Bell's spaceship paradox

Author: Jesús Miguel Ceresuela Torres and Advisor: Josep Llosa Carrasco
*Facultat de Física, Universitat de Barcelona, Diagonal 645, 08028 Barcelona, Spain.**

Abstract: The aim of this project is to find the "proper length" of the thread in both the tough and the mild variants of Bell's spaceship paradox. To do so, a method for measuring the proper length of an accelerated object needs to be proposed. It is developed and applied to both variants, and then, the results are compared.

I. INTRODUCTION

Although today it is known as "Bell's spaceship paradox", Dewan and Beran [1] were the first to propose this "Gedankenexperiment": Let R and F (rear and front) be two spaceships at rest in an inertial frame S with origin in R. The distance between the ships is h. The ships are tied by a thread from the tie of F to the head of R. At equal times in S both ships start accelerating with an identical acceleration profile $a(t)$ in the direction of the line that joins them. This constitutes a paradox due to the fact that

1. At any time both ships will have the same velocity measured from S, so their relative distance is always h in this frame. There is no reason for the thread to break.

2. According to S the resting length of the thread from R to F is h but, when the thread is travelling at a certain speed, it is Lorentz contracted and its length measured from S is shorter than h, and can't cover the distance between ships. The thread is thus stretched and it should break.

Bell uses this paradox in [2] to show how this kind of problems help the students to consolidate key concepts. He asserts that the thread would break, and exposes how this paradox caused a great deal of trouble in the CERN canteen.

Usually, a paradox takes place when there is a concept which is not properly understood or it is ambiguously defined. This is the case of the proper length. It corresponds to the length of an object measured in the object's comoving frame. In case the object is moving at a constant speed, we can get the proper length through Lorenz transformations, but in case the object has some acceleration profile then there is no such thing as a comoving inertial frame against what Franklin states in [3]. To measure the proper length of the accelerated thread, we will apply a similar idea to that of the clock hypothesis [4]. In each piece of the thread there exists an instantaneous comoving inertial frame, and in this frame we can measure a differential of length of the thread the same as in the non accelerated case. Then, the proper length of the whole thread will be the sum of every one of these differentials of length. For the addition to make sense, all elementary lengths added must be simultaneous.

We will solve the paradox when the acceleration programmes are constant proper acceleration a. For us, whenever the thread is stretched (increases its proper length) it breaks. As transverse dimensions are not relevant, we will restrict the problem to a $1+1$ Minkowski spacetime, this is, one spatial component and one time component. Whenever we refer to a (x,t) diagram, those will be coordinates of S.

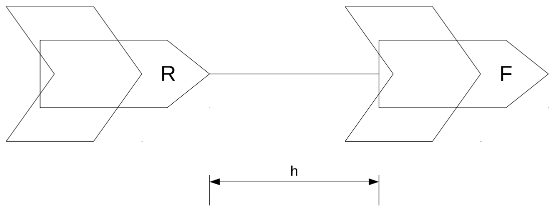

FIG. 1: A view from S of the ships at every instant of time.

For the sake of simplicity, we will assume that each point in the thread moves with constant proper acceleration a. This is not the most realistic consideration in the sense that the thread does not have a team of engines along it, and a more sensible assumption would have been to consider that the motion of the ships is transmitted to the thread elastically (this would be an interesting way to continue the study of the paradox). Yet, our assumption allows us to focus on the key concepts of the problem while keeping it reasonnably affordable.

We distinguish between the tough paradox, where the ships never stop accelerating, and the mild paradox, which considers that at a given time t^* (in S) both ships stop their engines and continue travelling through spacetime at a constant speed v^*.

