

Batalin-Vilkovisky analysis for covariant and noncovariant actions

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The equivalence between the covariant and the noncovariant versions of a constrained system is shown to hold after quantization in the framework of the field-antifield formalism. Our study covers the cases of electromagnetism and Yang-Mills fields and sheds light on some aspects of the Faddeev-Popov method, for both the covariant and noncovariant approaches, which have not been fully clarified in the literature.

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I. INTRODUCTION

In a recent paper [1] the Becchi-Rouet-Stora-Tyutin (BRST) quantization of a class of constrained dynamical systems was performed in the framework of the Batalin-Vilkovisky (BV) formalism [2]. These systems were specified by a Lagrangian which is quadratic in the velocities and such that only primary first-class constraints, linear in the momenta, appear in its Hamiltonian analysis. After solving the classical master equation, which is straightforward due to its closed algebra structure, the problem of the ambiguity inherent to the resolution of the full quantum master equation was addressed. It is well known [2] that this ambiguity, which can be drawn to the problem of defining the measure for the path integral, has no solution within the BV formalism by itself and one has to rely on other formulations, the operator formalism, for instance, to get the correct answer. In this sense our result was promising: the physical requirement of making contact with the reduced path integral quantization procedure, which is very close to ensuring unitarity, is equivalent to the internal requirement (i.e., without departing from the BV formalism) of choosing the solution of the quantum master equation that makes the path integral reparametrization invariant.

But a wide class of constrained systems do not fit within the theories just considered. Many relevant physical examples, such as electromagnetism (EM) and Yang-Mills (YM) fields, exhibit secondary as well as primary first-class Hamiltonian constraints. There is an important physical reason for the appearance of secondary constraints in these theories, and it is related to the way the Hamiltonian formalism, which is manifestly noncovariant, is able to provide us with gauge transformations which are Lorentz covariant. Consider, for instance, the

case of EM. The infinitesimal gauge transformation $\delta A_\mu = \partial\Lambda/\partial x^\mu$ shows how to get a vector $\partial\Lambda/\partial x^\mu$ from a scalar Λ : just by taking the gradient. The appearance of a time derivative of the arbitrary function Λ forces the Hamiltonian generator of the gauge transformation to have two pieces, one with the first time derivative of Λ and the other one without a time derivative. Let us be more specific; we know on theoretical grounds [3] that a generator of gauge transformations, acting through Poisson brackets, must have the form

$$G = \sum_{k=0}^N G_{N-k} \Lambda^k, \quad (1.1)$$

Λ^k being the k th time derivative of Λ and G_s an s -generation first-class constraint. In the case of EM, G is the sum of two pieces, coming from one primary and one secondary constraint (two generations). In fact, from the Lagrangian $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ we get a canonical Hamiltonian $H_c = \int d\mathbf{x}[\frac{1}{2}(\pi^2 + \mathbf{B}^2) + \pi \cdot \nabla A_0]$ and a primary constraint (coming just from the definition of the momenta) $\pi_0 \simeq 0$. The stability of this constraint under the Hamiltonian dynamics leads to the secondary constraint $\dot{\pi}_0 = \{\pi_0, H_c\} = \nabla \cdot \pi \simeq 0$. No more constraints arise. Both the primary and the secondary constraints are first class and allow us to write the gauge generator (1.1) in this case as

$$G = \int d\mathbf{x}[\dot{\Lambda}(\mathbf{x}, t)\pi^0(\mathbf{x}, t) + \Lambda(\mathbf{x}, t)\nabla \cdot \pi(\mathbf{x}, t)] \\ = \int d\mathbf{x} \left[\frac{\partial\Lambda}{\partial x^\mu} \pi^\mu \right],$$

where Λ is an infinitesimal arbitrary function. The gauge transformation of the gauge field is then $\delta A_\mu = \{A_\mu, G\} = \partial\Lambda/\partial x^\mu$. We see therefore that a primary and a secondary constraint are necessary to get the gauge field A_μ transformed covariantly.

So, in principle, we are faced with the problem that the class of theories studied in [1], which exhibit only primary constraints, seems to exclude the important physi-

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cal case of covariant theories. In fact this is not true, as we will see that our primary constraint theories can be conveniently covariantized simply by promoting the Lagrangian multipliers associated with the primary constraints to the status of dynamical variables, covering this way the cases of EM and YM fields [4]. With this covariant theory at hand we can proceed to study the new features that arise in this case and that were absent in the class of systems studied in [1], for instance the possibility of having an open gauge algebra. This process of covariantization and the study of the Hamiltonian and Lagrangian gauge algebra will be the topics of the next section. In Sec. III we perform the BV quantization of the covariant theory by solving explicitly the quantum master equation and, after that, implementing several gauge fixings. In this fashion we show the equivalence between this covariant quantization and the noncovariant approach used in [1]. Section IV is devoted to conclusions. Finally, an Appendix is introduced to show how the covariantization procedure described in Sec. II works in the YM case.

II. GAUGE ALGEBRA AND COVARIANTIZATION OF THE ACTION

A. General setting

The noncovariant Lagrangians¹ $L_{nc}(q, \dot{q})$ of interest to us are those for which the tangent space is free from any constraints, yet L_{nc} is still a singular Lagrangian. As is proven in [5] this is equivalent to having in phase space only primary first-class constraints. In this case, the canonical Hamiltonian $H_0(q, p)$ and the constraints $T_\alpha(q, p)$, $\alpha = 1, \dots, r$, satisfy

$$\{T_\alpha, T_\beta\} = -C_{\alpha\beta}^\gamma(q, p)T_\gamma, \quad \{T_\alpha, H_0\} = -V_\alpha^\beta(q, p)T_\beta. \quad (2.1)$$

