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$1/c$ expansion of a separable model of direct-interaction type

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The properties of a proposed model of N point particles in direct interaction are considered in the limit of small velocities. It is shown that, in this limit, time correlations cancel out and that Newtonian dynamics is recovered for the system in a natural way.

I. INTRODUCTION

During the last few years a great deal of interest has been focused on the elaboration of models based on direct-interaction theory[1] to describe the dynamics of relativistic interacting particles. Dirac[2] was the first to suggest a Hamiltonian framework to pursue this objective, based on the construction of a realization of the Poincaré group in terms of 12 parameters. The works of Bakamjian and Thomas[3] were the first attempt at an explicit construction of such a realization, but these authors were not able to define Poincaré-invariant world lines.

This unpleasant feature was, however, an early indication of the no-interaction theorem, later stated and demonstrated by Currie, Jordan, and Sudarshan.[4] This theorem shows that the requirements of Poincaré, world-line invariance, with a canonical realization of the Poincaré group are mutually exclusive.

The development of the study of systems with constraints,[5] where the constraints themselves define the dynamics of the system (singular Lagrangians, etc.) has explicitly shown a way to circumvent the consequences of the no-interaction theorem. In particular it has allowed for the construction of realizations of the Poincaré group of the Bakamjian-Thomas type, but recovering their full physical meaning through the prescriptions given to relate the physical positions to the canonical (with respect to Dirac brackets) variables. World lines are thus identified through such prescriptions. Explicit models have been proposed both for the instant form[6] and the front form[7] of dynamics.

Another important question is that of separability[8] or the cluster decomposition property, whose necessity was first raised by Foldy and later investigated by several authors. It is well known that this is the origin of difficult problems for systems containing more than two particles.

By separability, or the cluster decomposition property, we mean the following: if a system of N interacting particles breaks into two or more dynamically independent clusters, because of the finite range of the interactions or because the clusters are separated by a large spacelike separation so that their interaction vanishes, the set of constraints must likewise break into two or more corresponding subgroups, each one describing the separate dynamics of the corresponding cluster. An important consequence of the separability requirement is the need for many-body forces.[9]

The approaches to the problem can be classified into two groups. One of them starts with a set of N first-class constraints and was initiated by Todorov,[10] Komar,[11] and Droz-Vincent.[12] Generally speaking, in this kind of model it is necessary to specify a set of $N-1$ "gauge-fixing" constraints[13] in order to have a definite classical dynamics. A different approach, starting with both first- and second-class constraints[14,15] has been followed by other authors. The requirement of separability in the above-defined sense is only accomplished in some of these models in a perturbative way.[16–18]

In the present paper we want to discuss a model based on $2N-1$ Poincaré-invariant constraints which form a "quasi-second-class" set, i.e., the rank of the $(2N-1)\times(2N-1)$ matrix of the Poisson brackets of the constraints is one less than its dimension (rank $2N-2$), thereby making it possible to extract one linear combination which is first class. The explicit form of the constraints in this model is suggested by the Hamiltonian constraints of a two-particle system.[19]

A model of this kind has been proposed in Ref. 19, see also Ref. 20; here we want to show that this model, with a slight modification, admits a well-defined nonrelativistic limit (or, rather, a $1/c$ expansion), so that this model acquires a physical content.

The aim of this work is to show, by exhibiting a specific model, that the requirements of relativistic invariance (world-line conditions satisfied and so on) and cluster decomposition property are not mutually exclusive.

II. THE MODEL

In order to characterize the constraints which define our model it is convenient to recall the Lagrangian for the two-body problem:[21]

$$\mathscr{L}=-\{[m_1{}^2c^2-V(r^2)]\dot{x}_1{}^2\}^{1/2}-\{[m_2{}^2c^2-V(r^2)]\dot{x}_2{}^2\}^{1/2}\,, \quad (2.1)$$

m_1,m_2 are the rest masses of the particles, V is the interaction potential, and $r=x_2-x_1$ is the four-separation between the particles.

