Bargmann-Wigner method and $(6s+1)$-component wave equations

J. Gomis and J. M. Pons
Departament de Física Teòrica, Universitat Central de Barcelona, Diagonal 647, Barcelona-28, Spain
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A modified Bargmann-Wigner method is used to derive $(6s+1)$-component wave equations. The relation between different forms of these equations is shown.

An elegant method to construct wave equations for arbitrary spin was given by Bargmann and Wigner (BW).[1] Their equations are

$$\begin{aligned}&D\otimes I\otimes\cdots\otimes I\psi=0\,,\\&I\otimes D\otimes I\otimes\cdots\otimes I\psi=0\,,\\&\cdots\\&I\otimes I\otimes\cdots\otimes I\otimes D\psi=0\,,\end{aligned}\tag{1}$$

D being the Dirac operator $(i\gamma_\mu\partial^\mu-m)$. ψ is a symmetric multispinor of rank $2s$:

$$\psi(x)=\psi(x)^{\alpha_1\alpha_2\cdots\alpha_n}e_{\alpha_1}\otimes e_{\alpha_2}\otimes\cdots\otimes e_{\alpha_n}\,,$$
$$\alpha_i=1,\ldots,4$$

which describes a particle of spin s.

The Eqs. (1) are invariant under the transformation

$$\psi'(x')=\mathfrak{D}(\Lambda)\otimes D(\Lambda)\otimes\cdots\otimes\mathfrak{D}(\Lambda)\psi(x)\,,\tag{2}$$

where $\mathfrak{D}(\Lambda)$ is the usual Dirac representation $\mathfrak{D}^{(1/2,0)\otimes(0,1/2)}$ of the SO(3, 1) group. Unfortunately this approach has several drawbacks, so the set of BW equations cannot be derived from a Lagrangian, without introducing auxiliary fields. Another serious difficulty connected with the above is the internal inconsistency of Eq. (1) in the presence of an external electromagnetic field. A class of relativistic wave equations which does not suffer from this last inconsistency is one proposed by Hurley.[2] His wave functions, describing a massive particle of spin s, have $6s+1$ components which transform under the $(s,0)\oplus(s-\frac{1}{2},\frac{1}{2})$ representation of the Lorentz group SO(3, 1). These equations are a subclass of those of Fierz and Pauli[3] and were later studied by several authors.[4]

Now we want to show that Hurley's $6s+1$ theory is nothing but a modification of the Bargmann-Wigner set (1), that is to say,

$$\begin{aligned}&D\otimes\Gamma\otimes\cdots\otimes\Gamma\psi=0\,,\\&\Gamma\otimes D\otimes\Gamma\otimes\cdots\otimes\Gamma\psi=0\,,\\&\cdots\\&\Gamma\otimes\Gamma\otimes\cdots\otimes\Gamma\otimes D\psi=0\,,\end{aligned}\tag{3}$$

where D is the Dirac operator and Γ a 4×4 projector matrix whose expression depends on the representation of the γ matrices of the Dirac operator. If we choose the Kramers representation, as we shall do in the following, the matrix Γ is

$$\Gamma=\begin{pmatrix}I&0\\0&0\end{pmatrix}.$$

The equations in (3) are invariant under the transformation (2) due to the fact that D is an invariant operator, and Γ has been chosen so that it commutes with the Dirac representation $\mathfrak{D}^{(1/2,0)\oplus(0,1/2)}$. Obviously this theory describes a wave functions with $6s+1$ components.

An important difference between Eqs. (1) and (3) is that Eq. (3) can be derived from the single equation

$$\sum_{i=1}^{2s}(\Gamma\otimes\cdots\otimes\Gamma\otimes\underset{(i)}{D}\otimes\Gamma\otimes\cdots\otimes\Gamma)\psi=0\,,\tag{4}$$

the proof being similar to one presented by Hagen[5] in a Galilean context. In our case, the crucial point is that $\Gamma(\gamma\cdot P+m)\Gamma=m\Gamma$.

In order to see explicitly the equivalence between Hurley's equations and the modified BW set (3) and (4), we must introduce a new basis in the $(6s+1)$-dimensional subspace of symmetric multispinors where Eqs. (3) and (4) have a nontrivial content:

$$\begin{aligned}&|s,r\rangle_x=f(s,r)y_n(s-r)\,,\quad r=-s,-s+1,\ldots,s\\&|s,l\rangle_y=f(s,l)(2s)^{-1/2}[(s+l)y_n(s-l;e_3)+(s-l)y_n(s-l-1;e_4)]\,,\quad l=-s,-s+1,\ldots,s\\&|s-1,h\rangle_z=f(s-1,h)(2s-1)^{1/2}[-y_n(s-h;e_3)+y_n(s-h-1;e_4)]\,,\quad h=-(s-1),\ldots,s-1\\&f(s,r)=\left(\frac{(2s)!}{(s+r)!(s-r)!}\right)^{1/2},\end{aligned}\tag{5}$$

being

$$y_n(k)=\frac{1}{n!}\sum_{\sigma\in S_n}\sigma(e_1\otimes\cdots\otimes e_1\otimes e_2\otimes\cdots\otimes e_2)\,,$$

$$y_n(k;e_\alpha)=\frac{1}{n!}\sum_{\sigma\in S_n}\sigma(e_1\otimes\cdots\otimes e_1\otimes e_2\otimes\cdots\otimes e_2\otimes e_\alpha)\,,$$

where $\alpha=3,4$, k is the number of vectors e_2, and n the total number of vectors involved in the tensor product. S_n is the symmetric group, $n=2s$.