Let us now have a quick look at the issue of ‘‘covariantization.’’ There is a simple way to get, from a canonical theory with only primary first-class constraints, a classically equivalent theory with primary and secondary first-class constraints. This can be done by promoting the Lagrangian multipliers λ^α , associated with the original constraints, to the status of dynamical variables and to assume as the new primary constraints its canonical conjugate momenta π_α . Under such conditions the extended Hamiltonian will read

$$H(q, p; \lambda) = H_0(q, p) + \lambda^\alpha T_\alpha(q, p),$$

and the stability of the new primary constraints $\pi_\alpha \simeq 0$ will lead to the, now, secondary constraints

¹For the sake of simplicity we are going to use throughout this paper the language of discrete degrees of freedom. The switch to field theory language can be done, at least formally, by using DeWitt’s condensed notation. Also for the same reason of simplicity we will restrict ourselves to the case of classical bosonic variables, $\epsilon(q) = 0$.

$$\dot{\pi}_\alpha = \{\pi_\alpha, H\} = -T_\alpha(q, p) \simeq 0,$$

whose stability gives no new information:

$$\dot{T}_\alpha = \{T_\alpha, H\} = -V_\alpha^\beta T_\beta - C_{\alpha\beta}^\gamma \lambda^\beta T_\gamma \simeq 0.$$

Using the well known algorithms briefly described in the Introduction [3] it is straightforward to construct the gauge generator for this case,

$$G = \dot{\epsilon}^\alpha \pi_\alpha + \epsilon^\alpha [T_\alpha - (V_\alpha^\beta + C_{\alpha\gamma}^\beta \lambda^\gamma) \pi_\beta],$$

and, consequently, the Hamiltonian gauge transformations for the coordinates q^A, λ^α :

$$\begin{aligned} \delta_H q^A &= \{q^A, G\} = \{q^A, T_\alpha\} \epsilon^\alpha - \{q^A, (V_\alpha^\beta + C_{\alpha\gamma}^\beta \lambda^\gamma)\} \epsilon^\alpha \pi_\beta, \\ \delta_H \lambda^\alpha &= \{\lambda^\alpha, G\} = \dot{\epsilon}^\alpha - V_\beta^\alpha \epsilon^\beta - C_{\beta\gamma}^\alpha \lambda^\gamma \epsilon^\beta, \end{aligned} \quad (2.2)$$

where ϵ^α is an infinitesimal arbitrary function of time (or space-time, in the case of field theory).

As is well known, to perform the covariant quantization of a gauge theory within the framework of the field-antifield formalism, knowledge of the gauge structure of the classical Lagrangian theory is of fundamental importance. Since for the systems under consideration much of this information is already contained in the Hamiltonian gauge structure, in what follows we briefly describe the derivation of quantities and relations defining this structure at the Hamiltonian level.

To begin with, let us derive some relations involving the quantities $V_\beta^\alpha, C_{\beta\gamma}^\alpha$, the constraints T_α and the canonical Hamiltonian H_0 , which appear as consequences of some Jacobi identities. Consider, for instance, the following Jacobi identity involving the constraints T_α :

$$\{T_\alpha, \{T_\beta, T_\gamma\}\} + \text{cyclic perm. of } (\alpha, \beta, \gamma) = 0.$$

Using (2.1) we get

$$[C_{\sigma\gamma}^\mu C_{\alpha\beta}^\sigma - \{C_{\alpha\beta}^\mu, T_\gamma\} + \text{cyclic perm. of } (\alpha, \beta, \gamma)] T_\mu = 0.$$

The general solution of this equation,

$$[C_{\sigma\gamma}^\mu C_{\alpha\beta}^\sigma - \{C_{\alpha\beta}^\mu, T_\gamma\} + \text{cyclic perm. of } (\alpha, \beta, \gamma)] = B_{\alpha\beta\gamma}^{\mu\sigma}(q, p) T_\rho, \quad (2.3)$$

leads to the existence of a new function $B_{\alpha\beta\gamma}^{\mu\sigma}$ antisymmetric in its upper indices.

In much the same way, from the Jacobi identity among the constraints and the canonical Hamiltonian,

$$\{T_\alpha, \{T_\beta, H_0\}\} + \{H_0, \{T_\alpha, T_\beta\}\} + \{T_\beta, \{H_0, T_\alpha\}\} = 0,$$

a new relation is obtained,

$$\begin{aligned} [-C_{\gamma\beta}^\sigma V_\alpha^\gamma + C_{\gamma\alpha}^\sigma V_\beta^\gamma + C_{\alpha\beta}^\gamma V_\gamma^\sigma \\ + \{V_\alpha^\sigma, T_\beta\} - \{V_\beta^\sigma, T_\alpha\} - \{C_{\alpha\beta}^\sigma, H_0\}] = D_{\alpha\beta}^{\sigma\mu}(q, p) T_\mu, \end{aligned} \quad (2.4)$$

yielding the appearance of a new structure function $D_{\alpha\beta}^{\sigma\mu}$, antisymmetric in its upper and lower indices.

Continuing this procedure, that is, taking an increasing number of Poisson brackets among the constraints and the canonical Hamiltonian and antisymmetrizing them in

a convenient fashion, new quantities and new relations among the functions previously obtained are found. All these objects are the so-called structure functions and the relations among them determine the structure of the Hamiltonian gauge algebra. For a more exhaustive study of this Hamiltonian gauge structure we refer the reader to Ref. [6].

B. The model: Quadratic Lagrangians

So far we have established the most general setting for theories we are interested in. Now we will apply this framework to the case of noncovariant quadratic Lagrangians of the type [1]

$$L_{\text{nc}}(q, \dot{q}) = \frac{1}{2} \dot{q}^A G_{AB}(q) \dot{q}^B - V(q), \quad A, B = 1, \dots, N, \quad (2.5)$$

where $G_{AB}(q)$ is a singular metric such that its null vectors $U_\alpha^A(q)$, $G_{AB} U_\alpha^B = 0$, $\alpha = 1, \dots, k$, are Killing vectors for it, i.e.,

$$(\mathcal{L}_\alpha G)_{AB} = G_{AB,C} U_\alpha^C + G_{AC} U_{\alpha,B}^C + G_{BC} U_{\alpha,A}^C = 0, \quad (2.6)$$

and keep the potential V unchanged

$$U_\alpha^A \frac{\partial V}{\partial q^A} = 0.$$

In (2.6), \mathcal{L}_α stands for the Lie derivative in the \hat{U}_α direction. These last two conditions enforce the nonexistence of Lagrangian constraints [1].