The Hamiltonian primary constraints turn out to be generalized mass-shell equations

$$\Omega_i=p_i{}^2-m_i{}^2c^2+V(r^2)\approx 0,\quad i=1,2\,, \quad (2.2)$$

where the symbol $\approx$ is defined as in Ref. 5.

There is also a secondary constraint

$$(p_1+p_2,r)V'(r^2)\approx 0\,, \quad (2.3)$$

which deserves a comment. Whenever $V'(r^2)\neq 0$ the constraint $(P_1+P_2,r)\approx 0$ holds; it defines a correlation between points of both world lines corresponding to the same value of the evolution parameter (this correlation, however, can be eliminated in favor of an instantaneous force). On the other hand if $V'(r^2)=0$ (2.3) is automatically satisfied, and any correlation between worlds lines disappears; in this case these are straight lines and the evolution of each particle is completely independent of the other.

We see that the constraint (2.3) has nice properties with respect to the problem of separability, because it makes the system break into two pieces (in this case of one particle each) in a very natural way whenever their interaction vanishes. For a system of two particles this is a consequence of the separability of the Lagrangian (2.1).

For systems of three or more particles we will generalize the constraints (2.2) and (2.3) rather than the Lagrangian (2.1), in such a way as to keep the correct cluster decomposition properties. As a consequence of that we will have from the very beginning a canonical formulation in terms of a set of canonical variables x_i^μ,p_i^ν with the assumed symplectic structure

$$\{p_i^\mu,x_j^\nu\}=g^{\mu\nu}\delta_{ij}\quad (i,j=1,\ldots,N)\,. \quad (2.4)$$

It must be stressed that in this kind of model the variables x_i^μ are by definition the physical positions of the particles, in contrast with other approaches (see for instance, Ref. 12). The no-interaction theorem[4] which forbids the identification of physical position variables with canonical variables does not apply here, due to the presence of second-class constraints.

For the generalization to N particles we will need $2N-1$ constraints (plus a gauge-fixing constraint to be eventually added at the end). This is necessary in order to have a $6N$-dimensional realization of the Poincaré group.

The model is defined by the following set of constraints:

$$\Omega_i\equiv p_i{}^2-m_i{}^2c^2+\sum_{j\neq i}^{1\cdots N}V_{ij}(r_{ij}{}^2)\approx 0,\quad i=1,\ldots,N\,, \quad (2.5\text{a})$$

where

$$r_{ij}^\mu=x_i^\mu-x_j^\mu\,, \quad (2.6\text{a})$$

$$V_{ij}(r_{ij}{}^2)=V_{ji}(r_{ij}{}^2)\,, \quad (2.6\text{b})$$

and

$$B_i=\sum_{j=1}^{N}|\,V'_{ij}(r_{ij}{}^2)\,|\,(p_i+p_j,r_{ij})\approx 0,\quad i=1,\ldots,N\,. \quad (2.5\text{b})$$

The set (2.6) only contains $N-1$ independent constraints due to the identity $\sum_{i=1}^{N}B_i=0$.

A possible objection to this model is that the constraints Ω_i contain only two-body forces, so that they would not satisfy the requirement of separability; but it must be observed that Ω_i are not the first-class mass-shell constraints of the Todorov[10] and Komar[11] approach, as we have already stressed (for a discussion of this point see Ref. 19).

The presence of the absolute value in (2.6) is a modification with respect to the model proposed in Ref. 19. With this modification the conditions for separability are readily analyzed and the conditions for the application of the implicit function theorem, discussed in Sec. III, are satisfied.