II. PRELIMINARY CONCEPTS AND DEFINITIONS

A. Adapted coordinates

Let us consider a set of material points P, each one with worldline $X^\mu(P;\tau)$, τ being the proper time. Suppose these lines cover completely a region M in the $3+1$ Minkowski spacetime. Then, we can refer to each point

*Electronic address: jesusceresuela@gmail.com

of M as

$$X^\mu = X^\mu(\xi^1, \xi^2, \xi^3, \tau).$$

Assuming these worldlines are continuous and differentiable, and due to the fact that two different points P and Q cannot invade mutually, we get that the relation between X^μ y $(\xi^1, \xi^2, \xi^3, \tau)$ is a bijection between an open set from spacetime M and an open set in $\mathbb{R}^4$. Thus, we can take $(\xi^1, \xi^2, \xi^3, \tau)$ as a system of coordinates in M that is adapted to the motion of the material points.

It might be also useful to replace τ with another time coordinate $T = T(\xi^1, \xi^2, \xi^3, \tau)$, so that changes local clocks' rate ($\partial_\tau T \neq 1$) but always future-pointing ($\partial_\tau T > 0$). In these coordinates the invariant interval

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

with $\eta_{\mu\nu} = diag(+1+1+1-1)$, becomes

$$ds^2 = \eta_{\mu\nu} \frac{\partial x^\mu(\xi)}{\partial \xi^\alpha} \frac{\partial x^\nu(\xi)}{\partial \xi^\beta} d\xi^\alpha d\xi^\beta := g_{\alpha\beta}(\xi) d\xi^\alpha d\xi^\beta. \quad (1)$$

As usual, we use latin indices for spatial terms ($i = 1, 2, 3$), and greek ones for both spatial and temporal terms ($\mu = 1, 2, 3, 4$). Proper velocity in each point $P = (\xi^1, \xi^2, \xi^3, T = \xi^4)$ is now

$$\frac{\partial x^\mu(\tau, \xi^i)}{\partial \tau} = \frac{\partial x^\mu(T, \xi^i)}{\partial T} \frac{\partial T}{\partial \tau} = \frac{\partial x^\mu}{\partial \xi^4} \frac{\partial \xi^4}{\partial \tau}. \quad (2)$$

B. Simultaneity at a distance

Let us consider (as in [5]) two nearby space spots ξ^i y $\xi^i + d\xi^i$ and three different events A, B and C. In A, a beam of light is emitted from $\xi^i + d\xi^i$. In B the beam reaches ξ^i and reflects. In C the beam reaches again $\xi^i + d\xi^i$. The following table lists the coordinates of each event and displacements between them.

A	$(\xi^i + d\xi^i, \xi^4 + (d\xi^4)_1)$
B	(ξ^μ)
C	$(\xi^i + d\xi^i, \xi^4 + (d\xi^4)_2)$
$A \to B$	$B - A = -(d\xi^\mu)_1 = -(d\xi^i, (d\xi^4)_1)$
$B \to C$	$C - B = (d\xi^\mu)_2 = (d\xi^i, (d\xi^4)_2)$

TABLE I: Events' coordinates

For A is in B's past light cone we have $ds^2_{AB} = 0$ and $(d\xi^4)_1 < 0$. Developing the first one and taking into account (1), we get the following quadratic equation:

$$g_{ij}(\xi) d\xi^i d\xi^j + 2g_{i4}(\xi) d\xi^i (d\xi^4)_1 + g_{44}(d\xi^4)_1^2 = 0. \quad (3)$$

For C is in B's future light cone, we get the same equation before but replacing $(d\xi^4)_1$ with $(d\xi^4)_2$. This shows

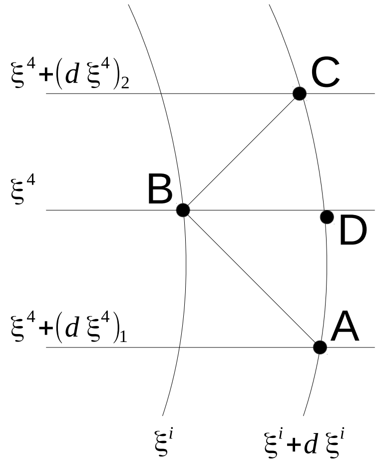

FIG. 2: Diagram with events and coordinates from table I

$(d\xi^4)_1 < 0$ and $(d\xi^4)_2 > 0$ are the two solutions to equation 3, the positive and the negative.