The primary Hamiltonian constraints for this system are easily found,

$$T_\alpha = U_{\alpha A}^A p_A, \quad (2.7)$$

and its first-class character, which is guaranteed by requirement (2.6), yields the commutation relations defining the structure functions $C_{\alpha\beta}^\gamma$,

$$U_{\alpha,B}^A U_\beta^B - U_{\beta,B}^A U_\alpha^B = -U_\gamma^A C_{\alpha\beta}^\gamma, \quad (2.8)$$

which in the present case depend only on the coordinates q^A .

On the other hand, the canonical Hamiltonian H_0 associated with (2.5) is

$$H_0(q, p) = \frac{1}{2} p_A M^{AB}(q) p_B + V(q), \quad (2.9)$$

where the metric M^{AB} is a symmetric nonsingular matrix satisfying

$$M^{AB} G_{AC} G_{BD} = G_{CD}. \quad (2.10)$$

Let us notice that the metric M^{AB} defined through (2.10) displays a certain degree of arbitrariness. This corresponds to the fact that the canonical Hamiltonian is only unambiguously defined on the primary constraint surface.²

Taking into account (2.10) and the fact that the vector fields $\hat{U}_\alpha = U_\alpha^A \partial / \partial q^A$ are Killing vectors for the metric G , (2.6), we immediately obtain

$$(\mathcal{L}_\alpha M)^{AB} G_{AC} G_{BD} = 0.$$

This result leads to the following form for $(\mathcal{L}_\alpha M)$:

$$(\mathcal{L}_\alpha M)^{AB} = U_\beta^A M_\alpha^{\beta B}(q) + U_\beta^B M_\alpha^{\beta A}(q). \quad (2.11)$$

Notice that the choice of $M_\alpha^{\beta A}(q)$ is again ambiguous. In fact there is a family of such possible objects, related to each other by

$$M_\alpha^{\beta A} \rightarrow M_\alpha^{\prime\beta A} = M_\alpha^{\beta A} + G_\alpha^{\beta\gamma} U_\gamma^A, \quad (2.12)$$

$G_\alpha^{\beta\gamma}$ being an arbitrary array of coefficients antisymmetric in its upper indices. In the next subsection we will take advantage of this arbitrariness.

From the above results the form of the structure functions V_β^α is easily worked out. Indeed, taking into account its definition, the form of the constraints T_α (2.7) and the Hamiltonian H_0 (2.9), we have

$$\{T_\alpha, H_0\} = -\frac{1}{2} p_A (\mathcal{L}_\alpha M)^{AB} p_B = -V_\alpha^\beta T_\beta,$$

where use of the relation (2.1) allows us to factorize the constraints and write the structure functions V_α^β as

$$V_\alpha^\beta(q, p) = M_\alpha^{\beta A}(q) p_A. \quad (2.13)$$

Finally, let us write down the consequences of the Jacobi identities for the constraints and the canonical Hamiltonian in our particular model. From relation (2.3) we obtain

$$C_{\sigma\gamma}^\mu C_{\alpha\beta}^\sigma - C_{\alpha\beta, A}^\mu U_\gamma^A + \text{cyclic perm. of } (\alpha, \beta, \gamma) = 0,$$

the structure functions $B_{\alpha\beta\gamma}^{\mu\sigma}$ vanishing in this case, due to the fact that $C_{\alpha\beta}^\gamma$ and U_α^A depend only on q^A . On the other hand (2.4) turns out to be

$$\begin{aligned} & [-C_{\gamma\beta}^\sigma V_\alpha^\gamma + C_{\gamma\alpha}^\sigma V_\beta^\gamma + C_{\alpha\beta}^\gamma V_\gamma^\sigma \\ & + \{V_\alpha^\sigma, T_\beta\} - \{V_\beta^\sigma, T_\alpha\} - C_{\alpha\beta, A}^\sigma M^{AB} p_B] = D_{\alpha\beta}^{\sigma\mu} T_\mu, \end{aligned} \quad (2.14)$$

where now the structure functions $D_{\alpha\beta}^{\sigma\mu}$ can be chosen to depend only on q^A , as it is seen if the linear dependence of the constraints T_α and the structure functions V_β^α on p_A is taken into account.

This analysis could be carried on to determine the higher order structure functions. Nevertheless, since these quantities will not appear in the situation we will consider, we do not pursue this direction any further. Rather, in what follows, we are going to undertake the study of the Lagrangian gauge structure using as background the above results.

C. Covariantization and Lagrangian gauge structure

Having studied the Hamiltonian gauge algebra we are ready for ‘‘covariantization.’’ Using the Lagrangian multipliers as new variables, the extended Hamiltonian reads, in our particular case,

²As is proven in [5] and [7], this arbitrariness has no effect when a reduced (classical elimination of the gauge degrees of freedom) quantization is performed.

$$\begin{aligned} H(q, p; \lambda) &= H_0(q, p) + \lambda^\alpha T_\alpha(q, p) \\ &= \frac{1}{2} p_A M^{AB}(q) p_B + V(q) + \lambda^\alpha U_\alpha^A p_A . \end{aligned}$$

To obtain the associated Lagrangian we should eliminate the momenta p_A in terms of the velocities \dot{q}^A . Use of the equations of motion yields

$$\dot{q}^A = \frac{\partial H}{\partial p_A} = M^{AB} p_B + U_\alpha^A \lambda^\alpha ,$$

and since M^{AB} is invertible, we have

$$p_A(q, \dot{q}; \lambda) = M_{AB}(\dot{q}^B - U_\beta^B \lambda^\beta) ,$$

with $M_{AB} M^{BC} = \delta_A^C$.