The set (2.5) with (2.6) has the cluster decomposition property, in the sense that it yields the corresponding set of constraints when some coupling goes to zero. Even in the case of partial open chain configurations the correct set of constraints is obtained. As an example let us write (2.5b) for $N=4$:

$$|\,V'_{12}\,|\,(p_1+p_2,r_{12})+|\,V'_{13}\,|\,(p_1+p_3,r_{13})+|\,V'_{14}\,|\,(p_1+p_4,r_{14})\approx 0\,, \quad (2.7\text{a})$$

$$|\,V'_{21}\,|\,(p_2+p_1,r_{21})+|\,V'_{23}\,|\,(p_2+p_3,r_{13})+|\,V'_{24}\,|\,(p_2+p_4,r_{24})\approx 0\,, \quad (2.7\text{b})$$

$$|\,V'_{31}\,|\,(p_3+p_1,r_{31})+|\,V'_{32}\,|\,(p_3+p_2,r_{32})+|\,V'_{34}\,|\,(p_3+p_4,r_{34})\approx 0\,. \quad (2.7\text{c})$$

(It is not necessary to write the fourth.)

If, for example, $V'_{i4}=0$ we get three constraints, of which only two are independent, and they are precisely (2.6) for $N=3$.

III. 1/c EXPANSION

In this section we will calculate the nonrelativistic limit of the model proposed in the previous section. This will be done by studying the series expansion of energies and relative times in inverse powers of c. In the limit for $c\rightarrow\infty$ we will recover the characteristic features of Newtonian mechanics.

Since there are $N-1$ relative times and N energies, we look for a total of $2N-1$ series expansions; these are provided in an implicit way by the $2N-1$ constraints (2.5a) and (2.5b). These are written in terms of the variables x_i^μ,p_i^ν of the phase space; since we have assumed that x_i^μ are true physical events we must write

$$x_i^\mu=(ct_i,\mathbf{x}_i),\quad i=1,\ldots,N\,. \quad (3.1)$$

Hence we have necessarily

$$p_i^\mu = \left[\frac{E_i}{c}, \mathbf{p}_i \right], \quad i=1,\ldots,N , \tag{3.2}$$

E_i being energies, due to the canonical relations (2.4).

One can think of the set of constraints (2.5) as a set of equations for the N unknowns E_i, or rather for the kinetic energies $\mathscr{E}_i$,

$$\mathscr{E}_i = E_i - m_i c^2 , \tag{3.3}$$

and the $N-1$ independent relative times $t_i - t_j$ or, rather, the combinations

$$\alpha_{ij} = c(t_i - t_j) . \tag{3.4}$$

A convenient choice of a basis for these α_{ij} has to be made. We adopt the α_{1i} ($i=2,\ldots,N$) for this purpose; hence

$$\alpha_{ij} = \alpha_{1j} - \alpha_{1i} . \tag{3.5}$$

Let us rewrite the constraints in terms of the new notation:

$$\Omega_i = \frac{\mathscr{E}_i^{\,2}}{c^2} + 2m_i \left[\mathscr{E}_i - \frac{\mathbf{p}_i^{\,2}}{2m_i} \right] + \sum_{j\neq i}^{1\cdots N} V_{ij}(\alpha_{ij}^{\;2} - \mathbf{r}_{ij}^{\;2}) , \tag{3.6}$$

$$B_i = \frac{1}{c} \sum_{j=1}^{N} |\, V'_{ij}(\alpha_{ij}^{\;2} - \mathbf{r}_{ij}^{\;2}) \,| \left[\left[m_{ij} + \frac{\mathscr{E}_{ij}}{c^2} \right] \alpha_{ij} - \frac{\mathbf{p}_{ij}\cdot \mathbf{r}_{ij}}{c} \right] , \tag{3.7}$$

where

$$m_{ij} = m_i + m_j, \quad \mathscr{E}_{ij} = \mathscr{E}_i + \mathscr{E}_j, \quad \mathbf{p}_{ij} = \mathbf{p}_i + \mathbf{p}_j . \tag{3.8}$$