$$(d\xi^4)_{1,2} = \frac{-g_{i4} d\xi^i \pm \sqrt{(g_{i4} d\xi^i)^2 - g_{44} g_{ij} d\xi^i d\xi^j}}{g_{44}}. \quad (4)$$

Let us introduce a new event D, which corresponds with the event that takes place in $\xi^i + d\xi^i$ simultaneously with B. We can parametrize it with $(\xi^i + d\xi^i, \xi^4 + d\xi^4)$. Telegraphist protocol states that

$$\begin{aligned} d\xi^4 = \frac{(d\xi^4)_1 + (d\xi^4)_2}{2} &= -\frac{g_{i4} d\xi^i}{g_{44}} \Rightarrow \\ g_{4\mu}(\xi) d\xi^\mu &= 0 \end{aligned} \quad (5)$$

must hold for D and B to be simultaneous. Thus, two neighbouring events ξ^μ and $\xi^\mu + d\xi^\mu$ are simultaneous if (5) holds. This can be seen as a definition of local simultaneity in a non-inertial system of coordinates.

C. Proper time and coordinate time

A standard clock stationary at a point with coordinates ξ^i indicates the proper time of its worldline. If we write this line in terms of coordinates $(\xi^1, \xi^2, \xi^3, \xi^4 = T)$ we will get the expressions $\xi^i = ct$ and $\xi^4 = T$. Thus, velocity vector with these parameters is $v^\mu = \frac{d\xi^\mu}{dT} = \delta^\mu_4$. Now, proper time is given by the invariant interval (clock's hypothesis):

$$-d\tau^2 = ds^2 = g_{\mu\nu}(v^\mu dT)(v^\nu dT) \Rightarrow d\tau^2 = -g_{44} dT^2$$

$$d\tau = \sqrt{-g_{44}} dT. \quad (6)$$

A coordinate system is said to be synchronous if two simultaneous events have the same time coordinate. Let's consider the nearby events ξ^i and $\xi^i + d\xi^i$. Due to equation 5, the coordinate system $(\xi_1, \xi_2, \xi_3, \xi_4 = T)$ is synchronous if

$$g_{4\mu}(\xi) d\xi^\mu = 0 \Leftrightarrow d\xi^4 = 0.$$

That is $g_{4i} = 0$ for $i = 1, 2, 3$.

D. Distance between neighbouring points

When the beam of light in section II B goes $A \to B \to C$ travels two times the distance dl between ξ^i and $\xi^i + d\xi^i$. The time gap between A an C is, including equation (4)

$$\Delta\xi^4 = (d\xi^4)_2 - (d\xi^4)_1 = \frac{2\sqrt{(g_{i4}g_{j4} - g_{44}g_{ij})\, d\xi^i d\xi^j}}{|g_{44}|}. \tag{7}$$

This depends on the time coordinate we choose, which is not, in general, a standard clock and so it is not a scalar (invariant under coordinate changes). It is convenient to consider the proper invariant gap using equation (6), which is a scalar.

$$\Delta\tau = \sqrt{-g_{44}}\Delta\xi^4. \tag{8}$$

The time gap for the radar signal round trip from A to C through B measured by a stationary clock at $\xi^j + d\xi^j$ is then

$$\Delta\tau = 2\sqrt{\left(g_{ij} - \frac{g_{4i}g_{4j}}{g_{44}}\right) d\xi^i d\xi^j}. \tag{9}$$

Recalling $c = 1$, radar infinitesimal distance between ξ^i and $\xi^i + d\xi^i$ is

$$dl^2 = \left(g_{ij} - \frac{g_{4i}g_{4j}}{g_{44}}\right) d\xi^i d\xi^j. \tag{10}$$

This dl is called *Born's length* [6] and stands for the length between ξ^j and $\xi^j + d\xi^j$ measured in the instantaneously inertial comoving frame with ξ^j. By integrating Born's length through a set of points which are simultaneous, we will get the proper length of an accelerated body.

