superstrings has also been done.[14–17]

In the case of the closed bosonic string,[11] the situation is more involved because of the presence of the constraints $(L_0-\bar{L}_0)\phi=0$ and its generalization for the supplementary fields. Neveu *et al.* also constructed an action with finite[8] and infinite[9] number of supplementary fields, where the condition $(L_0-\bar{L}_0)\phi=0$ is put by hand. Additional symmetries also exist for these actions[18] and a larger action with more supplementary fields was constructed. There is a difficulty in writing this action in terms of the Becchi-Rouet-Stora operator of the first-quantized closed string due to the presence of two sets of anticommuting zero modes, which imply the appearance of four possible vacua. If one only considers two vacua, identifying two anticommuting zero modes, it is possible to write the action in terms of a modified BRS charge.[17,18] In the case when one considers the whole set of vacua,[18] it is possible to write an action containing the BRS operator Q such that the constraints $(L_0-\bar{L}_0)\phi=0$ and the ones related to it are obtained as equations of motion. The difficulty of this action is that the gauge parameters are restricted. The Ramond-Neveu-Schwarz and Ramond-Ramond sectors of closed superstrings have the same difficulties of the closed bosonic string, plus the problems due to the existence of zero commuting modes.

In this paper we will construct the gauge-invariant action for the open- and closed-bosonic-string field theories without any restriction on the gauge parameters. The departure point will be the equation of motion in the conformal gauge and the gauge transformation of the string field.[10]

A suitable action in the conformal gauge is introduced using the cohomology operators of Banks and Peskin.[4] This action, denoted by $A_0^{(1)}$, is not invariant under gauge transformations, so then we will modify it introducing supplementary fields and appropriate gauge transformations for them in such a way that the resulting action, $A^{(1)}$, is gauge invariant under the whole set of transformations.

The equations of motion in the conformal gauge should be obtained from $A^{(1)}$ by performing a suitable gauge transformation. This is not possible at this level because there exists a piece of the supplementary field that is gauge invariant.[16,19] This leads to the introduction of a new equation of motion which does not follow from $A^{(1)}$. Therefore a new action $A_0^{(2)}$ is needed. On the other hand, one realizes that a subset of the equations given by $A^{(1)}$ has additional gauge invariances.[16] Of course, neither $A^{(1)}$ nor $A_0^{(2)}$ are invariant under them. As before we will introduce more supplementary fields and construct a new action $A^{(2)}$ which has the same kind of difficulties as $A^{(1)}$. Therefore we need to iterate the procedure up to infinity. The final action can be written as $\langle \varphi | Q | \varphi \rangle$ for the open string[9,13] whereas for the closed string the expression is more involved. It is important to point out that in the later case the action gives as equation of motion $(L_0-\bar{L}_0)\phi=0$ and its generalizations without restriction on the gauge parameters. This new action is related to the one that was first proposed by Ballestrero and Maina.[20]

II. OPEN BOSONIC STRING

In the conformal gauge, the equations of motion for the string functional $\phi[x(\sigma)]$ are known to be

$$(L_0-1)\phi=0\ , \tag{1a}$$

$$L_n\phi=0,\quad n>0\ . \tag{1b}$$

As suggested by Siegel, the unfixed gauge theory must be invariant under

$$\delta\phi=L_{-k}\Lambda^k,\quad k>0 \tag{2}$$

where the Λ^k are arbitrary string functionals.

In order to simplify our expressions, it will be useful to introduce $\binom{a}{b}$-forms,[21] which belong to the exterior algebra constructed on the infinite-dimensional space of functions $x^\mu(\sigma)$. We can define on these forms the cohomology operations k, d, ∂, $\downarrow$ of Banks and Peskin,[4] which obey the algebra

$$[k,d]=[k,\partial]=[k,\phi]=0\ , \tag{3a}$$

$$\{d,\downarrow\}=\{\partial,\downarrow\}=0\ , \tag{3b}$$

$$[d,\partial]=\downarrow k\ , \tag{3c}$$

as well as the nilpotency conditions

$$d^2=\partial^2=0\ . \tag{4}$$

Relation (3c) is true only when the dimension of the spacetime is equal to 26. Operation d, called the exterior derivative, and operation ∂, the divergence, are the adjoint one of the other with respect to the scalar product

$$(f,g)=\int\mathscr{D}[x^\mu(\sigma)]f^\dagger[x^\mu(\sigma)]g[x^\mu(\sigma)] \tag{5}$$

defined on the space of string functionals. Explicitly stated, we have

$$(\partial f,g)=(f,dg)\ . \tag{6}$$

The string functional can be thought of as a $\binom{0}{0}$-form, while the arbitrary functionals Λ^k can be arranged into a $\binom{1}{0}$ form denoted by Λ^1. Then Eqs. (1) are written as

$$K\phi=0\ , \tag{7}$$

$$d\phi=0\ , \tag{8}$$

and the transformation (2) becomes

$$\delta\phi=\partial\Lambda^1\ . \tag{9}$$

Obviously, Eqs. (7) and (8) are not invariant under (9). Equation (7) is a "true" equation of motion, in the sense that it contains second-order derivatives, while (B) is usually interpreted as a gauge condition.

In order to construct a gauge-invariant action, we depart from

$$A_0^{(1)}=\tfrac{1}{4}(\phi,k\phi)+(S^1,d\phi)\ , \tag{10}$$

where we have introduced an auxiliary $\binom{1}{0}$-form s^1, which incorporates the gauge condition (8). Now, following the method of Ref. 10 we transform $A_0^{(1)}$ under (9):

$$\delta A_0^{(1)}=\tfrac{1}{2}(\phi,k\partial\Lambda^1)+(\delta s^1,d\phi)+(s^1,d\partial\Lambda^1)\ .$$

We choose

$$\delta s^1=-\tfrac{1}{2}k\Lambda^1 \tag{11}$$

and then

$$\delta A_0^{(1)}=(s^1,d\partial\Lambda^1)=(s^1,\partial d\Lambda^1)+(s^1,\downarrow k\Lambda^1)$$
$$=(s^1,\partial d\Lambda^1)-\delta(s^1,\downarrow s^1)\ .$$

Therefore, if we define

$$A_1^{(1)}=A_0^{(1)}+(s^1,\downarrow s^1) \tag{12}$$

we get

$$\delta A_1^{(1)}=(s^1,\partial d\Lambda^1)=-(s^1,\delta\partial\phi_1^1)\ ,$$

where we have introduced a $\binom{1}{1}$-form ϕ_1^1 which transforms as

$$\delta\phi_1^1=-d\Lambda^1\ . \tag{13}$$

Then

$$\delta A_1^{(1)}=-\delta(s^1,\partial\phi_1^1)+(\delta s^1,\partial\phi_1^1)$$

and defining

$$A_2^{(1)}=A_1^{(1)}+(s^1,\partial\phi_1^1) \tag{14}$$

we obtain

$$\begin{aligned}\partial A_2^{(1)}&=(\delta s^1,\partial\phi_1^1)\\&=-\tfrac{1}{2}(K\Lambda^1,\partial\phi_1^1)\\&=-\tfrac{1}{2}(d\Lambda^1,k\phi_1^1)\\&=\tfrac{1}{2}(\delta\phi_1^1,k\phi_1^1)\\&=\tfrac{1}{4}\delta(\phi_1^1,k\phi_1^1)\end{aligned}$$

so finally $\delta A^{(1)}=0$ with $A^{(1)}$ being defined by

$$\begin{aligned}A^{(1)}&=A_2^{(1)}-\tfrac{1}{4}(\phi_1^1,k\phi_1^1)\\&=\tfrac{1}{4}(\phi,k\phi)-\tfrac{1}{4}(\phi_1^1,k\phi_1^1)+(s^1,d\phi+\partial\phi_1^1+\downarrow s^2)\ .\end{aligned} \tag{15}$$

This action gives the following equations of motion:

$$\tfrac{1}{2}k\phi+\partial s^1=0\ , \tag{16a}$$

$$-\tfrac{1}{2}k\phi_1^1+ds^1=0\ , \tag{16b}$$

$$d\phi+\partial\phi_1^1+2\downarrow s^1=0\ . \tag{16c}$$

They are invariant under

$$\delta\phi=\partial\Lambda^1\ , \tag{17a}$$

$$\delta s^1=-\tfrac{1}{2}k\Lambda^1\ , \tag{17b}$$

$$\delta\phi_1^1=-d\Lambda^1\ . \tag{17c}$$

The auxiliary functional s^1 can be eliminated from equation (17c). ϕ_1^1 is a Stuckelberg functional and contains the Stuckelberg fields necessary to build a gauge theory with massive physical fields. They cannot be eliminated in a local way; that is, their elimination produces a gauge theory with a nonlocal action for the physical fields.