The natural meaning of the $\mathscr{E}_i$ is that of nonrelativistic energies in the limit $c^{-1} \rightarrow 0$, so we assume that in this limit the $\mathscr{E}_i$ are finite; the explicit calculation below will show the consistency of this hypothesis. We can now use the system of equations (3.7) to find α_{ij}. According to the implicit function theory (see, e.g., Apostol[22]) this can be done whenever a set of numbers $\alpha_{ij}^{(0)}$ exists such that, if $c^{-1}=0$, for $\alpha_{ij} = \alpha_{ij}^{(0)}$

$$B_i = 0 \tag{3.9}$$

and

$$\det \left[\frac{\partial B_i}{\partial \alpha_{1j}} \right] \neq 0 . \tag{3.10}$$

If this is the case, we are assured of the existence of the asymptotic expansion

$$\alpha_{ij} = \alpha_{ij}^{(0)} + \frac{\alpha_{ij}^{(1)}}{c} + \frac{\alpha_{ij}^{(2)}}{c^2} + \cdots . \tag{3.11}$$

The coefficients $\alpha_{ij}^{(1)}, \alpha_{ij}^{(2)}, \ldots$ can be calculated by a very simple iterative procedure, which will be outlined later, once the $\alpha_{ij}^{(0)}$ are known.

Conditions (3.9) and (3.10) are easily worked out in this particular case. They yield, respectively,

$$\sum_{j=2}^{N} A_{ij} \alpha_{ij}^{(0)} = 0 , \tag{3.12}$$

$$\Delta_N \equiv \det(A_{ij}) \neq 0 , \tag{3.13}$$

where

$$A_{ij} \equiv a_{ij} - \delta_{ij} \sum_{k=1}^{N} a_{ik} = A_{ji} \tag{3.14}$$

and

$$a_{ij} \equiv m_{ij} \,|\, V'_{ij}(\alpha_{ij}^{(0)2} - \mathbf{r}_{ij}^{\;2}) \,| = a_{ji} \geq 0 . \tag{3.15}$$

It is apparent from Eqs. (3.12) and (3.13) that the only solution to this system of equations is

$$\alpha_{ij}^{(0)} = 0, \quad j=2,\ldots,N , \tag{3.16}$$

as long as these conditions do not force $\det(A_{ij})$ to vanish, since the coefficients A_{ij} depend on $\alpha_{ij}^{(0)}$. This actually is the only restriction which the potentials V_{ij} have to satisfy.

The solution (3.16) is very convenient because it implies that the series expansion for α_{ij} is of the form

$$\alpha_{ij} = \frac{\alpha_{ij}^{(1)}}{c} + \frac{\alpha_{ij}^{(2)}}{c^2} + \cdots \tag{3.17}$$

or that the expansion of the relative times is

$$t_i - t_j = \frac{\alpha_{ij}^{(1)}}{c^2} + \frac{\alpha_{ij}^{(2)}}{c^3} + \cdots . \tag{3.18}$$

It can be further shown that $\alpha_{ij}^{(2)} = 0$ and, in general, that only terms with even powers of c^{-1} will appear in (3.18) because Eqs. (2.5) only depend on c^2 when expressed in terms of the unknown $\mathscr{E}_i$ and $t_i - t_j$. Hence

$$t_i - t_j = \frac{\alpha_{ij}^{(1)}}{c^2} + \frac{\alpha_{ij}^{(3)}}{c^4} + \cdots . \tag{3.19}$$

Likewise we see that

$$\mathscr{E}_i = \mathscr{E}_i^{(0)} + \frac{\mathscr{E}_i^{(1)}}{c^2} + \frac{\mathscr{E}_i^{(2)}}{c^4} + \cdots . \tag{3.20}$$

Now according to the discussion given in the Appendix, the determinant Δ_N of (3.13) may only vanish when the system of particles is composed of at least two clusters with no interaction at all between them. But in that case the set of constraints (2.5) breaks into as many uncoupled sets as there are clusters, and the previous arguments must be applied to each one separately. The time correlations α_{ij} between particles in different clusters remain arbitrary, while those in the same cluster are found in the way just outlined above.