The corresponding ‘‘covariant’’ Lagrangian L_C is

$$L_C(q, \dot{q}; \lambda) = \frac{1}{2} (\dot{q}^A - U_\alpha^A \lambda^\alpha) M_{AB}(q) (\dot{q}^B - U_\beta^B \lambda^\beta) - V(q) . \quad (2.15)$$

In the Appendix we show that in the case of Yang-Mills theories (2.15) is the standard covariant Lagrangian for these systems.

This Lagrangian $L_C(q, \dot{q}; \lambda)$ will be the starting point of our analysis. First we can check that the pullback of the transformations (2.2), given by

$$\begin{aligned} \delta q^A &= U_\alpha^A(q) \varepsilon^\alpha , \\ \delta \lambda^\alpha &= \dot{\varepsilon}^\alpha - M_\beta^{\alpha A} p_A(q, \dot{q}; \lambda) \varepsilon^\beta - C_{\beta\gamma}^\alpha(q) \lambda^\gamma \varepsilon^\beta , \end{aligned} \quad (2.16)$$

are indeed, as was expected, gauge transformations for L_C :

$$\delta L_C = 0 .$$

As we have said, the structure of the algebra of the Lagrangian gauge transformations plays a crucial role in solving the master equation in the field-antifield approach, which is the subject of the next section. In our case, for $\delta_1 := \delta[\varepsilon_1]$, $\delta_2 := \delta[\varepsilon_2]$, we obtain

$$[\delta_1, \delta_2] q^A = \delta [C_{\alpha\beta}^\gamma \varepsilon_2^\beta \varepsilon_1^\alpha] q^A ,$$

and, after a lengthy calculation,

$$\begin{aligned} [\delta_1, \delta_2] \lambda^\alpha &= \delta [C_{\sigma\beta}^\gamma \varepsilon_2^\beta \varepsilon_1^\sigma] \lambda^\alpha \\ &+ \{ [M_\mu^{\alpha A} M_{AB} M_\beta^{\sigma B} - (\mu \leftrightarrow \beta)] \\ &+ D_{\beta\mu}^{\alpha\sigma} \} \frac{\partial L_C}{\partial \lambda^\sigma} \varepsilon_1^\mu \varepsilon_2^\beta , \end{aligned} \quad (2.17)$$

where $D_{\beta\mu}^{\alpha\sigma}$ are the (pullback of the) Hamiltonian structure functions defined through relation (2.14) and $\partial L_C / \partial \lambda^\alpha$ the equations of motion for the fields λ^α derived from the covariant Lagrangian L_C (2.15), given by

$$\frac{\partial L_C}{\partial \lambda^\alpha} = -U_\alpha^A M_{AB} (\dot{q}^B - U_\beta^B \lambda^\beta) . \quad (2.18)$$

In the study of this gauge algebra we meet for the first time the new features introduced in the theory by the process of covariantization. Indeed, observe that the structure of (2.17) is, in general, that of an *open algebra*. This fact makes the computation of the proper solution

of the master equation rather cumbersome and we will try to circumvent this problem. To this end we will use a result from Ref. [8], to wit: *any open algebra of gauge transformations may be closed by the addition of the appropriate antisymmetric combinations³ of the equations of motion*. In our specific case, since the openness of the algebra only concerns the λ sector and, moreover, its open algebra part [see (2.17)] only exhibits the equations of motion for the λ 's, $[L_C]_{\lambda^\alpha} = \partial L_C / \partial \lambda^\alpha$, (2.18), it seems to be very *plausible* that we can get the closed algebra structure just by leaving δq^A unchanged and modifying $\delta \lambda^\alpha$ (2.16) as follows:

$$\delta \lambda^\alpha \rightarrow \delta' \lambda^\alpha = \delta \lambda^\alpha + F_\gamma^{\alpha\beta} \frac{\partial L_C}{\partial \lambda^\beta} \varepsilon^\gamma ,$$

with an appropriate $F_\gamma^{\alpha\beta}$ antisymmetric in its upper indices. Using (2.16) and the explicit expression for the equations of motion of λ , (2.18), this can also be written as

$$\delta' \lambda^\alpha = \dot{\varepsilon}^\alpha - M_\beta^{\alpha A} p_A(q, \dot{q}; \lambda) \varepsilon^\beta - C_{\beta\gamma}^\alpha(q) \lambda^\gamma \varepsilon^\beta ,$$

with

$$M_\beta^{\alpha A} = M_\beta^{\alpha A} + F_\beta^{\alpha\gamma} U_\gamma^A .$$

This last expression simply displays the freedom in the choice of $M_\beta^{\alpha A}$ we discovered in (2.12). We can therefore conclude that it is plausible that the freedom described by (2.12) allows for a choice of $\delta' \lambda^\alpha$ which satisfies the closed algebra structure. Strictly speaking we have not proven this, although it is very plausible, as we have argued.

Actually we may have considered from the beginning a more restrictive case: the assumption [9], for instance, that the regular metric tensor M^{AB} be such that the action of the gauge group leads to isometries, i.e., $(\mathcal{L}_\alpha M)^{AB} = 0$, $\alpha = 1, \dots, k$. In fact, this Killing condition implies that the vector fields \hat{U}_α form a Lie algebra ($C_{\alpha\beta}^\gamma = \text{const}$). Indeed, Eqs. (2.8) with $C_{\alpha\beta}^\gamma = \text{const}$ are the integrability conditions for $(\mathcal{L}_\alpha M)^{AB} = 0$, $\alpha = 1, \dots, k$. In that case, the treatment of the system greatly simplifies: from (2.11) we see that the quantities $M_\alpha^{\beta B}$ and, as a consequence, the structure functions V_α^β of (2.13) can be chosen to be zero. Then $D_{\alpha\beta}^{\sigma\mu}$ defined in (2.14) can be put to zero as well. All these results together lead to the closure of the gauge algebra of (2.17) in this particular case. Note that the important cases of EM and YM, for which the structure functions $C_{\alpha\beta}^\gamma$ are constants, fall into this last category and have a closed gauge algebra.