The equations of motion (16a) and (16c) should reduce to Eqs. (7) and (8) by means of a suitable gauge transformation. In fact, there is no problem to set $s^1=0$ and then one is left with

$$k\phi=0,\quad k\phi_1^1=0,\quad d\phi+\partial\phi_1^1=0\ .$$

Now, one would have $\partial\phi_1^1=0$, but from $k\phi_1^1=0$ and (3c) this also implies $d\phi_1^1=0$. Difficulties arise when one observes[19] that from (17c) and the nilpotency property it follows

$$\delta(d\phi_1^1)=0\ ,$$

that is, $d\phi_1^1$ is invariant under gauge transformation, so one is not free to gauge reduce (16c) in order to get $d\phi=0$. To solve the problem we should look for an action incorporating $d\phi_1^1=0$ as an equation of motion:

$$A_0^{(2)}=A^{(1)}+(s_1^2,d\phi_1^1)\ , \tag{18}$$

where we have introduced a new auxiliary $\binom{2}{1}$-form. This action is invariant under (17) if one simply takes $\delta s_1^2=0$. But there is another way which solves the problem and enlarges the set of gauge invariances. We start by noticing that Eqs. (16a) and (16c) have two additional gauge invariances:[16]

$$\delta\phi_1^1=\partial\Lambda_1^2+2\iota\chi^2,\quad \delta s^l=\partial\chi^2,\quad \delta\phi=0\ , \tag{19}$$

where Λ_1^2 and χ^2 are arbitrary $\binom{2}{1}$- and $\binom{2}{0}$-forms, respectively. The appearance of these extra invariances is related to the nonindependent character of the first-class constraints of the Hamiltonian theory.[23] Next we transform $A_0^{(2)}$ under (19). A little algebra gives

$$\delta A_0^{(2)}=-\tfrac{1}{2}(\phi_1^1,k\partial\Lambda_1^2-2\partial d\chi^2)+(\delta s_1^2,d\phi_1^1)+(s_1^2,d(\partial\Lambda_1^2+2\iota\chi^2))\ .$$

We choose

$$\delta s_1^2=\tfrac{1}{2}k\Lambda_1^2-d\chi^2 \tag{20}$$

and then

$$\begin{aligned}\delta A_0^{(2)}&=(s_1^2,d[\partial\Lambda_1^2+2\iota\chi^2])\\&=(s_1^2,\partial d\Lambda_1^2)+(s_1^2,\iota k\Lambda_1^2-2\iota d\chi^2)\\&=(s_1^2,\partial d\Lambda_1^2)+\delta(s_1^2,\iota s_1^2)\ .\end{aligned}$$

So if we define

$$A_1^{(2)}=A_0^{(2)}-(s_1^2,\iota s_1^2)\ , \tag{21}$$

then

$$\delta A_1^{(2)}=(s_1^2,\partial d\Lambda_1^2)=-(s_1^2,\delta\partial\phi_2^2)\ ,$$

where

$$\delta\phi_2^2=-d\Lambda_1^2\ . \tag{22}$$

Following the same steps as before we write

$$\delta A_1^{(2)}=-\delta(s_1^2,\partial\phi_2^2)+(\delta s_1^2,\partial\phi_2^2)$$

and define

$$A_2^{(2)}=A_1^{(2)}+(s_1^2,\partial\phi_2^2) \tag{23}$$

which satisfies

$$\begin{aligned}\delta A_2^{(2)}&=(\delta s_1^2,\partial\phi_2^2)=(\tfrac{1}{2}K\Lambda_1^2-d\chi^2,\partial\phi_2^2)\\&=(\tfrac{1}{2}k\Lambda_1^2,\partial\phi_2^2)=\tfrac{1}{2}(d\Lambda_1^2,k\phi_2^2)=-\tfrac{1}{4}\delta(\phi_2^2,k\phi_2^2)\ .\end{aligned}$$

So finally $\delta A^{(2)}=0$ with

$$A^{(2)}=A_2^{(2)}+\tfrac{1}{4}(\phi_2^2,k\phi_2^2)=\tfrac{1}{4}(\phi,k\phi)-\tfrac{1}{4}(\phi_1^1,k\phi_1^1)+\tfrac{1}{4}(\phi_2^2,K\phi_2^2)+(s^1,d\phi+\partial\phi_1^1+\iota s^1)+(s_1^2,d\phi_1^1+\partial\phi_2^2-\iota s_1^2)\ . \tag{24}$$

From this action one gets the equations of motion

$$\tfrac{1}{2}k\phi+\partial s^1=0\ , \tag{25a}$$

$$-\tfrac{1}{2}k\phi_1^1+ds^1+\partial s_1^2=0\ , \tag{25b}$$

$$\tfrac{1}{2}K\phi_2^2+ds_1^2=0\ , \tag{25c}$$

$$d\phi+\partial\phi_1^1+2\iota s^1=0\ , \tag{25d}$$

$$d\phi_1^1+\partial\phi_2^2-2\iota s_1^2=0\ , \tag{25e}$$

which are invariant under

$$\delta\phi=\partial\Lambda^1\ , \tag{26a}$$

$$\delta\phi_1^1=-d\Lambda^1+\partial\Lambda_1^2+2\iota\chi^2\ , \tag{26b}$$

$$\delta\phi_2^2=-d\Lambda_1^2\ , \tag{26c}$$

$$\delta s^1=-\tfrac{1}{2}k\Lambda^1+\partial\chi^2\ , \tag{26d}$$

$$\delta s_1^2=\tfrac{1}{2}k\Lambda_1^2-d\chi^2\ . \tag{26e}$$

Again, Eqs. (25a) and (25d) should be gauge reduced to $k\phi=0$ and $d\phi=0$. First, one sets $s^1=s_1^2=0$.

Now $d\phi_1^1$ is no more gauge invariant, so one can get $\partial\phi_1^1=0$ (which, together with $k\phi_1^1=0$, implies $d\phi_1^1=0$) by means of a gauge transformation. But all this reduced Eqs. (25c) and (25e) to $k\phi_2^2=0$ and $\partial\phi_2^2=0$, so $d\phi_2^2=0$ also, while from (26c) one can see that

$$\delta(d\phi_2^2)=0\ .$$

Therefore, the gauge transformations we have carried out are not consistent, and again one is led to search for an action giving $d\phi_2^2=0$ as an equation of motion. One can also see that Eqs. (25), except (25c), have additional gauge invariances which now leave ϕ, ϕ_1^1, and s^1 unchanged. Hence we can repeat the same steps that led from $A^{(1)}$ to $A^{(2)}$ and the process follows without end. As shown in the Appendix one gets the action

$$A=\sum_{i=0}^{\infty}\left[\frac{(-1)^i}{4}(\phi_i^i,k\phi_i^i)+[S_i^{i+1},d\phi_i^i+\partial\phi_{i+1}^{i+1}+(-1)^i\iota S_i^{i+1}]\right] \tag{27}$$

which is invariant under

$$\delta\phi_i^i = -d\Lambda_{i-1}^i + \partial\Lambda_i^{i+1} + 2(-1)^{i-1}\chi_{i-1}^{i+1}\ , \tag{28a}$$

$$\delta s_i^{i+1} = -\frac{(-1)^i}{2}k\Lambda_i^{i+1} - d\chi_{i-1}^{i+1} + \partial\chi_i^{i+2}\ , \tag{28b}$$

for $i=0,\ldots,\infty$ (unless an index is <0).