III. TOUGH VARIANT

Our aim is now to find an expression for the proper length of the thread in the tough variant, and to see how it changes with time. We assume that the ships start moving with constant proper acceleration in the direction that joins them. The worldlines of the ships are given by

$$\begin{aligned} x_R &= \sqrt{t^2 + a^{-2}} - a^{-1}, \\ x_F &= \sqrt{t^2 + a^{-2}} - a^{-1} + h. \end{aligned} \tag{11}$$

These equations correspond to an hyperbolic movement, only different at the starting point. We are not going to work with Lorentz coordinates. Instead we will solve the problem in another set of coordinates (ξ, τ) given by

$$\begin{aligned} x &= a^{-1}(\cosh(a\tau) - 1) + \xi, \\ t &= a^{-1}\sinh(a\tau). \end{aligned} \tag{12}$$

These new coordinates will not only allow us to describe the movement of the ships in a much easier way but can also describe whatever point of space-time in the region where our problem takes place. In these coordinates the worldlines of the thread points are $\xi = constant$, and for the ships

$$\xi_R = 0 \qquad \xi_F = h. \tag{13}$$

The metric considering the new coordinates is the following

$$g_{\alpha\beta} = \begin{pmatrix} 1 & \sinh(a\tau) \\ \sinh(a\tau) & -1 \end{pmatrix}. \tag{14}$$

The condition (5) for two events to be simultaneous in the new coordinates is

$$d\xi = \frac{d\tau}{\sinh(a\tau)}. \tag{15}$$

By integrating (15), we will obtain a curve joining space-time points which are locally simultaneous with respect to the thread. We will call this a simultaneity curve. The integral is easy to solve and results

$$\xi = a^{-1}\log\left(\frac{e^{a\tau} - 1}{e^{a\tau} + 1}\right) + \xi_0, \tag{16}$$

where ξ_0 is an integration constant that will be determined later.

Now, substituting the elements of the metric in (10) we get Born's length:

$$dl = \cosh(a\tau)d\xi. \tag{17}$$

The distance we are looking for will be obtained by adding this differential pieces of the thread (dl) between two boundary points (τ_1 and τ_2) that will be determined later. Using (15) and (17) we get

$$\begin{aligned} \int_{\tau_1}^{\tau_2} dl &= \int_{\tau_1}^{\tau_2} \cosh(a\tau)d\xi = \\ \int_{\tau_1}^{\tau_2} \coth(a\tau)d\tau &= \left(a^{-1}\log|\sinh(a\tau)|\right)_{\tau_1}^{\tau_2}. \end{aligned} \tag{18}$$

To find our limits τ_1 and τ_2 for the integral (18) we will find the intersections between the worldlines of our spaceships and the simultaneity curve. Equation (16) has ξ_0 as an integration constant. We determine it by imposing that when $\tau = \tau_1$ the simultaneity curve intersects $\xi = 0$, and we obtain

$$\xi_0(\tau_1) = a^{-1}\log\left(\frac{e^{a\tau_1} + 1}{e^{a\tau_1} - 1}\right). \tag{19}$$

So the final expression for the simultaneity curve starting the measure in τ_1 (we will call this curve Σ_{τ_1}) is

$$\xi = \Sigma_{\tau_1}(\tau) = a^{-1}\log\left(\frac{e^{a\tau} - 1}{e^{a\tau} + 1} \cdot \frac{e^{a\tau_1} + 1}{e^{a\tau_1} - 1}\right). \tag{20}$$

This means that when we measure the proper length, the rear ship is in $(\xi, \tau) = (0, \tau_1)$. Taking different values for τ_1, we are moving the time when the measure is done. Now we find the expression for $\tau_2 = \tau_2(\tau_1)$ such that when $\tau = \tau_2$, equation (20) intersects $\xi = h$, this is, solving

$$h = a^{-1} \log \left(\frac{e^{a\tau_2} - 1}{e^{a\tau_2} + 1} \cdot \frac{e^{a\tau_1} + 1}{e^{a\tau_1} - 1} \right). \tag{21}$$