From now on, whatever the case we are dealing with, we will assume that we have met the conditions to get the gauge algebra in the closed form

$$[\delta_1, \delta_2](q^A, \lambda^\alpha) = \delta [C_{\alpha\beta}^\gamma \varepsilon_2^\beta \varepsilon_1^\alpha] (q^A, \lambda^\alpha) .$$

This assumption greatly simplifies the determination of the solutions of the classical and quantum master equa-

³It should be noted that in a general case with both bosons and fermions, these combinations will have a graded antisymmetry.

tions, which we are going to undertake in the next section.

III. BV QUANTIZATION OF THE COVARIANT ACTION

In the case of an irreducible closed algebra of gauge transformations, the field-antifield formalism starts by enlarging the original configuration space with the introduction of a ghost c^α for each gauge parameter ε^α , of opposite parity. These ghosts, together with the classical fields $\{\phi^a\} = \{q^A, \lambda^\alpha\}$, form the minimal sector of fields $\{\phi^i\}$. A new set of ‘‘antifields,’’ $\{q^*_A, \lambda^*_\alpha, c^*_\alpha\} = \{\phi^*_i\}$, with parities opposite to those of its associated fields, is introduced as well. Then, in the space of functionals of the fields and their antifields, some new structures, the antibrackets

$$(A, B) = \frac{\partial_r A}{\partial \phi^i} \frac{\partial_l B}{\partial \phi^*_i} - \frac{\partial_r A}{\partial \phi^*_i} \frac{\partial_l B}{\partial \phi^i},$$

and the BRST ‘‘Laplacian’’

$$\Delta = \frac{\partial_r}{\partial \phi^i} \frac{\partial_l}{\partial \phi^*_i},$$

are defined (sum over continuous indices is understood in both structures). The quantum master equation is then formulated as an equation for a functional \mathcal{W} , the full quantum action:

$$(\mathcal{W}, \mathcal{W}) - 2i\hbar\Delta\mathcal{W} = 0. \quad (3.1)$$

The usual way to solve the above equation consists in expanding the quantum action \mathcal{W} in powers of \hbar :

$$\mathcal{W} = S + \sum_{m=1}^{\infty} \hbar^m W_m,$$

so that (3.1) splits into the classical master equation

$$(S, S) = 0, \quad (3.2)$$

and the equations for the ‘‘quantum corrections’’

$$(W_1, S) = i\Delta S, \quad (3.3)$$

$$(W_p, S) = i\Delta W_{p-1} - \frac{1}{2} \sum_{q=1}^{p-1} (W_q, W_{p-q}), \quad p \geq 2.$$

For an irreducible, closed gauge algebra the classical master equation (3.2) has the well known (minimal) proper solution

$$S = S_0(\phi) + \phi^*_A R^A c^\alpha + \frac{1}{2} c^*_\alpha T^{\alpha}_{\beta\gamma} c^\beta c^\gamma,$$

where S_0 , R^A , and $T^{\alpha}_{\beta\gamma}$ are the classical action, the generators of the gauge transformations and the structure functions of the gauge algebra, respectively. In our case, taking into account the results obtained in the preceding section and assuming to have met the conditions to get the algebra in closed form, we obtain the following expression for the proper solution:

$$S = S_C(q, \lambda) + q^*_A U^A c^\alpha + \lambda^*_\alpha (\dot{c}^\alpha - V^{\alpha}_{\beta\gamma} c^\beta - C^{\alpha}_{\beta\gamma} \lambda^\gamma c^\beta) - \frac{1}{2} c^*_\alpha C^{\alpha}_{\beta\gamma} c^\beta c^\gamma,$$

where now S_C is the classical action associated with the covariant Lagrangian (2.15), $U^A_\alpha(q)$ the gauge generators for the fields q^A and $V^{\alpha}_{\beta\gamma}, C^{\alpha}_{\beta\gamma}$ the pullback of the corresponding Hamiltonian structure functions (2.8) and (2.13).

Let us consider now the equations for the quantum corrections (3.3). Since for the type of theories we are analyzing the proper solution is linear in the antifields, the quantity ΔS does not depend on them. As a consequence, the term W_1 can be chosen to be a function of the classical fields only and Eqs. (3.3) are immediately solved by taking $W_p = 0$ for $p \geq 2$. Therefore, no higher order terms in \hbar appear apart from the W_1 term.

To compute the first quantum correction W_1 we need the explicit expression of ΔS . In our case it is easily seen that

$$\Delta S = (U^A_{\alpha, A} + M^{\beta A}_{\alpha} M_{AB} U^B_{\beta}) c^\alpha.$$

Use of the Lie derivative of M^{AB} in the \hat{U}_α direction (2.11) together with the symmetry property of this metric allows us to write the last term of the above expression as

$$\begin{aligned} (M^{\beta A}_{\alpha} M_{AB} U^B_{\beta}) c^\alpha &= [\frac{1}{2} M_{AB} (\mathcal{L}_\alpha M)^{AB}] c^\alpha \\ &= [-\frac{1}{2} M^{AB} (\mathcal{L}_\alpha M)_{AB}] c^\alpha \\ &= -(\frac{1}{2} M^{AB} M_{AB, C} U^C_{\alpha} + U^A_{\alpha, A}) c^\alpha, \end{aligned}$$

where in the last equation use has been made of the definition of $(\mathcal{L}_\alpha M)_{AB}$. Therefore, after these manipulations, ΔS can be written in the more useful form

$$\begin{aligned} \Delta S &= [-\frac{1}{2} \hat{U}_\alpha (\text{tr} \ln M_{AB})] c^\alpha \\ &= [-\frac{1}{2} \hat{U}_\alpha (\ln \det M_{AB})] c^\alpha \\ &= -[\ln(\det M_{AB})^{1/2}, S]. \end{aligned} \quad (3.4)$$

Notice that the trace over continuous indices imply, in our case of a local gauge theory, that ΔS is proportional to $\delta(0)$. Therefore, in order to make sense out of this construction, some scheme to regularize the above expression must be considered.