It is known that action (27) can be formulated in terms of the BRST charge operator $\hat{Q}$ of the first-quantized theory:

$$A = \tfrac{1}{2}\langle\varphi|\hat{Q}|\varphi\rangle\ , \tag{29}$$

where the "total" field $|\varphi\rangle$ is decomposed as

$$|\varphi\rangle = (\phi + s\beta_0)|-\rangle\ , \tag{30}$$

where both the "minus" vacuum and $|\varphi\rangle$ have ghost number $-\frac{1}{2}$, and ϕ and s have ghost number 0 and -1, respectively. When spanned in terms of a basis of oscillators, ϕ and s give the components of the various forms ϕ_i^i and s_i^{i+1}. The BRST charge $\hat{Q}$ can be written as[11]

$$\hat{Q} = \hat{d} + \hat{\partial} + \hat{k}\beta_0 - \hat{\uparrow}\bar{\beta}_0\ , \tag{31}$$

where $\hat{d}$, $\hat{\partial}$, $\hat{\uparrow}$, and $\hat{k}$ are operators related to the cohomology operations. For $D=26$ one has

$$\hat{Q}^2 = 0 \tag{32}$$

and the invariance of A under

$$\delta|\varphi\rangle = \hat{Q}|\epsilon\rangle\ , \tag{33}$$

$|\epsilon\rangle$ being an arbitrary state, follows immediately. Because of the fact that $\hat{Q}$ has ghost number $+1$, $|\epsilon\rangle$ must have ghost number $-\frac{3}{2}$ and can be spanned as

$$|\epsilon\rangle = (\Lambda + \chi\beta_0)|-\rangle\ , \tag{34}$$

where Λ and χ have ghost numbers -1 and -2, respectively. It must be also noticed that if

$$|\epsilon\rangle = \hat{Q}|\epsilon'\rangle\ ,$$

then $\delta|\varphi\rangle = 0$. This leads to the appearance of the "ghosts for ghosts" and in fact an infinite chain of "ghosts for ghosts. . .for ghosts" is obtained.[11] All this is associated to the nonindependence of the first-class constraints of the Hamiltonian theory.[22,23] Finally, we would like to point out that the partial actions $A^{(1)},\ldots,A^{(n)}$ can also be formulated in terms of the same BRST charge $\hat{Q}$ if one takes only the first terms in the expansions of ϕ and s (Ref. 24). Nevertheless, their gauge transformations do not have the form (33).

III. CLOSED BOSONIC STRING

The equations of motion for the closed-string functional $\phi[x^\mu(\sigma)]$ in the conformal gauge are

$$(L_0 + \bar{L}_0 - 2)\phi = 0\ , \tag{35a}$$

$$L_n\phi = 0,\ \ \bar{L}_n\phi = 0,\ \ n>0\ , \tag{35b}$$

$$(L_0 - \bar{L}_0)\phi = 0\ . \tag{35c}$$

The last equation implements the symmetry between right and left movers. We expect the unfixed gauge theory to be invariant under

$$\delta\phi = L_{-n}\Lambda^n + \bar{L}_{-n}\Lambda^n,\ \ n>0\ , \tag{36}$$

where both Λ^n and $\bar{\Lambda}^n$ are arbitrary string functionals. Usually, Eq. (35c) is preserved under (36) by demanding

$$(L_0 - \bar{L}_0)\Lambda^n = (L_0 - \bar{L}_0)\bar{\Lambda}^n = 0 \tag{37}$$

but we prefer to have (35b) and (35c) in an equal footing and therefore condition (37) will not be imposed in our construction.

In a similar way as we did for the open string, we introduce $\binom{a\,\bar{c}}{b\,\bar{d}}$-forms which carry two kinds of indexes. We also introduce two sets of cohomology operations $d,\partial,k,\downarrow$ and $\bar{d},\bar{\partial},\bar{k},\bar{\downarrow}$ each of them obeying the algebra (3) and the nilpotency conditions (4), while operations belonging to different sets do commute. When acting on a form, every set works on the corresponding indexes.

The string functional can be considered as a $\binom{0\,\bar{0}}{0\,\bar{0}}$-form, while the Λ^n and $\bar{\Lambda}^n$ are the components of an $\binom{1\,\bar{0}}{0\,\bar{0}}$- and a $\binom{0\,\bar{1}}{0\,\bar{0}}$-form, denoted, respectively, by Λ^1 and $\Lambda^{\bar{1}}$.

Now, Eqs. (35) are written as

$$(k+\bar{k})\phi = 0\ , \tag{38}$$

$$d\phi = 0,\ \ \bar{d}\phi = 0\ , \tag{39}$$

$$(k-\bar{k})\phi = 0\ , \tag{40}$$

and the gauge transformation (36) is

$$\delta\phi = \partial\Lambda^1 + \bar{\partial}\Lambda^{\bar{1}}\ . \tag{41}$$

Equations (38), (39), and (40) are not invariant under (41). In order to construct the gauge-invariant theory, we start with

$$A_0^{(1)} = \tfrac{1}{8}(\phi,(k+\bar{k})\phi) + (T,(k-\bar{k})\phi) + (s^1,d\phi) + (s^{\bar{1}},\bar{d}\phi)\ , \tag{42}$$

where auxiliary forms T, S^1, and $s^{\bar{1}}$ have been introduced. The variation of $A_0^{(1)}$ under (41) gives

$$\delta A_0^{(1)} = \tfrac{1}{4}(\phi,(k+\bar{k})(\partial\Lambda^1+\bar{\partial}\Lambda^{\bar{1}})) + (\delta s^1,d\phi) + (\delta s^{\bar{1}},\bar{d}\phi) + (\delta T,(k-\bar{k})\phi) + (T,(k-\bar{k})(\partial\Lambda^1+\bar{\partial}\Lambda^{\bar{1}})) + (s^1,d(\partial\Lambda^1+\bar{\partial}\Lambda^{\bar{1}})) + (s^{\bar{1}},\bar{d}(\partial\Lambda^1+\bar{\partial}\Lambda^{\bar{1}}))\ .$$

We choose

$$\delta s^1 = -\tfrac{1}{4}(k+\bar{k})\Lambda^1,\ \ \delta s^{\bar{1}} = -\tfrac{1}{4}(k+\bar{k})\Lambda^{\bar{1}},\ \ \delta T = 0 \tag{43}$$

so then

$$\delta A_0^{(1)}=(s^1,d(\partial\Lambda^1+\bar\partial\Lambda^{\bar 1}))+(s^{\bar 1},\bar d(\partial\Lambda^1+\bar\partial\Lambda^{\bar 1}))+(T,(k-\bar k)(\partial\Lambda^1+\bar\partial\Lambda^{\bar 1}))\ .$$

Let us work with the last term

$$(T,(k-\bar k)(\partial\Lambda^1+\bar\partial\Lambda^{\bar 1}0))=(T,(k-\bar k)(\partial\delta\alpha^1+\bar\partial\delta\alpha^{\bar 1}))=\delta(T,(k-\bar k)(\partial\alpha^1+\bar\partial\alpha^{\bar 1}))\ ,$$

where we have introduced new forms α^1, $\alpha^{\bar 1}$ such that

$$\delta\alpha^1=\Lambda^1,\quad \delta\alpha^{\bar 1}=\Lambda^{\bar 1}\ . \tag{44}$$

Then, if

$$A_1^{(1)}=A_0^{(1)}-(T,(k-\bar k)(\partial\alpha^1+\bar\partial\alpha^{\bar 1})) \tag{45}$$

we get

$$\begin{aligned}\delta A_1^{(1)}&=(s^1,d(\partial\Lambda^1+\bar\partial\Lambda^{\bar 1}))+(s^{\bar 1},\bar d(\partial\Lambda^1+\bar\partial\Lambda^{\bar 1}))\\&=(s^1,\partial d\Lambda^1+\bar\partial d\Lambda^{\bar 1})+(s^{\bar 1},\partial\bar d\Lambda^1+\bar\partial\bar d\Lambda^{\bar 1})+(s^1,k\downarrow\Lambda^1)+(s^{\bar 1},\bar k\downarrow\Lambda^{\bar 1})\ .\end{aligned}$$

Consider the last two terms

$$\begin{aligned}(s^1,\downarrow k\Lambda^1)+(s^{\bar 1},\bar{\downarrow}\bar k\Lambda^{\bar 1})&=\tfrac{1}{2}(s^1,\downarrow(k+\bar k)\Lambda^1)+\tfrac{1}{2}(s^{\bar 1},\bar{\downarrow}(k+\bar k)\Lambda^{\bar 1})+\tfrac{1}{2}(s^1,\downarrow(k-\bar k)\Lambda^1)-\tfrac{1}{2}(s^{\bar 1},\bar{\downarrow}(k-\bar k)\Lambda^{\bar 1})\\&=-\delta(s^1,\downarrow s^1)-\delta(s^{\bar 1},\bar{\downarrow}s^{\bar 1})+\tfrac{1}{2}(s^1,\downarrow(k-\bar k)\Lambda^1)-\tfrac{1}{2}(s^{\bar 1},\bar{\downarrow}(k-\bar k)\Lambda^{\bar 1})\ .\end{aligned}$$