To calculate the remaining coefficients we only have to substitute (3.17) into (3.7) and equate to zero the terms containing like powers of c^{-1}. Since $\alpha_{ij}^{\;2}$ is of order c^{-2} it is simple to evaluate $\alpha_{ij}^{(1)}$; indeed

$$\sum_{j=2}^{N} A_{ij} \alpha_{1j}^{(1)} = \sum_{j=2}^{N} |\, V'_{ij}(-\mathbf{r}_{ij}^{\;2}) \,| (\mathbf{p}_{ij} \cdot \mathbf{r}_{ij}) . \tag{3.21}$$

This is a linear system which can be inverted because (3.13) is supposed to hold. It is easily seen that the same kind of linear system arises for higher-order terms, always leading to unique solutions for the corresponding coefficients. This is, in fact, a consequence of the implicit-function theorem.

Let us now calculate the coefficients $\mathscr{E}_i^{(0)}, \mathscr{E}_i^{(1)}, \ldots,$ in (3.20). From Eq. (3.6) it is easily seen that

$$\mathscr{E}_i^{(0)} = \frac{1}{2m_i}\left[\mathbf{p}_i^2 - \sum_{j\neq i} V_{ij}(-\mathbf{r}_{ij}^2)\right], \tag{3.22}$$

and, to next order,

$$\mathscr{E}_i^{(1)} = -\frac{1}{2m_i}\left[\mathscr{E}_i^{(0)2} + \sum_{j\neq i} V'_{ij}(-\mathbf{r}_{ij}^2)\alpha_{ij}^{(1)2}\right], \tag{3.23}$$

an expression in which the previously calculated $\mathscr{E}_i^{(0)}$ and $\alpha_{ij}^{(1)}$ must be used.

Observe that the kinetic energies $\mathscr{E}_i$ can be calculated without any reference to the cluster structure of the system of particles. Indeed the time separations appearing in (3.23) remain arbitrary, but their presence does not affect the value of the energy because the coefficients $V_{ij}(-\mathbf{r}_{ij}^2)$ cancel out their contribution whenever the indices i and j refer to different clusters. Therefore the energy of each particle only depends on its interactions with other particles in the same cluster.

IV. NONRELATIVISTIC LIMIT

In the previous section we studied the asymptotic expansions of energies and relative times, derived from the structure of the constraints. Now we have to analyze how these expansions lead, at the lowest order, to nonrelativistic dynamics.

To this end let us consider the lowest-order expressions of the constraints (2.5)

$$\mathscr{E}_i = \frac{1}{2m_i}\left[\mathbf{p}_i^2 - \sum_{j\neq i} V_{ij}(-\mathbf{r}_{ij}^2)\right], \tag{4.1a}$$

$$t_{ij} = 0 . \tag{4.1b}$$

This set of equations is a set of constraints for the nonrelativistic dynamics. We want to show that it corresponds to a Newtonian dynamics for the spatial coordinates, $\mathbf{p}_i$ and $\mathbf{r}_{ij}$ in terms of a unique time t, in our inertial reference frame.

Let us put

$$\phi_i \equiv \mathscr{E}_i - \frac{1}{2m_i}\left[\mathbf{p}_i^2 - \sum_{j\neq i} V_{ij}(\mathbf{r}_j^2)\right], \tag{4.2}$$

$$\psi_j \equiv t_1 - t_{j+1} , \tag{4.3}$$

with

$$i = 1, \ldots, N; \quad j = 1, \ldots, N-1 ,$$

the algebra of constraints is

$$\{\phi_i, \phi_j\} = -\frac{1}{m_i m_j} V'_{ij}(-\mathbf{r}_{ij}^2)(\mathbf{p}_{ij}\cdot\mathbf{r}_{ij}) , \tag{4.4}$$