Expression (3.4) is already spelling out the formal solution for W_1 . Indeed, we can simply take

$$W_1 = -i \ln(\det M_{AB})^{1/2} + \text{BRST-invariant terms}, \quad (3.5)$$

where ‘‘BRST-invariant terms’’ we mean terms with vanishing antibrackets with S . As we have said, the above ambiguity in W_1 is inherent to the field-antifield formalism. In the present case we solve this ambiguity just by dropping the second term in the left-hand side (LHS) of (3.5). As will be shown below, this is the correct choice that makes contact with the reduced path integral formalism.

Now, to proceed to fix the gauge within the field-antifield approach, some auxiliary fields, $\{\bar{c}^\alpha, B^\alpha\}$, and its corresponding antifields, $\{\bar{c}^*_\alpha, B^*_\alpha\}$, are introduced. After that, the minimal proper solution S is modified by the addition of an extra term in these new fields as

$$S_{\text{NM}} = S + \bar{c}^*_\alpha B^\alpha.$$

Then, if we call Φ^i the whole set of fields, a ‘‘gauge fermion’’ Ψ will eliminate the antifields through the requirement

$$\Phi_i^* = \frac{\partial \Psi}{\partial \Phi^i} .$$

The Batalin-Vilkovisky path integral is then defined as

$$Z_\Psi = \int [Dq][D\lambda][D\bar{c}][Dc][DB] \exp \left\{ \frac{i}{\hbar} W_\Sigma \right\} , \quad (3.6)$$

where W_Σ stands for $W(\Phi, \Phi^* = \partial\Psi/\partial\Phi)$. It should be noted that, in our case, as W_1 does not have any dependence on the antifields Φ^* , its expression will not depend on the choice of the gauge fixing. Therefore, we will have

$$W_\Sigma = S \left[\Phi, \Phi^* = \frac{\partial \Psi}{\partial \Phi} \right] + \hbar W_1(\Phi) .$$

Now, a customary lattice regularization for $\delta(0)$, $\delta(0) \rightarrow 1/\varepsilon$, $\varepsilon \rightarrow 0$, allows us to write part of the exponential in (3.6) as

$$\exp\{iW_1\} = \prod_t (\det M_{AB})^{1/2} ,$$

so that Z_Ψ becomes

$$Z_\Psi = \int [Dq][D\lambda][D\bar{c}][Dc][DB] (\det M_{AB})^{1/2} \exp \left\{ \frac{i}{\hbar} S_\Sigma \right\} . \quad (3.7)$$

Therefore, from the above expression for Z_Ψ it is evident that while the gauge fixed proper solution of the classical master equation (3.2) represents the classical effective action of the theory, the W_1 term stands for quantum corrections to the naive measure. In this way, the determinant $(\det M_{AB})^{1/2}$ modifies the naive measure $[Dq] \cdots [DB]$ yielding a BRST-invariant measure.

At this point, different choices of the gauge-fixing fermion are possible. The physical equivalence of the different gauges, i.e., the invariance of the path integral (3.6) under deformations of the gauge-fixing fermion, is a well known result in the context of the field-antifield formalism, and has been proven by Batalin and Vilkovisky in [2]. In fact, what they do in this reference is to prove the theorem for gauges that differ infinitesimally. More suited to our purposes, a direct proof of this invariance under arbitrary deformations of Ψ for the case of theories with closed, irreducible gauge algebras, can be found in Ref. [10] and will not be repeated here. From now on, we will take for granted this invariance of the path integral (3.6) under changes of the gauge-fixing fermion.⁴

One of the main purposes of this paper is to make contact with the noncovariant path integral quantization presented in [1]. To this end, we will use a gauge-fixing fermion implementing unitary or, more generally, nonco-

variant gauge-fixing conditions. However, other types of gauge-fixing conditions, for instance ‘‘covariant’’ gauges, can be chosen as well. In what follows, we are going to work out the form of the Batalin-Vilkovisky path integral in both classes of gauges.

Unitary or noncovariant gauge conditions are necessary in order to make contact with the reduced path integral quantization. In this formulation, unitarity is obvious once the usual assumptions about the positivity of the spectrum of the reduced theory are made. Gauge-fixing fermions which implement these gauges are taken to be of the form $\Psi_1 = \bar{c}_\alpha F^\alpha(q)$, where the gauge-fixing conditions $F^\alpha(q)$ do not involve the Lagrange multipliers λ^α . For such gauge-fixing fermions S_{Σ_1} becomes

$$S_{\Sigma_1} = S_C + \bar{c}_\alpha \frac{\partial F^\alpha}{\partial \varepsilon^\beta} c^\beta + B_\alpha F^\alpha ,$$

where $\partial F^\alpha / \partial \varepsilon^\beta = \hat{U}_\beta(F^\alpha)$ measures the rate of change of F^α under the action of the gauge generators \hat{U}_β . After this, straightforward integration of c, \bar{c} and B in (3.7) yields

$$Z_{\Psi_1} = \int [Dq][D\lambda] (\det M_{AB})^{1/2} \det \left[\frac{\partial F^\alpha}{\partial \varepsilon^\beta} \right] \times \delta(F^\alpha) \exp \left\{ \frac{i}{\hbar} S_C \right\} . \quad (3.8)$$