Defining

$$A_2^{(1)}=A_1^{(1)}+(s^1,\downarrow s^1)+(s^{\bar 1},\bar{\downarrow}s^{\bar 1}) \tag{46}$$

we have

$$\delta A_2^{(1)}=-(s^1,\partial\delta\phi_1^1+\bar\partial\delta\phi_1^{\bar 1})-(s^{\bar 1},\partial\delta\phi_{\bar 1}^1+\bar\partial\delta\phi_{\bar 1}^{\bar 1})+\tfrac{1}{2}(s^1,\downarrow(k-\bar k)\Lambda^1)-\tfrac{1}{2}(s^{\bar 1},\bar{\downarrow}(k-\bar k)\Lambda^{\bar 1})\ ,$$

where we have introduced new forms such that

$$\delta\phi_1^1=-d\Lambda^1,\quad \delta\phi_1^{\bar 1}=-d\Lambda^{\bar 1},\quad \delta\phi_{\bar 1}^1=-\bar d\Lambda^1,\quad \delta\phi_{\bar 1}^{\bar 1}=-\bar d\Lambda^{\bar 1}\ . \tag{47}$$

Therefore, if we consider

$$A_3^{(1)}=A_2^{(1)}+(s^1,\partial\phi_1^1+\bar\partial\phi_1^{\bar 1})+(s^{\bar 1},\partial\phi_{\bar 1}^1+\bar\partial\phi_{\bar 1}^{\bar 1})\ , \tag{48}$$

we get

$$\delta A_3^{(1)}=(-\tfrac{1}{4}(k+\bar k)\Lambda^1,\partial\phi_1^1+\bar\partial\phi_1^{\bar 1})+(-\tfrac{1}{4}(k+\bar k)\Lambda^{\bar 1},\partial\phi_{\bar 1}^1+\bar\partial\phi_{\bar 1}^{\bar 1})+\tfrac{1}{2}(s^1,\downarrow(k-\bar k)\Lambda^1)-\tfrac{1}{2}(s^{\bar 1},\bar{\downarrow}(k-\bar k)\Lambda^{\bar 1})\ .$$

As for the open string, the first two terms are canceled by introducing new kinetic terms:

$$A_4^{(1)}=A_3^{(1)}-\tfrac{1}{8}(\phi_1^1,(k+\bar k)\phi_1^1)-\tfrac{1}{4}(\phi_1^{\bar 1},(k+\bar k)\phi_{\bar 1}^1)-\tfrac{1}{8}(\phi_{\bar 1}^{\bar 1},(k+\bar k)\phi_{\bar 1}^{\bar 1}) \tag{49}$$

and then

$$\delta A_4^{(1)}=\tfrac{1}{2}(s^1,\downarrow(k-\bar k)\Lambda^1)-\tfrac{1}{2}(s^{\bar 1},\bar{\downarrow}(k-\bar k)\Lambda^{\bar 1})=\tfrac{1}{2}(s^1,(k-\bar k)\delta R_1)+\tfrac{1}{2}(s^{\bar 1},(k-\bar k)\delta R_{\bar 1})\ ,$$

where new forms transforming as

$$\delta R_1=\downarrow\Lambda^1,\quad \delta R_{\bar 1}=-\bar{\downarrow}\Lambda^{\bar 1} \tag{50}$$

have been introduced. Defining

$$A_5^{(1)}=A_4^{(1)}-\tfrac{1}{2}(s^1,(k-\bar k)R_1)-\tfrac{1}{2}(s^{\bar 1},(k-\bar k)R_{\bar 1})\ , \tag{51}$$

we get

$$\begin{aligned}\delta A_5^{(1)}&=-\tfrac{1}{2}(-\tfrac{1}{4}(k+\bar k)\Lambda^1,(k-\bar k)R_1)-\tfrac{1}{2}(-\tfrac{1}{4}(k+\bar k)\Lambda^{\bar 1},(k-\bar k)R_{\bar 1})\\&=\tfrac{1}{8}((k+\bar k)\delta\alpha^1,(k-\bar k)R_1)+\tfrac{1}{8}((k+\bar k)\delta\alpha^{\bar 1},(k-\bar k)R_{\bar 1})\end{aligned}$$

and now, if

$$A_6^{(1)}=A_5^{(1)}-\tfrac{1}{8}((k+\bar k)\alpha^1,(k-\bar k)R_1)-\tfrac{1}{8}((k+\bar k)\alpha^{\bar 1},(k-\bar k)R_{\bar 1})\ , \tag{52}$$

then

$$\delta A_6^{(1)} = -\tfrac{1}{8}((k+\bar{k})\alpha^1,(k-\bar{k})\downarrow\Lambda^1)+\tfrac{1}{8}((k+\bar{k})\alpha^{\bar{1}},(k-\bar{k})\bar{\downarrow}\Lambda^{\bar{1}})$$
$$= -\tfrac{1}{16}\delta((k+\bar{k})\alpha^1,(k-\bar{k})\downarrow\alpha^1)-\tfrac{1}{16}\delta((k+\bar{k})\alpha^{\bar{1}},(k-\bar{k})\bar{\downarrow}\alpha^{\bar{1}}) \; .$$

Finally, if

$$A^{(1)} = A_6^{(1)}+\tfrac{1}{16}((k+\bar{k})\alpha^1,(k-\bar{k})\downarrow\alpha^1)-\tfrac{1}{16}((k+\bar{k})\alpha^{\bar{1}},(k-\bar{k})\bar{\downarrow}\alpha^{\bar{1}}) \tag{53}$$

we obtain $\delta A^{(1)}=0$. The complete expression for $A^{(1)}$ is

$$\begin{aligned} A^{(1)} = {} & \tfrac{1}{8}(\phi,(k+\bar{k})\phi)-\tfrac{1}{8}(\phi_1^1,(k+\bar{k})\phi_1^1)-\tfrac{1}{4}(\phi_1^{\bar{1}},(k+\bar{k})\phi_{\bar{1}}^1)-\tfrac{1}{8}(\phi_{\bar{1}}^{\bar{1}},(k+\bar{k})\phi_{\bar{1}}^{\bar{1}})+((k-\bar{k})\phi,T) \\ & +(s^1,d\phi+\partial\phi_1^1+\bar{\partial}\phi_1^{\bar{1}}+\downarrow s^1)+(s^{\bar{1}},\bar{d}\phi+2\phi_{\bar{1}}^1+\bar{\partial}\phi_{\bar{1}}^{\bar{1}}+\bar{\downarrow}s^{\bar{1}})-\tfrac{1}{2}((k-\bar{k})s^1,R_1)-\tfrac{1}{2}((k-\bar{k})s^{\bar{1}},R_{\bar{1}}) \\ & -\tfrac{1}{8}((k-\bar{k})R_1,(k+\bar{k})\alpha^1)-\tfrac{1}{8}((k-\bar{k})R_{\bar{1}},(k+\bar{k})\alpha^{\bar{1}})-((k-\bar{k})T,\partial\alpha^1+\bar{\partial}\alpha^{\bar{1}}) \\ & +\tfrac{1}{16}((k+\bar{k})\alpha^1,(k-\bar{k})\downarrow\alpha^1)-\tfrac{1}{16}((k+\bar{k})\alpha^{\bar{1}},(k-\bar{k})\bar{\downarrow}\alpha^{\bar{1}}) \; . \end{aligned} \tag{54}$$

Its equations of motion are

$$\tfrac{1}{4}(k+\bar{k})\phi+\partial s^1+\bar{\partial}s^{\bar{1}}+(k-\bar{k})T=0 \; , \tag{55}$$

$$-\tfrac{1}{4}(k+\bar{k})\phi_1^1+ds^1=0 \; , \tag{56a}$$

$$-\tfrac{1}{4}(k+\bar{k})\phi_{\bar{1}}^1+\bar{d}s^1=0 \; , \tag{56b}$$

$$-\tfrac{1}{4}(k+\bar{k})\phi_1^{\bar{1}}+ds^{\bar{1}}=0 \; , \tag{56c}$$

$$-\tfrac{1}{4}(k+\bar{k})\phi_{\bar{1}}^{\bar{1}}+\bar{d}s^{\bar{1}}=0 \; , \tag{56d}$$

$$d\phi+\partial\phi_1^1+\bar{\partial}\phi_1^1+2\downarrow s^1-\tfrac{1}{2}(k-\bar{k})R_1=0 \; , \tag{57a}$$

$$\bar{d}\phi+\partial\phi_{\bar{1}}^1+\bar{\partial}\phi_{\bar{1}}^{\bar{1}}+2\bar{\downarrow}s^{\bar{1}}-\tfrac{1}{2}(k-\bar{k})R_{\bar{1}}=0 \; , \tag{57b}$$

$$-\tfrac{1}{2}(k-\bar{k})s^1-\tfrac{1}{8}(k-\bar{k})(k+\bar{k})\alpha^1=0 \; , \tag{58a}$$