After some algebraic manipulation we get the expression

$$\tau_2 = a^{-1} \log \left(\frac{\alpha + e^{ah}}{\alpha - e^{ah}} \right) \qquad \alpha = \frac{e^{a\tau_1} + 1}{e^{a\tau_1} - 1}. \tag{22}$$

Now it is time to evaluate the integral (18) and obtain the proper length L^{tough} in terms of τ_1.

$$L^{tough}(\tau_1) = h + \log \left(\frac{4e^{a\tau_1}}{(e^{a\tau_1} + 1)^2 - e^{2ah}(e^{a\tau_1} - 1)^2} \right). \tag{23}$$

Now that the calculus has been done, we take a look

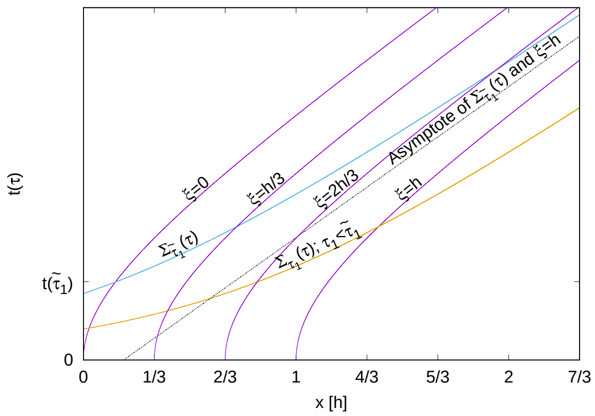

FIG. 3: Tough version's (x, t) diagram. Purple lines are worldlines of different pieces of the thread. Blue and yellow are simultaneity curves, and the black one is an asymptote.

on the range of values τ_1 can take. The simultaneity curve in (20) is given as a $\xi = \Sigma_{\tau_1}(\tau)$ function, and it is convenient to observe that

$$\lim_{\tau \to \infty} \Sigma_{\tau_1}(\tau) = \xi_0(\tau_1). \tag{24}$$

Equation (24) means that the simultaneity curve never intersects the worldline $\xi_0(\tau_1)$. Let us analyse how ξ_0 depends on τ_1 by equation (19). It is a continuous and decreasing function with

$$\begin{aligned} \lim_{\tau_1 \to 0} \xi_0(\tau_1) &= \infty, \\ \lim_{\tau_1 \to \infty} \xi_0(\tau_1) &= 0. \end{aligned} \tag{25}$$

This means that we will always find a value $\widetilde{\tau_1}$ that makes $\xi_0(\widetilde{\tau_1}) = h$. For $\tau_1 < \widetilde{\tau_1}$ we have $\xi_0(\tau_1) > h$ so the simultaneity curve intersects $\xi = h$ in τ_2 and L is a finite distance. If $\tau_1 \geq \widetilde{\tau_1}$ then $\xi_0(\tau_1) \leq h$ so the simultaneity curve will never intersect $\xi = h$ and the distance L becomes infinite.

IV. MILD VARIANT

For $t < t^*$, the worldlines of both ships are given by (12) but in $t = t^*$ the ships stop the engines, so for $t > t^*$ their worldlines become straight lines:

$$\begin{aligned} x_{R/F} &= mt + n_{R/F} \\ m &= \frac{at^*}{\sqrt{1 + (at^*)^2}} \\ n_{R/F} &= x_{R/F}(t^*) - \frac{a(t^*)^2}{\sqrt{1 + (at^*)^2}}. \end{aligned} \tag{26}$$

which are the tangent lines to (12) in t^*. This means after t^*, both ships continue travelling with constant velocity

$$v^* := v(t^*) = \frac{at^*}{\sqrt{1 + (at^*)^2}}. \tag{27}$$

Thus, for $t > t^*$ we can define a new framework S' that moves with the ships and has its origin in R. S and S' are inertial frames connected by a Lorentz transformation. We know from Minkowski diagram that simultaneity curves of S' in (x, t) diagram are straight lines with slope v^*. Given an event (t_0, x_0) with $t_0 > t^*$, the event (x, t), $t > t^*$ simultaneous with it holds

$$t - t_0 = v^*(x - x_0). \tag{28}$$

We will continue our explanation considering coordinates (ξ, τ) given by (12). Let τ^* be the proper time coordinate when the ships stop their engines, that corresponds to t^* through (12). Depending on whether $\tau < \tau^*$