$$\{\phi_i, \psi_j\} = \delta_{i1} - \delta_{ij+1} , \tag{4.5}$$

$$\{\psi_i, \psi_j\} = 0 . \tag{4.6}$$

The Dirac Hamiltonian

$$H_D = \sum_{i=1}^{N} \lambda_i \phi_i + \sum_{j=1}^{N-1} \mu_j \psi_j \tag{4.7}$$

is determined by

$$\{\phi_i, H_D\} \approx \sum_{e=1}^{N} -\lambda_e \frac{V'_{ie}(-\mathbf{r}_{ie}^2)}{m_i m_e} \mathbf{p}_{ie}\cdot\mathbf{r}_{ie} + \sum_{k=1}^{N-1} \mu_k(\delta_{i1} - \delta_{ik+1}) \approx 0 \tag{4.8}$$

and

$$\{\psi_j, H_D\} \approx \lambda_1 - \lambda_{j+1} \approx 0 . \tag{4.9}$$

From this last set of equations we have

$$\lambda_1 \approx \lambda_2 \approx \cdots \approx \lambda_N \equiv \lambda_0 \tag{4.10}$$

so H_D is of the form

$$H_D \approx \lambda_0 \left[\sum_{i=1}^{N}\left[\mathscr{E}_i - \frac{\mathbf{p}_i^2}{2m_i}\right] + V\right] + \sum_{j=1}^{N-1} \mu_j \psi_j , \tag{4.11}$$

where

$$V \equiv \sum_{\substack{i,j=1\\(i\neq j)}}^{N} \frac{1}{2m_i} V_{ij}(-\mathbf{r}_{ij}^2) . \tag{4.12}$$

From (4.11) the equations of motion for $\mathbf{x}_i$ and $\mathbf{p}_i$ are

$$\dot{\mathbf{x}}_i \approx -\lambda_0\{\mathbf{x}_i, H\} \quad (i = 1, \ldots, N) ,$$
$$\dot{\mathbf{p}}_i \approx -\lambda_0\{\mathbf{p}_i, H\} , \tag{4.13}$$

with

$$H \equiv \sum_{i=1}^{N} \frac{1}{2m_i}\mathbf{p}_i^2 + V , \tag{4.14}$$

which is the Newtonian Hamiltonian. (Notice that V is a function of the spatial $\mathbf{r}_{ij}$ only.) On the other hand, from the equations of motion for t_i, we have $(\{\mathscr{E}_i, t_j\} = \delta_{ij})$

$$\dot{t}_i \approx \lambda_0\{t_i, \mathscr{E}_1 + \cdots + \mathscr{E}_N\} = -\lambda_0 . \tag{4.15}$$

From this, with the use of constraints $\psi_j = 0$, we get

$$\dot{t} = -\lambda_0, \quad t = t_1 \approx t_2 \approx \cdots \approx t_N \tag{4.16}$$

so, if we choose the parametrization

$$\lambda_0 = -1, \ \text{ i.e., } t = \tau + \text{const} , \tag{4.17}$$

we have the equations of motion (4.13) in Newtonian form:

$$\dot{\mathbf{x}}_i = \{\mathbf{x}_i, H\}, \quad \dot{\mathbf{p}}_i = \{\mathbf{p}_i, H\} , \tag{4.18}$$

with $t_1 = t_2 = \cdots = t_n \equiv t$.

What remains a complicated task is to find the equations of motion for the $\mathscr{E}_i$, which are determined by the μ_j, determined in turn by Eqs. (4.8):

$$\dot{\mathscr{E}}_i \approx \sum_{j=1}^{N-1} \mu_j(\delta_{i1} - \delta_{ij+1}) .$$

However, for the purposes of our discussion it is not necessary to solve these equations, since, in any case, we observe that

$$\mathscr{E} \equiv \mathscr{E}_1 + \mathscr{E}_2 + \cdots + \mathscr{E}_N \tag{4.19}$$

is, as expected, a constant of motion since it has zero Poisson brackets with the constraints (4.2) and (4.3), and hence it is in weak involution with H_D.