$$-\tfrac{1}{2}(k-\bar{k})s^{\bar{1}}-\tfrac{1}{8}(k-\bar{k})(k+\bar{k})\alpha^{\bar{1}}=0 \; , \tag{58b}$$

$$(k-\bar{k})\phi-(k-\bar{k})(\partial\alpha^1+\bar{\partial}\alpha^{\bar{1}})=0 \; , \tag{59}$$

$$-dT-\tfrac{1}{8}(k+\bar{k})R_1+\tfrac{1}{8}(k+\bar{k})\downarrow\alpha^1=0 \; , \tag{60a}$$

$$-\bar{d}T-\tfrac{1}{8}(k+\bar{k})R_{\bar{1}}-\tfrac{1}{8}(k+\bar{k})\bar{\downarrow}\alpha^{\bar{1}}=0 \; , \tag{60b}$$

and the gauge invariances are

$$\begin{aligned} & \delta\phi=\partial\Lambda^1+\bar{\partial}\Lambda^{\bar{1}},\;\; \delta\phi_1^1=-d\Lambda^1,\;\; \delta\phi_1^{\bar{1}}=-d\Lambda^{\bar{1}},\;\; \delta\phi_{\bar{1}}^1=-\bar{d}\Lambda^1,\;\; \delta\phi_{\bar{1}}^{\bar{1}}=-\bar{d}\Lambda^{\bar{1}},\;\; \delta s^1=-\tfrac{1}{4}(k+\bar{k})\Lambda^1 \; , \\ & \delta s^{\bar{1}}=-\tfrac{1}{4}(k+\bar{k})\Lambda^{\bar{1}} \; ,\;\; \delta R_1=\downarrow\Lambda^1,\;\; \delta R_{\bar{1}}=-\bar{\downarrow}\Lambda^{\bar{1}},\;\; \delta T=0,\;\; \delta\alpha^1=\Lambda^1,\;\; \delta\alpha^{\bar{1}}=\Lambda^{\bar{1}} \; . \end{aligned} \tag{61}$$

As for the open string, there are some quantities involving $\phi_1^1,\phi_{\bar{1}}^1,\phi_1^{\bar{1}},\phi_{\bar{1}}^{\bar{1}}$ that are gauge invariant:

$$\delta(d\phi_1^1)=\delta(\bar{d}\phi_{\bar{1}}^1)=\delta(d\phi_1^{\bar{1}})=\delta(\bar{d}\phi_{\bar{1}}^{\bar{1}})=\delta(d\phi_{\bar{1}}^1-\bar{d}\phi_1^1)=\delta(d\phi_{\bar{1}}^{\bar{1}}-\bar{d}\phi_1^{\bar{1}})=0 \; . \tag{62}$$

In order to gauge reduce (55), (57), and (59) to the corresponding equations in the conformal gauge, we must demand the above quantities to be zero by the equations of motion, so then we consider

$$A_0^{(2)}=A^{(1)}+(s_1^2,d\phi_1^1)+(s_{\bar{1}}^{\bar{2}}\bar{d}\phi_{\bar{1}}^1)+(s_{\bar{1}}^2,d\phi_1^{\bar{1}})+(s_{\bar{1}}^{\bar{2}},\bar{d}\phi_{\bar{1}}^{\bar{1}})+(s_1^{1\bar{1}},d\phi_{\bar{1}}^1-\bar{d}\phi_1^1)+(s_{\bar{1}}^{1\bar{1}},d\phi_{\bar{1}}^{\bar{1}}-\bar{d}\phi_1^{\bar{1}}) \; . \tag{63}$$

Following the same strategy that proved to be successful for the open string, we notice that Eqs. (55) and (57) have additional gauge invariances:

$$\begin{aligned} & \delta\phi=(k-\bar{k})\rho,\;\; \delta T=-\tfrac{1}{4}(k+\bar{k})\rho-\partial\omega^1-\bar{\partial}\omega^{\bar{1}},\;\; \delta\alpha^1=\delta\alpha^{\bar{1}}=0\;,\delta\phi_1^1=\partial\Lambda_1^2-\bar{\partial}\Lambda_1^{1\bar{1}}+2\downarrow\chi^2,\;\; \delta\phi_1^{\bar{1}}=\bar{\partial}\Lambda_1^{\bar{2}}+\partial\Lambda_1^{1\bar{1}}+2\downarrow\chi^{1\bar{1}} \; , \\ & \delta\phi_{\bar{1}}^1=\partial\Lambda_{\bar{1}}^2-\bar{\partial}\Lambda_{\bar{1}}^{1\bar{1}}-2\downarrow\chi^{1\bar{1}},\;\; \delta\phi_{\bar{1}}^{\bar{1}}=\bar{\partial}\Lambda_{\bar{1}}^{\bar{2}}+\partial\Lambda_{\bar{1}}^{1\bar{1}}+2\bar{\downarrow}\chi^{\bar{2}} \; , \\ & \delta s^1=\partial\chi^2-\bar{\partial}\chi^{1\bar{1}}+(k-\bar{k})\omega^1,\;\; \delta s^{\bar{1}}=\bar{\partial}\chi^{\bar{2}}+\partial\chi^{1\bar{1}}+(k-\bar{k})\omega^{\bar{1}}\;,\delta R_1=2d\rho+4\downarrow\omega^1,\;\; \delta R_{\bar{1}}=2\bar{d}\rho+4\downarrow\omega^{\bar{1}} \; , \end{aligned} \tag{64}$$

where $\Lambda_1^2, \ldots, \Lambda_{\bar{1}}^{\bar{2}}, \chi^2, \chi^{1\bar{1}}, \chi^{\bar{1}}, \rho, \omega^1$, and $\omega^{\bar{1}}$ are new gauge parameters, and now we can repeat what we did for the open string. But in the present case, in order to have an action invariant under (64) we must introduce an enormous number of forms:

$$\begin{aligned}&\phi^2_2,\phi^{1\bar1}_2,\phi^{\bar2}_2,\phi^2_{\bar2},\phi^{1\bar1}_{\bar2},\phi^{\bar2}_{\bar2},\phi^2_{1\bar1}\phi^{1\bar1}_{1\bar1},\phi^{\bar2}_{1\bar1};s^2_1,s^{\bar2}_1,s^{1\bar1}_1,s^2_{\bar1},s^{\bar2}_{\bar1},s^{1\bar1}_{\bar1};\ R^1_2,R^1_{\bar2},R^1_{1\bar1},R^{\bar1}_2,R^{\bar1}_{\bar2},R^1_{1\bar1};\\&T^1_1,T^{\bar1}_1,T^1_{\bar1},T^{\bar1}_{\bar1};\ \alpha^2_1,\ \alpha^{\bar2}_1,\ \alpha^2_{\bar1}\,\alpha^{\bar2}_{\bar1}\,\alpha^{1\bar1}_1\,\alpha^{1\bar1}_{\bar1};\ \beta^2,\ \beta^{1\bar1},\ \beta^{\bar2};\ \mu;\ \nu^1,\ \nu^{\bar1}\ .\end{aligned}\tag{65}$$