This basis has been constructed in order to have good behavior under SO(3), i.e., $|s,r\rangle_x$ and $|s,l\rangle_y$ are objects of spin s, and $|s-1,h\rangle_z$ is an object of spin $s-1$. Let us note that $(e_1,\ldots,e_4)$ is the basis of the four-dimensional space where the Dirac representation acts; this representation becomes $D^{1/2}(R)\oplus D^{1/2}(R)$ when reduced to SO(3). In this case (e_1,e_2) are the up-down vectors of the first $D^{1/2}(R)$ representation, whereas (e_3,e_4) are the same for the second representation.

Let us expand $\psi(x)$ in terms of our new basis:

$$\psi(x)=\sum_{r=-s}^{+s}X_r(x)|s,r\rangle_x+\sum_{l=-s}^{+s}Y_l(x)|s,l\rangle_y+\sum_{h=-s+1}^{s-1}Z_h(x)|s-1,h\rangle_z+\cdots\,. \quad (6)$$

Now we want to rewrite Eq. (4) in terms of the expression (6) of the multispinor ψ. Owing to the appearance of the projector Γ, this equation gives only information about the $(6s+1)$-dimensional space spanned by the vectors (5). Therefore we have now equations of motion only for the components X, Y, Z. Making some trivial but tedious algebra we arrive at the equations

$$\left\{\begin{bmatrix}0&I&0\\I&0&0\\0&0&0\end{bmatrix}E+\frac{1}{s}\begin{bmatrix}0&-\vec{S}&\vec{K}^\dagger\\\vec{S}&0&0\\-\vec{K}&0&0\end{bmatrix}\vec{P}-mI\right\}\begin{bmatrix}\vec{X}\\\vec{Y}\\\vec{Z}\end{bmatrix}=0\,, \quad (7)$$

where $\vec{S}$ are the matrices of generators of SO(3) in the spin-s representation, and $\vec{K}$ are rectangular $(2s-1)\times(2s+1)$ matrices satisfying

$$S_iS_j+K_i^\dagger K_j=is\epsilon_{ijk}S_k+s^2\delta_{ij}\,.$$

Equation (7) is exactly the one proposed by Hurley. Therefore we have shown, by using the changes of basis (5), the equivalence of formulation (3) [or (4)] and the one of Hurley. Let us remark that, if one introduces interaction with an external field, the equivalence only holds between Eqs. (4) and (7).

The equivalence between (3) and (4) suggests the possibility of a Lagrangian formulation; in fact the Lagrangian

$$\mathcal{L}=\psi^*(\gamma^0\otimes\gamma^0\otimes\cdots\otimes\gamma^0)\times\left(\sum_{i=1}^{2s}(\Gamma\otimes\cdots\otimes\underset{(i)}{D}\otimes\cdots\otimes\Gamma)\right)\psi+\text{H.c.} \quad (8)$$

gives the equation

$$\sum_{i=1}^{2s}(\Gamma\otimes\cdots\otimes\Gamma\otimes\underset{(i)}{D}\otimes\Gamma\otimes\cdots\otimes\Gamma)\psi=0\,, \quad (4)$$

but it also appears as a new equation,

$$\sum_{i=1}^{2s}(W\otimes\cdots\otimes W\otimes\underset{(i)}{D}\otimes W\otimes\cdots\otimes W)\psi=0\,, \quad (9)$$

where $W=\left(\begin{smallmatrix}0&0\\0&I\end{smallmatrix}\right)$. The appearance of this new equation is due to the fact that the first term on the right-hand side of (8) is not real. Notice that (4) is an equation for an object whose components have at most only one dotted index, whereas (9) describes an independent object, whose components have all the indices dotted, except at most one. Therefore, in order to have a Lagrangian, we must double the number of components of the theory. If we want to have $6s+1$ components, we cannot have a Lagrangian; in the same way, we cannot implement parity.

Let us briefly comment on the Galilean case. The decompositions (6) can be used in the same way as we have done before, in order to show the equivalence between the $6s+1$ Galilean Hagen equations[5]

$$G\otimes\Gamma\otimes\cdots\otimes\Gamma\psi=0\,,$$
$$\Gamma\otimes G\otimes\Gamma\otimes\cdots\otimes\Gamma\psi=0\,, \quad (10)$$
$$\Gamma\otimes\Gamma\otimes\cdots\otimes\Gamma\otimes G\psi=0\,,$$

where G is the usual Levy-Leblond operator, Γ is the projector used before, and the $(6s+1)$ Hurley equations[6]

$$\left\{\begin{bmatrix}I&0&0\\0&0&0\\0&0&0\end{bmatrix}E+\frac{1}{s}\begin{bmatrix}0&\vec{S}&\vec{K}^\dagger\\\vec{S}&0&0\\\vec{K}&0&0\end{bmatrix}\vec{P}+\begin{bmatrix}0&0&0\\0&2m&0\\0&0&2m\end{bmatrix}\right\}\begin{bmatrix}\vec{X}\\\vec{Y}\\\vec{Z}\end{bmatrix}=0\,.$$

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