V. CONCLUSIONS

What we have proposed in this work is a model which satisfies relativistic invariance and the cluster decomposition property, as we have defined it. We have shown that it has a well-defined nonrelativistic limit, which is nothing but the usual Newtonian mechanics. From this point of view it has a clear physical content.

Some words should be said about the definition we gave of the cluster decomposition property and what is meant of it by other authors. It could be observed that, with no other restrictions on the potential functions $V_{ij}(r_{ij}^2)$, timelike interactions between the particles, out of the nonrelativistic limit, could occur. Even so, a vanishing of the potentials results in a decoupling of the constraint equations as we have seen, so any separate cluster has a separate dynamics.

If we require a more strict definition of separability, that is, that it should occur for a great spacelike separation between clusters of particles, a more specific choice of the potentials must be made; namely, we must take the potentials to vanish for $r_{ij}^2 > 0$. (Let us remark that the usual definition of force makes sense for spacelike distances only.)

With this last choice, the definition of separability, as defined, for instance, in Ref. 8, is recovered.

For potentials not satisfying this property, more general kinds of models can be devised, and an enlargement of the class of models can be achieved by relaxing the requirement we have made that the potentials be of positive derivative. In this last case models without a well-defined nonrelativistic limit are obtained.

APPENDIX

In this appendix we study the $(N-1)$th-order determinant

$$\Delta_N = \det(A_{ij});\quad i=1,\ldots,N-1;\quad j=2,\ldots,N\ .$$

Explicitly

$$\Delta_N = \begin{vmatrix} a_{12} & a_{13} & \cdots & a_{1N} \\ -(a_{21}+\cdots+a_{2N}) & a_{23} & \cdots & a_{2N} \\ a_{32} & -(a_{31}+\cdots a_{3N}) & \cdots & a_{3N} \\ \vdots & \vdots & & \vdots \\ a_{N-1,2} & a_{N-1,3} & \cdots & a_{N-1,N} \end{vmatrix}, \tag{A1}$$

where $a_{ij} = a_{ij}$.

First note that if we add all the other columns to the last, change the sign of the first $N-2$ columns and then transpose the last column in the first place, Δ_N can be written in the more symmetric form:

$$\Delta_N = \begin{vmatrix} a_{12}+\cdots+a_{1N} & -a_{12} & \cdots & -a_{1,N-1} \\ -a_{21} & a_{21}+\cdots+a_{2N} & & -a_{2,N-2} \\ \vdots & \vdots & & \vdots \\ -a_{N-1,1} & -a_{N-1,2} & & a_{N-1,1}+\cdots+a_{N-1,N} \end{vmatrix}. \tag{A2}$$

In what follows we will show that Δ_N is equal to the sum of all possible products of $N-1$ a_{ij}, with positive sign, corresponding to all possible trees (connected graphs without loops) of N vertices and $N-1$ branches (Cayley trees):

$$\Delta_N = \sum_t \delta_t,\quad t=1,\ldots,N^{N-2}\ , \tag{A3}$$

where δ_t is the contribution to Δ_N of the tth tree and $\delta_t \geq 0$. The total number of Cayley trees with N vertices in N^{N-2} (see below).

A single term δ_t can be obtain from Δ_N by equating to zero all the a_{ij} corresponding to the branches which do not contribute to the tth tree.

From (A3) it follows that $\Delta_N = 0$ if and only if the a_{ij} (that is, essentially the V'_{ij}) correspond to a system of particles *separated* in at least two *noninteracting clusters.* If the N particles constitute a unique cluster of interacting particles, we have $\Delta_N \neq 0$, which is the result used in Sec. III. Further, Δ_N is a symmetric *unsignant* determinant,[23] due to the circumstance that in the expanded form all its terms are positive.