After a tedious calculation one gets the action

$$\begin{aligned}A^{(2)}=A^{(1)}&+\tfrac18(\phi^2_2,(k+\bar k)\phi^2_2)+\tfrac14(\phi^2_{1\bar1},(k+\bar k)\phi^{1\bar1}_2)+\tfrac14(\phi^{\bar2}_2,(k+\bar k)\phi^2_{\bar2})\\
&+\tfrac14(\phi^{\bar2}_{1\bar1},(k+\bar k)\phi^{1\bar1}_{\bar2})+\tfrac18(\phi^{\bar2}_{\bar2},(k+\bar k)\phi^{\bar2}_{\bar2})+\tfrac18(\phi^{1\bar1}_{1\bar1},(k+\bar k)\phi^{1\bar1}_{1\bar1})\\
&+(s^2_1,d\phi^1_1+\partial\phi^2_2-\bar\partial\phi^{1\bar1}_2-\downarrow s^1)+(s^{\bar2}_1,\bar d\phi^1_{\bar1}+\partial\phi^2_{\bar2}-\bar\partial\phi^{1\bar1}_{\bar2}+2\bar\downarrow s^{1\bar1}_{\bar1})\\
&+(s^2_{\bar1},d\phi^{\bar1}_1+\bar\partial\phi^{\bar2}_2+\partial\phi^{1\bar1}_2-2\downarrow s^{1\bar1}_1)+(s^{\bar2}_{\bar1},\bar d\phi^{\bar1}_{\bar1}+\bar\partial\phi^{\bar2}_{\bar2}+\partial\phi^{1\bar1}_{\bar2}-\bar\downarrow s^{\bar2}_{\bar1})\\
&+(s^{1\bar1}_1,d\phi^1_{\bar1}-\bar d\phi^1_1+\partial\phi^2_{1\bar1}-\bar\partial\phi^{1\bar1}_{1\bar1}-\bar\downarrow s^{1\bar1}_1)+(s^{1\bar1}_{\bar1},d\phi^{\bar1}_{\bar1}-\bar d\phi^{\bar1}_1+\bar\partial\phi^{\bar2}_{1\bar1}+\partial\phi^{1\bar1}_{1\bar1}-\downarrow s^{1\bar1}_{\bar1})\\
&-\tfrac12((k-\bar k)s^2_1,R^1_2)-\tfrac12((k-\bar k)s^{\bar2}_1,R^1_{\bar2})-\tfrac12((k-\bar k)s^2_{\bar1},R^{\bar1}_2)\\
&-\tfrac12((k-\bar k)s^{\bar2}_{\bar1},R^{\bar1}_{\bar2})-\tfrac12((k-\bar k)s^{1\bar1}_1,R^1_{1\bar1})-\tfrac12((k-\bar k)s^{1\bar1}_{\bar1},R^{\bar1}_{1\bar1})\\
&+((k-\bar k)\phi^1_1,T^1_1)+((k-\bar k)\phi^1_{\bar1},T^{\bar1}_1)+((k-\bar k)\phi^{\bar1}_1,T^1_{\bar1})+((k-\bar k)\phi^{\bar1}_{\bar1},T^{\bar1}_{\bar1})\\
&+\tfrac12((k-\bar k)R_1,\partial\beta^2-\bar\partial\beta^{1\bar1}+\tfrac14(k-\bar k)\nu^1)+\tfrac12((k-\bar k)R_{\bar1},\bar\partial\beta^{\bar2}+\partial\beta^{1\bar1}+\tfrac14(k-\bar k)\nu^{\bar1})\\
&-((k-\bar k)R^1_2,-\tfrac18(k+\bar k)\alpha^2_1+\tfrac12 d\beta^2)-((k-\bar k)R^1_{\bar2},-\tfrac18(k+\bar k)\alpha^{\bar2}_1+\tfrac12 d\beta^{\bar2})\\
&-((k-\bar k)R^{\bar1}_2,-\tfrac18(k+\bar k)\alpha^2_{\bar1}+\tfrac12\bar d\beta^2)-((k-\bar k)R^{\bar1}_{\bar2},-\tfrac18(k+\bar k)\alpha^{\bar2}_{\bar1}+\tfrac12\bar d\beta^{\bar2})\\
&-((k-\bar k)R^1_{1\bar1},-\tfrac18(k+\bar k)\alpha^{1\bar1}_1+\tfrac12 d\beta^{1\bar1})-((k-\bar k)R^{\bar1}_{1\bar1},-\tfrac18(k+\bar k)\alpha^{1\bar1}_{\bar1}+\tfrac12\bar d\beta^{1\bar1})-((k-\bar k)T,\tfrac12(k-\bar k)\mu)\\
&-((k-\bar k)T^1_1,-d\alpha^1+\partial\alpha^2_1-\bar\partial\alpha^{1\bar1}_1+2\downarrow\beta^2)-((k-\bar k)T^{\bar1}_1,-\bar d\alpha^1+\partial\alpha^2_{\bar1}-\bar\partial\alpha^{1\bar1}_{\bar1}-2\bar\downarrow\beta^{1\bar1})\\
&-((k-\bar k)T^1_{\bar1},-d\alpha^{\bar1}+\bar\partial\alpha^{\bar2}_1+\partial\alpha^{1\bar1}_1+2\bar\downarrow\beta^{1\bar1})-((k-\bar k)T^{\bar1}_{\bar1},-\bar d\alpha^{\bar1}+\bar\partial\alpha^{\bar2}_{\bar1}+\partial\alpha^{1\bar1}_{\bar1}+2\bar\downarrow\beta^{\bar2})\\
&-\tfrac12((k-\bar k)\alpha^1,\downarrow\partial\beta^2-\downarrow\bar\partial\beta^{1\bar1}+\tfrac14(k-\bar k)\downarrow\nu^1)-\tfrac12((k-\bar k)\alpha^{\bar1},-\bar\downarrow\partial\beta^{1\bar1}-\bar\downarrow\bar\partial\beta^{\bar2}-\tfrac14(k-\bar k)\bar\downarrow\nu^{\bar1})\\
&-((k-\bar k)\alpha^2_1,\tfrac1{16}(k+\bar k)\downarrow\alpha^2_1-\tfrac12\downarrow d\beta^2)-((k-\bar k)\alpha^{1\bar1}_1,\tfrac1{16}(k+\bar k)(\downarrow\alpha^2_{\bar1}-\bar\downarrow\alpha^{1\bar1}_1)-\tfrac12(\downarrow\bar d\beta^2-\bar\downarrow d\beta^{1\bar1}))\\
&-((k-k)\alpha^{\bar2}_1,\tfrac1{16}(k+\bar k)\bar\downarrow\alpha^{2\bar1}_{\bar1}-\tfrac12\bar\downarrow\bar d\beta^{1\bar1})-((k-\bar k)\alpha^2_{\bar1},\tfrac1{16}(k+\bar k)\downarrow\alpha^{1\bar1}_1-\tfrac12\downarrow d\beta^{1\bar1})\\
&-(k-\bar k)\alpha^{1\bar1}_{\bar1},\tfrac1{16}(k+\bar k)(\downarrow\alpha^{1\bar1}_{\bar1}+\bar\downarrow\alpha^{\bar2}_1)-\tfrac12(\downarrow\bar d\beta^{1\bar1}+\bar\downarrow d\beta^{\bar2}))-((k-\bar k)\alpha^{\bar2}_{\bar1},-\tfrac1{16}(k+\bar k)\bar\downarrow\alpha^{\bar2}_{\bar1}+\tfrac12\bar\downarrow\bar d\beta^{\bar2})\\
&-\tfrac12((k-\bar k)\beta^2,\downarrow\downarrow\beta^2)+\tfrac12((k-\bar k)\beta^{\bar2},\bar\downarrow\bar\downarrow\beta^{\bar2})\\
&-((k-\bar k)\mu,\tfrac1{32}(k+\bar k)(k-\bar k)\mu+\tfrac18(k-\bar k)(\partial\nu^1+\bar\partial\nu^{\bar1}))\\
&-\tfrac1{16}((k-\bar k)\nu^1,(k-\bar k)\downarrow\nu^1)-\tfrac1{16}((k-\bar k)\nu^{\bar1},(k-\bar k)\bar\downarrow\nu^{\bar1})\ .\end{aligned}\tag{66}$$