The noncovariant formulation is recovered by integrating out the Lagrange multipliers λ , which appear quadratically in S_C . Once this is done we get

$$Z_{\Psi_1} = \int [Dq] \frac{(\det M_{AB})^{1/2}}{(\det \theta_{\alpha\beta})^{1/2}} \det \left[\frac{\partial F^\alpha}{\partial \varepsilon^\beta} \right] \times \delta(F^\alpha) \exp \left\{ \frac{i}{\hbar} S'_0 \right\} , \quad (3.9)$$

where $\theta_{\alpha\beta}$ is defined by

$$\theta_{\alpha\beta} = U_\alpha^A M_{AB} U_\beta^B ,$$

and the new classical action S'_0 is

$$S'_0 = \int dt \left\{ \frac{1}{2} \dot{q}^A M_{AB} \dot{q}^B - \frac{1}{2} (\dot{q}^A M_{AC} U_\alpha^C) (\theta^{-1})^{\alpha\beta} (\dot{q}^B M_{BD} U_\beta^D) + V(q) \right\} .$$

It can be shown that the metric defining the kinetic term of S'_0 is nothing but the one which appears in the original quadratic noncovariant Lagrangian (2.5), i.e.,

$$G_{AB} = M_{AB} - M_{AC} U_\alpha^C (\theta^{-1})^{\alpha\beta} M_{BD} U_\beta^D .$$

Therefore, we conclude that S'_0 is equal to S_{nc} , the action for the noncovariant Lagrangian (2.5), and that expression (3.9) for Z_{Ψ_1} in noncovariant gauges is in complete agreement with the one obtained in [1]. This result proves that the equivalence between the noncovariant (with variables q only) and the covariant (with variables q and λ) formulations, which is easily seen at the classical level, still holds after quantization of the theory within the framework of the field-antifield formalism when a

⁴This will be true as long as the theory is free from gauge anomalies. We assume that this is the case in this paper. See, however, Ref. [11].

noncovariant gauge fixing is imposed.

At this point it is worth comparing the two versions, (3.8) and (3.9), of the path integral Z_{Ψ_1} . In fact, they are two different, although equivalent, Faddeev-Popov (FP) formulas. On the one hand, expression (3.8) for the covariant theory corresponds to the standard FP formula as used in the literature (with two extensions: the presence of a nontrivial determinant in the measure and also the fact that we are dealing with the so-called quasigroup structure [10] rather than a Lie group). On the other hand, the equivalent expression (3.9) uses a noncovariant action and corresponds to the correct FP formula for systems with first-class primary constraints only (strictly speaking, systems with quadratic kinetic term and constraints linear in the momenta), as it was proven in [7]. In this second case, it should be noted the presence of a new determinant, $(\det\theta_{\alpha\beta})^{1/2}$, which makes the path integral invariant under rescaling of the constraints. In summary, the above discussion points out that *the structure of the constraints of the theory [primary constraints in (3.9); primary and secondary in (3.8)] makes a difference with regard to the final form of the FP formula.*

In connection with the measure of our path integral (3.7) another comment is in order. As we have said in the preceding section, there is a certain amount of arbitrariness in the selection of the metric M_{AB} satisfying (2.10). One may then wonder how this arbitrariness affects the path integral (3.7). In fact, using expression (3.9) of Z_{Ψ} in a noncovariant gauge, one can see that it does not affect it at all. This expression was obtained in [7] starting from the reduced path integral quantization, in which this kind of ambiguity was not present.⁵ Therefore, in spite of this apparent dependence of (3.7) on the particular choice of M_{AB} , the measure and the action depend on it in such a way that (3.7), in the end, does not suffer from this arbitrariness. In the measure, this feature is neatly displayed as a cancellation of the dependence on the gauge part of $(\det M_{AB})^{1/2}$, namely, $(\det M_{\alpha\beta})^{1/2}$, and the similar dependence in $(\det\theta_{\alpha\beta})^{1/2}$ [in [7] it was shown that $(\det M_{AB})^{1/2}$ factorizes into a physical piece, dependent only on gauge invariant degrees of freedom, times $(\det M_{\alpha\beta})^{1/2}$. This was achieved in the so-called adapted coordinates; the gauge-dependent piece of $(\det\theta_{\alpha\beta})^{1/2}$ is also explicitly displayed this way].

To conclude, for the sake of completeness, let us study the form of Z_{Ψ} (3.7) in covariant gauges. As is well known, covariant gauge fixings are more convenient in obtaining Feynman rules which describe the perturbative sector of the quantum theory. This class of gauges is constructed so that all the fields become propagating. Gauge-fixing fermions enforcing covariant gauges are usually of the form

$$\Psi_2 = \bar{c}^\alpha [\dot{\lambda}^\alpha + F(q) + \frac{1}{2}\omega^{\alpha\beta}B_\beta],$$

where the maximum rank metric $\omega^{\alpha\beta}$ is usually taken to

be independent of the fields. The gauge fixed action S_{Σ_2} reads in this case

$$S_{\Sigma_2} = S_C + \bar{c}_\alpha \frac{\partial F^\alpha}{\partial \epsilon^\beta} c^\beta - \dot{c}_\alpha (\dot{c}^\alpha - V_\beta^\alpha c^\beta - C_{\beta\gamma}^\alpha \lambda^\gamma c^\beta) \\ + B_\alpha [\dot{\lambda}^\alpha + F(q) + \frac{1}{2}\omega^{\alpha\beta}B_\beta].$$

The auxiliary fields B_α can be integrated out of the path integral or, equivalently, eliminated algebraically in terms of their equations of motion

$$\frac{\partial L_{\Sigma_2}}{\partial B_\alpha} = \dot{\lambda}^\alpha + F(q) + \omega^{\alpha\beta}B_\beta \\ = 0 \implies B_\alpha = -\omega_{\alpha\beta}[\dot{\lambda}^\beta + F^\beta(q)],$$

where $\omega_{\alpha\beta}$ is the inverse of the metric $\omega^{\alpha\beta}$, yielding in this way a gauge fixed action of the form

$$S_{\Sigma_2} = S_C + \bar{c}_\alpha \frac{\partial F^\alpha}{\partial \epsilon^\beta} c^\beta - \dot{c}_\alpha (\dot{c}^\alpha - V_\beta^\alpha c^\beta - C_{\beta\gamma}^\alpha \lambda^\gamma c^\beta) \\ + \frac{1}{2}[\dot{\lambda}^\alpha + F^\alpha(q)]\omega_{\alpha\beta}[\dot{\lambda}^\beta + F^\beta(q)],$$

in which the kinetic terms of all the fields are invertible, so that they become propagating. This is an important feature which distinguishes the covariant formulation (i.e., with variables λ) from the noncovariant one.