In order to demonstrate (A3) we will prove that Δ_N has the following properties:

(a) it is invariant for any permutation of the indices $1,2,\ldots,N$;

(b) in the expanded form all its terms are positive (as long as the a_{ij} are positive, $i,j=1,\ldots,N$);

(c) it has an expansion formed of distinct terms, as long as the a_{ij} are distinct;

(d) if we represent each term of its expansion by means of a graph of N vertices and $N-1$ branches (where the branches correspond to the a_{ij} and the vertices to the indices), then no disconnected graphs, nor graphs with loops, contribute to such expansion, as long as all the a_{ij} are different from zero;

(e) and finally that the number of terms in its expansion is equal to the number of Cayley trees, that is, N^{N-2}.

Property (a) can be proved by adding all the other rows to the first row and all the other columns to the first column; by changing the sign of the first row and after that of the first column we get again Δ_N with the labels 1 and N exchanged. In an analogous way we can exchange any two labels (see Ref. 23, p. 390).

Let us denote by $\Delta_{N-r}^{(i_1 \cdots i_r)}$ the determinant Δ_{N-r} where the a_{iN} are replaced by $a_{ii_1}+a_{ii_2}+\cdots+a_{ii_r}$. Property (b) can be proved by induction using the following expansion of Δ_N (see Muir, Ref. 23, p. 390)

$$\Delta_N=\sum_{i=1}^{N-1} a_{iN}\Delta_{N-1}+\sum_{i,j=1}^{N-1} a_{iN}a_{jN}\Delta_{N-2}^{(ij)} +\cdots+a_{1N}a_{2N}\cdots a_{N-1,N} \; . \tag{A4}$$

Property (c) can be proved in the same way.

Each term in the expression of Δ_N is a product of $N-1$ a_{ij}'s ($i,j=1,\ldots,N$), so it can be represented by a graph of N vertices and $N-1$ branches corresponding to the a_{ij}'s. Now, if a graph of this kind has a loop, it is necessarily disconnected and vice versa. If indeed it has a loop, we will not have enough branches to build a connected graph. Conversely, if it is disconnected, we will have at least one branch in excess, which will necessarily form a loop. So, in order to demonstrate property (d) we have only to show that when all the $a_{ij}\neq 0$ no disconnected graphs can be present in the expansion of Δ_N. This follows trivially since if a disconnected graph does contribute to Δ_N it must give a nonvanishing contribution even when we put to zero the a_{ij}, with i belonging to one piece of the graph and j to another. The structure of Δ_N in this case, as can be seen by explicitly writing it, is the following: let us choose $i=1,2,\ldots,r$ and $j=r+1,\ldots,n$. Putting $a_{ij}=0$ we have that Δ_N is the product of two determinants, one of which is

$$\begin{vmatrix} a_{12}+\cdots+a_{1r} & \cdots & -a_{1r} \\ \vdots & & \vdots \\ -a_{r1} & \cdots & a_{r1}+\cdots+a_{r,r-1} \end{vmatrix}$$

which is zero. It follows that in this case $\Delta_N=0$.

We can conclude from this that only connected graphs without loops, that is, Cayley trees, can contribute to Δ_N. Further, if we show that the number of terms in the expansion of Δ_N is N^{N-2}, that is, the number of Cayley trees of N vertices and $N-1$ branches (see, for instance, Aigner[24]), then Eq. (A3) is proved. The number of terms in the expansion of Δ_N can be calculated by putting all the a_{ij} equal to 1. We get in this way the $(N-1)$th-order determinant:

$$\begin{vmatrix} N-1 & -1 & \cdots & -1 \\ -1 & N-1 & \cdots & -1 \\ \vdots & \vdots & & \vdots \\ -1 & -1 & \cdots & N-1 \end{vmatrix}$$

which can be easily seen to be equal to N^{N-2} (add all other rows to the first and subtract the first column so obtained from all the others).

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