The new fields transform as

$$\begin{aligned}&\delta\phi^2_2=-d\Lambda^2_1,\ \delta\phi^{1\bar1}_2=-d\Lambda^{1\bar1}_1,\ \delta\phi^2_{\bar2}=-\bar d\Lambda^2_{\bar1},\delta\phi^{1\bar1}_{\bar2}=-\bar d\Lambda^{1\bar1}_{\bar1},\ \delta\phi^{\bar2}_2=-d\Lambda^{\bar2}_1,\ \delta\phi^{\bar2}_{\bar2}=-\bar d\Lambda^{\bar2}_{\bar1}\ ,\\
&\delta\phi^2_{1\bar1}=-(d\Lambda^2_{\bar1}-\bar d\Lambda^2_1),\ \delta\phi^{1\bar1}_{1\bar1}=-(d\Lambda^{1\bar1}_{\bar1}-\bar d\Lambda^{1\bar1}_1),\ \delta\phi^{\bar2}_{1\bar1}=-(d\Lambda^{\bar2}_{\bar1}-\bar d\Lambda^{\bar2}_1)\ ;\\
&\delta s^2_1=\tfrac14(k+\bar k)\Lambda^2_1-d\chi^2,\ \delta s^{\bar2}_{\bar1}=\tfrac14(k+\bar k)\Lambda^{\bar2}_{\bar1}-\bar d\chi^{\bar2}\ ,\ \delta s^{\bar2}_1=\tfrac14(k+\bar k)\Lambda^{\bar2}_1-d\chi^{\bar2},\ \delta s^{1\bar1}_1=\tfrac14(k+\bar k)\Lambda^{1\bar1}_1-d\chi^{1\bar1}\ ,\\
&\delta s^2_{\bar1}=\tfrac14(k+\bar k)\Lambda^2_{\bar1}-\bar d\chi^2,\ \delta s^{1\bar1}_{\bar1}=\tfrac14(k+\bar k)\Lambda^{1\bar1}_{\bar1}-\bar d\chi^{1\bar1}\ ;\\
&\delta R^1_2=\downarrow\Lambda^2_1,\delta R^1_{\bar2}=\bar\downarrow\Lambda^{1\bar1}_{\bar1},\quad \delta R^{\bar1}_2=\downarrow\Lambda^{1\bar1}_1,\ \delta R^{\bar1}_{\bar2}=-\bar\downarrow\Lambda^{\bar2}_{\bar1}\ ,\delta R^1_{1\bar1}=\downarrow\Lambda^2_{\bar1}-\bar\downarrow\Lambda^{1\bar1}_1,\ \delta R^{\bar1}_{1\bar1}=\downarrow\Lambda^{1\bar1}_{\bar1}+\bar\downarrow\Lambda^{\bar2}_1\ ;\\
&\delta T^1_1=-d\omega^1-\tfrac12\downarrow\chi^2,\ \delta T^{\bar1}_1=-d\omega^{\bar1}-\tfrac12\downarrow\chi^{1\bar1}\ ,\delta T^1_{\bar1}=-\bar d\omega^1-\tfrac12\bar\downarrow\chi^{1\bar1},\quad \delta T^{\bar1}_{\bar1}=-\bar d\omega^{\bar1}+\tfrac12\bar\downarrow\chi^{\bar2}\ ;\\
&\delta\alpha^2_1=\Lambda^2_1,\ \delta\alpha^{1\bar1}_1=\Lambda^{1\bar1}_1,\ \delta\alpha^{\bar2}_1=\Lambda^{\bar2}_1,\ \delta\alpha^2_{\bar1}=\Lambda^2_{\bar1},\ \delta\alpha^{1\bar1}_{\bar1}=\Lambda^{1\bar1}_{\bar1},\ \delta\alpha^{\bar2}_{\bar1}=\Lambda^{\bar2}_{\bar1}\ ,\\
&\delta\beta^2=\chi^2,\ \delta\beta^{1\bar1}=\chi^{1\bar1},\ \delta\beta^{\bar2}=\chi^{\bar2},\ \delta\mu=2\rho,\ \delta\nu^1=4\omega^1,\ \delta\nu^{\bar1}=4\omega^{\bar1}\ .\end{aligned}\tag{67}$$

The algorithm could follow up to infinity as we did for the open string, but we will not carry it out due to the complexity of the higher steps. Rather, we propose the action

$$
\begin{aligned}
A=&\tfrac{1}{8}(\phi,(k+\bar{k})\phi)+(s,(\downarrow+\bar{\downarrow})s)+(s,\Omega\phi)+((k-\bar{k})\phi,T)-\tfrac{1}{2}((k-\bar{k})s,R)\\
&+((k-\bar{k})R,-\tfrac{1}{8}(k+\bar{k})\alpha+\tfrac{1}{2}\Omega\beta+\tfrac{1}{8}(k-\bar{k})\nu)-((k-\bar{k})T,\Omega\alpha-2(\downarrow+\bar{\downarrow})\beta+\tfrac{1}{2}(k-\bar{k})\mu)\\
&-((k-\bar{k})\alpha,-\tfrac{1}{16}(k+\bar{k})(\downarrow-\bar{\downarrow})\alpha+\tfrac{1}{2}(\downarrow-\bar{\downarrow})\Omega\beta+\tfrac{1}{8}(k-\bar{k})(\downarrow-\bar{\downarrow})\nu)\\
&-((k-\bar{k})\beta,-\tfrac{1}{2}(\downarrow-\bar{\downarrow})(\downarrow+\bar{\downarrow})\beta+\tfrac{1}{4}(k-\bar{k})(\downarrow-\bar{\downarrow})\mu)\\
&-((k-\bar{k})\mu,\tfrac{1}{32}(k+\bar{k})(k-\bar{k})\mu+\tfrac{1}{8}(k-\bar{k})\Omega\nu)-\tfrac{1}{16}((k-\bar{k})\nu,(k-\bar{k})(\downarrow+\bar{\downarrow})\nu)\ ,
\end{aligned}
\tag{68}
$$

where $\phi,s,R,T,\alpha,\beta,\mu,\nu$ are to be expanded in an oscillator basis to give the different forms we have worked with. The operators $k,\bar{k},\downarrow,\bar{\downarrow}$ and $\Omega=d+\bar{d}+\partial+\bar{\partial}$ have also expansions in terms of the oscillators.[18] The first terms in these expansions give the actions we have built. The action (68) has the gauge invariances

$$
\begin{aligned}
&\delta\phi=(k-\bar{k})\rho+\Omega\Lambda-2(\downarrow+\bar{\downarrow})\chi,\ \ \delta s=-\tfrac{1}{4}(k+\bar{k})\Lambda+(k-\bar{k})\omega+\Omega\chi\ ,\\
&\delta R=2\Omega\rho+4(\downarrow+\bar{\downarrow})\omega+(\downarrow-\bar{\downarrow})\Lambda\ ,\\
&\delta T=-\tfrac{1}{4}(k+\bar{k})\rho-\Omega\omega-\tfrac{1}{2}(\downarrow-\bar{\downarrow})\chi,\ \ \delta\alpha=\Lambda,\ \ \delta\beta=\chi,\delta\mu=2\rho,\ \ \delta\nu=4\omega\ .
\end{aligned}
\tag{69}
$$

If we perform the change

$$
T\to T-\tfrac{1}{4}(\downarrow-\bar{\downarrow})\beta-\tfrac{1}{16}(k+\bar{k})\mu-\tfrac{1}{8}\Omega\nu,\ \ R\to R+\tfrac{1}{2}(\downarrow-\bar{\downarrow})\alpha+\tfrac{1}{2}\Omega\mu+\tfrac{1}{2}(\downarrow+\bar{\downarrow})\nu\ ,
\tag{70}
$$

we get

$$
\begin{aligned}
A=&\tfrac{1}{8}(\phi,(k+\bar{k})\phi)+(s,(\downarrow+\bar{\downarrow})s)+(s,\Omega\phi)+((k-\bar{k})\phi,T)-\tfrac{1}{2}((k-\bar{k})s,R)\\
&+((k-\bar{k})\phi,-\tfrac{1}{4}(\downarrow-\bar{\downarrow})\beta-\tfrac{1}{16}(k+\bar{k})\mu-\tfrac{1}{8}\Omega\nu)+((k-\bar{k})s,-\tfrac{1}{4}(\downarrow-\bar{\downarrow})\alpha-\tfrac{1}{4}\Omega\mu-\tfrac{1}{4}(\downarrow+\bar{\downarrow})\nu)\\
&+((k-\bar{k})R,-\tfrac{1}{8}(k+\bar{k})\alpha+\tfrac{1}{2}\Omega\beta+\tfrac{1}{8}(k-\bar{k})\nu)+((k-\bar{k})T,-\Omega\alpha+2(\downarrow+\bar{\downarrow})\beta-\tfrac{1}{2}(k-\bar{k})\mu)\ .
\end{aligned}
\tag{71}
$$

This last action is not new. It was first proposed by Ballestrero and Maina[20] in terms of the BRST charge operator of the first quantized theory:

$$
\hat{Q}=C_0^+K^++C_0^-K^-+\Omega-\bar{C}_0^+\downarrow^+-\bar{C}_0^-\downarrow^-\ ,
\tag{72}
$$

where

$$
K^\pm=\frac{1}{\sqrt{2}}(k\pm\bar{k}),\ \ \downarrow^\pm=\frac{1}{\sqrt{2}}(\downarrow\pm\bar{\downarrow}),\ \ \Omega=d+\bar{d}+\partial+\bar{\partial}\ ,
$$

and the zero-mode operators obey

$$
\{C_0^+,\bar{C}_0^+\}=\{C_0^-,\bar{C}_0^-\}=1
$$

all other anticommutators vanishing. The vacuum of the theory, denoted by $|-+\rangle$, is such that

$$
\begin{aligned}
&C_0^-\,|-+\rangle=\bar{C}_0^+\,|-+\rangle=0\ ,\\
&\langle -+|\,C_0^+\bar{C}_0^-\,|-+\rangle=i\ ,
\end{aligned}
$$

all other zero-mode vacuum expectation values being equal to zero. The action is then given by

$$
A=\frac{i}{2}\langle\varphi|[\bar{C}_0^-,\hat{Q}]|\varphi\rangle+\frac{i}{2}\langle\varphi|\hat{Q}k^-|2\rangle-\frac{i}{2}\langle\eta|k^-\hat{Q}|\varphi\rangle\ ,
\tag{73}
$$

where the fields $|\varphi\rangle$ and $|\eta\rangle$ are decomposed as

$$
|\varphi\rangle=(\phi+sc_0^++R\bar{C}_0^-+TC_0^+\bar{C}_0^-)\,|-+\rangle,\ \ |\eta\rangle=(\alpha+\beta C_0^++\mu\bar{C}_0^-+\nu C_0^+\bar{C}_0^-)\,|-+\rangle\ .
\tag{74}
$$

The action A has ghost number zero, so then $|\varphi\rangle$ has also ghost number zero because $\bar{C}_0^-$ and $\hat{Q}$ have ghost numbers -1 and $+1$, respectively. One can also see that $|\eta\rangle$ must have ghost number -1. The vacuum is taken to have ghost number zero and then the ghost numbers of the various components are $(0,-1,1,0)$ for $|\varphi\rangle$ and $(-1,-2,0,-1)$ for $|\eta\rangle$.