Finally, the partition function in covariant gauges is written as

$$Z_{\Psi_2} = \int [Dq][D\lambda][D\bar{c}][Dc](\det M_{AB})^{1/2} \exp \left\{ \frac{i}{\hbar} S_{\Sigma_2} \right\},$$

this expression being the starting point in the construction of covariant Feynman rules.

IV. CONCLUSIONS

In this paper we have extended to Yang-Mills-type systems some previous work on the quantization of constrained systems which exhibited, in the canonical formalism, only primary first-class constraints linear in the momenta. This extension can be understood as the covariantization of the original system by introducing new degrees of freedom to it. At this point it is worth noticing that the marriage of covariance (for a constrained system such as YM, for instance) with the Hamiltonian formalism immediately implies the appearance of secondary constraints. Because of this fact, there are some differences in the application of the Batalin-Vilkovisky formalism to both the noncovariant and the covariant case which deserve some specific comments. The main difference is perhaps that the algebra of gauge transformations will be generally open in the covariant case, even though it was closed in the noncovariant one. In this paper we have dealt with this eventuality by arguing that it is possible to set up the covariant formalism in such a way that the algebra is still closed, and this is in fact the only case we have studied and where the equivalence with the noncovariant formulation has been shown.

Another difference, which can be traced to the different structure of the constraints in both cases, is the following:

⁵See footnote 2 in relation with the effects of this arbitrariness at the quantum level.

in the noncovariant case (which in our terminology corresponds to a system without Lagrangian constraints or, in other words [5], with only primary first-class Hamiltonian constraints), there appears [7] in the Faddeev-Popov formula a new determinant, unrelated to the gauge-fixing procedure, that keeps the path integral invariant under rescaling of the constraints (which we emphasize are linear in the momenta). Instead, in the covariant case (which is achieved by promoting the old Lagrangian multipliers to the status of dynamical variables, thus creating two generations, primary and secondary, of constraints), the new determinant is absorbed in the definition of the covariant Lagrangian, and the usual Faddeev-Popov formula is obtained. Our result, however, is an extension of the Faddeev-Popov formula because now the generators of the gauge group do not span a Lie algebra. The structure defined by these generators, whose commutation relations give rise to structure functions, unlike the structure constants that appear in a Lie algebra, has been called a quasigroup [10].

In conclusion, our results establish the equivalence, at the quantum level, of the noncovariant and the covariant version of a constrained dynamical system of Yang-Mills type. This equivalence is a fundamental issue because, in terms of path integrals, unitarity is best checked in the reduced quantization (classical elimination of the gauge degrees of freedom). This reduced quantization corresponds, as it is proven in [7], to the quantization of the noncovariant version of the system.

During the preparation stages of this manuscript we received a paper by Epp *et al.* [12], which deals with some of the topics raised here, as well as with other work by some of us. We completely agree with their results. The crucial point first raised in [7], and clarified to some extent in [12] using scalar QED as an example, is the need to distinguish between different forms of the Faddeev-Popov ansatz when both primary and secondary constraints are present classically (i.e., before the Lagrange multipliers are integrated out) and when only primary constraints are present (i.e., after the Lagrange multipliers are integrated out). Thus for example the usual, covariant form of the Faddeev-Popov ansatz [and consequently formula (3.31) of [13]] is correct only when

secondary constraints are present. Reference [7] dealt specifically with primary constraints only, while the present work extends the results to the case in which secondary constraints are also present.

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APPENDIX

Here we derive the standard covariant Yang-Mills Lagrangian from its noncovariant version as an example of the procedure of “covariantizing” (2.5) to get (2.15). Actually we can directly start from the Hamiltonian H_0 of (2.9) which for Yang-Mills takes the form

$$H_0 = \frac{1}{2} \pi_a^k \pi_a^k + \frac{1}{4} F_a^{ij} F_a^{ij}.$$

We hence identify, in the notation of Sec. II,

$$M_{AB} = \delta(\mathbf{x} - \mathbf{y}) \delta_{ab}, \quad V(A_i) = \frac{1}{4} F_a^{ij} F_a^{ij},$$

and, from Eqs. (2.16)

$$\begin{aligned} \delta A_i^a &= \partial_i \varepsilon^a - f^{acb} A_i^c \varepsilon^b = \mathcal{D}_i^{ab} \varepsilon^b \equiv U_\alpha^A \varepsilon^\alpha, \\ \delta A_0^a &= \partial_0 \varepsilon^a + f^{abc} A_0^b \varepsilon^c = \mathcal{D}_0^{ab} \varepsilon^b, \end{aligned}$$

where we have used the notation $\lambda^a = A_0^a$ for the Lagrange multipliers, and taken into account that now $V_\alpha^B(p, q)$, (2.13), can be chosen to be zero.

Then, we have, for (2.15),

$$\begin{aligned} L(A_i, \dot{A}_i, A_0) &= \int d^3x \left[\frac{1}{2} (\dot{A}_i^a - \mathcal{D}_i^{ab} A_0^b) (\dot{A}_i^a - \mathcal{D}_i^{ab} A_0^b) \right. \\ &\quad \left. - \frac{1}{4} F_a^{ij} F_a^{ij} \right], \end{aligned}$$

and since $(\dot{A}_i^a - \mathcal{D}_i^{ab} A_0^b) = F_{0i}^a$, we finally get

$$\begin{aligned} \int L dx^0 &= - \int d^4x \left[\frac{1}{4} F_{0i}^a F^{0ia} + \frac{1}{4} F_a^{ij} F_a^{ij} \right] \\ &= \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right], \end{aligned}$$

that is, the covariant action for Yang-Mills theories.

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