The action is invariant under

$$
\delta|\varphi\rangle=\hat{Q}|\epsilon\rangle,\ \ \delta|\eta\rangle=|\epsilon\rangle\ ,
\tag{75}
$$

due to $\hat{Q}^2=0$ (for $D=26$) and $\{\hat{Q},\bar{C}_0^-\}=k^-$. The arbitrary field $|\epsilon\rangle$ is decomposed as

$$|\epsilon\rangle=(\Lambda+\chi C_0^+ +\rho\bar{C}_0^- +\omega C_0^+\bar{C}_0^-)|-+\rangle \tag{76}$$

and its ghost numbers are the same as for $|\eta\rangle$.

When explicitly computed, expressions (73) and (75) give, up to global and field rescaling factors, the action (71), and its corresponding gauge transformations.

Finally, we should notice that the action of Ballestrero and Maina can be obtained from that of Neveu *et al.*[18]

$$A_N=\frac{i}{2}\langle\varphi|[\bar{C}_0^-,\hat{Q}]|\varphi\rangle \tag{77}$$

by a simple variation algorithm. Action A_N is invariant under $\delta|\varphi\rangle=\hat{Q}|\epsilon\rangle$ provided that $k^-|\epsilon\rangle=0$. This restriction can be removed if one introduces an auxiliary field:

$$\delta A_N=\frac{i}{2}\langle\epsilon|\hat{Q}(\bar{C}_0^-\hat{Q}-\hat{Q}\bar{C}_0^-)|\varphi\rangle+\frac{i}{2}\langle\varphi|(\bar{C}_0^-\hat{Q}-\hat{Q}\bar{C}_0^-)\hat{Q}|\epsilon\rangle$$

$$=\frac{i}{2}\langle\epsilon|k^-\hat{Q}|\varphi\rangle-\frac{i}{2}\langle\varphi|Qk^-|\epsilon\rangle=\frac{i}{2}\langle\delta\eta|k^-\hat{Q}|\varphi\rangle-\frac{i}{2}\langle\varphi|Qk^-|\delta\eta\rangle=\frac{i}{2}\delta\langle\eta|k^-\hat{Q}|\varphi\rangle-\frac{i}{2}\delta\langle\varphi|Qk^-|\eta\rangle$$

and so we recover the action (73). We can also set

$$\delta|\eta\rangle=k^-|\epsilon\rangle$$

and then we get an alternative expression for the action:

$$A=\frac{i}{2}\langle\varphi|[\bar{C}_0^-,\hat{Q}]|\varphi\rangle+\frac{i}{2}\langle\varphi|\hat{Q}|\eta\rangle-\frac{i}{2}\langle\eta|\hat{Q}|\varphi\rangle .$$

IV. CONCLUSIONS

We have developed an algorithm to construct gauge-invariant actions for string fields from the knowledge of the equation of motion in the conformal gauge and the gauge transformation for the physical functional using the cohomology operations of Banks and Peskin.

For the open string case we obtain the well-known action written in terms of the BRST charge operator, while in the closed string case we obtain actions related to the one recently proposed by Ballestrero and Maina, incorporating the equation $(L_0-\bar{L}_0)\phi=0$ and its generalization without restriction on the gauge parameters.

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APPENDIX

Here we want to generalize the procedure carried out in Sec. II for the two first action levels. At a given level we have the action

$$A^{(n)}=\sum_{i=0}^{n}\frac{(-1)^i}{4}(\phi_i^i,k\phi_i^i)+\sum_{i=0}^{n-1}(s_i^{i+1},d\phi_i^i+\partial\phi_{i+1}^{i+1}+(-1)^i\iota s_i^{i+1}) \tag{A1}$$

which is invariant under

$$\delta\phi_i^i=-d\Lambda_{i-1}^i+\partial\Lambda_i^{i+1}+2(-1)^{i-1}\chi_{i-1}^{i+1},\quad i=0,\ldots,n-1 , \tag{A2a}$$

$$\delta\phi_i^n=-d\Lambda_{n-1}^n , \tag{A2b}$$

$$\delta s_i^{i+1}=-\frac{(-1)^i}{2}k\Lambda_i^{i+1}-d\chi_{i-1}^{i+1}+\partial\chi_i^{i+2},\quad i=0,\ldots,n-2 , \tag{A2bc}$$

$$\delta s_{n-1}^n=-\frac{(-1)^{n-1}}{2}k\Lambda_{n-1}^n-d\chi_{n-2}^n \tag{A2d}$$

(when an index is <0, the corresponding term is supposed to be zero).

One observes that $\delta(d\phi_n^n)=0$ and we are led to consider the action

$$A_0^{(n+1)}=A^{(n)}+(s_n^{n+1},d\phi_n^n) . \tag{A3}$$

We also realize that all the equations of motion of $A^{(n)}$, except the one for ϕ_n^n, have the additional symmetries

$$\delta\phi_n^n=\partial\Lambda_n^{n+1}+2(-1)^{n-1}\iota\chi_{n-1}^{n+1} , \tag{A4a}$$

$$\delta s_{n-1}^n=\partial\chi_{n-1}^{n+1} . \tag{A4b}$$

Under (A4), $A_0^{(n+1)}$ transforms as

$$\delta A_0^{(n+1)}=\frac{(-1)^n}{2}(k\Lambda_n^{n+1},d\phi_n^n)+(\chi_{n-1}^{n+1},\partial d\phi_n^n)+(\delta s_n^{n+1},d\phi_n^n)+(s_n^{n+1},d(\partial\Lambda_n^{n+1}+2(-1)^{n-1}\iota\chi_{n-1}^{n+1}))\ .$$

As expected we choose

$$\delta s_n^{n+1}=-\tfrac{1}{2}(-1)^n k\Lambda_n^{n+1}-d\chi_{n-1}^{n+1} \tag{A5}$$

and then further manipulations give

$$\delta A_0^{(n+1)}=(s_n^{n+1},\partial d\Lambda_n^{n+1})-(-1)^n\delta(s_n^{n+1},\iota s_n^{n+1})\ .$$

If

$$A_1^{(n+1)}=A_0^{(n+1)}+(-1)^n(s_n^{n+1},\iota s_n^{n+1})\ , \tag{A6}$$

then

$$\delta A_1^{(n+1)}=(s_n^{n+1},\partial d\Lambda_n^{n+1})=-(s_n^{n+1},\partial\delta\phi_{n+1}^{n+1})\ ,$$

where

$$\delta\phi_{n+1}^{n+1}=-d\Lambda_n^{n+1} \tag{A7}$$

so then

$$\delta A_1^{(n+1)}=-\delta(s_n^{n+1},\partial\phi_{n+1}^{n+1})+(\delta s_n^{n+1},\partial\phi_{n+1}^{n+1})\ .$$

If we define

$$A_2^{(n+1)}=A_1^{(n+1)}+(s_n^{n+1},\partial\phi_{n+1}^{n+1}) \tag{A8}$$

its variation will be given by

$$\begin{aligned}\delta A_2^{(n+1)}&=(-\tfrac{1}{2}(-1)^n k\Lambda_n^{n+1}-d\chi_{n-1}^{n+1},\partial\phi_{n+1}^{n+1})\\&=\tfrac{1}{4}(-1)^n\delta(\phi_{n+1}^{n+1},k\phi_{n+1}^{n+1})\end{aligned}$$

and then $\delta A^{(n+1)}=0$ with

$$A^{(n+1)}=A_i^{(n+1)}+\tfrac{1}{4}(-1)^{n+1}(\phi_{n+1}^{n+1},k\phi_{n+1}^{n+1})\ . \tag{A9}$$

$A^{(n+1)}$ has the same structure as $A^{(n)}$, so then we can go to $n\to\infty$ arriving at the action (27).

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