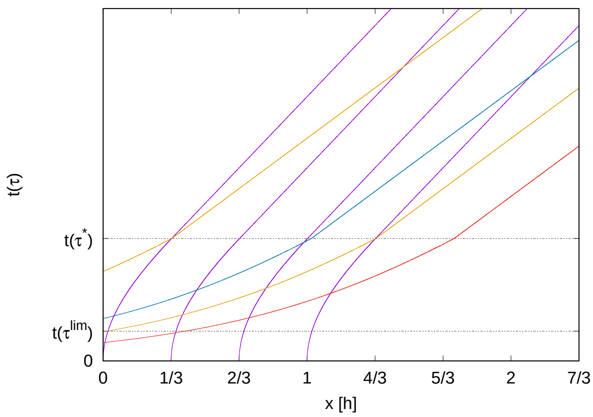

FIG. 4: Mild version's (x, t) diagram. Purple lines are worldlines. Simultaneity curves $\Sigma_{\tau^{lim}}$ and Σ_{τ^*} are plotted in yellow

or $\tau > \tau^*$, we have different kinds of simultaneity curves. Let τ^{lim} be such that $(\xi, \tau) = (0, \tau^{lim})$ is simultaneous with (h, τ^*). Its expression in terms of τ^* is

$$\tau^{lim} = a^{-1} \log \left(\frac{e^{ah} + \alpha}{e^{ah} - \alpha} \right) \qquad \alpha = \frac{e^{a\tau^*} - 1}{e^{a\tau^*} + 1}. \tag{29}$$

As in the tough version, τ_1 indicates the proper time in the rear ship when we make the measure. Then, if $\tau_1 \leq \tau^{lim}$ the points $(\xi, \tau) = (0, \tau_1)$ and (h, τ_2) that are simultaneous are joined by (20) and so, the distance between them is given by (23), the same as in the tough variant. This is the red simultaneity curve in figure 4.

If $\tau_1 \geq \tau^*$, the points $(0, \tau_1)$ and (h, τ_2) that are simultaneous are joined by (28). The proper length in this case is easy to obtain, since the frames S' and S are inertial. Due to Lorentz contraction, the gap between the ships will be γh in S', and it is constant no matter what time $\tau_1 \geq \tau^*$ the measure is performed.

For τ_1 values between τ^{lim} and τ^* simultaneity curves will be defined piecewise. They will be given by (20) until they reach $\tau = \tau^*$ and from that point, they will continue as (28). The length L^{mild} will be the sum of two different contributions: L_I, the integral of (20) from τ_1 to τ^*, and L_{II}, the integral of (28) from τ^* to τ_2. This is the case of the blue simultaneity curve in figure 4.

The first part is easy to obtain, by merely making $\tau_2 = \tau^*$ in (18) so that

$$L_I = a^{-1} \log \frac{\sinh(a\tau^*)}{\sinh(a\tau_1)}. \tag{30}$$

For the second part, we use the following reasoning: If the simultaneity curve $\Sigma_{\tau_1}(\tau)$ given by (20) intersects $\tau = \tau^*$ in (h, τ^*) then $L_{II} = 0$, and if it intersects $\tau = \tau^*$ in $(0, \tau^*)$ it should be $L_{II} = \gamma h$. Besides, this relation between $L_{II} \in [0, \gamma h]$ and $\Sigma_{\tau_1}(\tau^*) \in [0, h]$ has to be lineal, since we are working with triangles in a flat spacetime. Hence

$$L_{II} = \gamma(h - \Sigma_{\tau_1}(\tau^*)). \tag{31}$$

Recalling (20), the expression for the proper length of the thread will be

$$\begin{aligned} L^{mild} &= L_I + L_{II} = a^{-1} \log \frac{\sinh(a\tau^*)}{\sinh(a\tau_1)} + \\ & \gamma \left(h - a^{-1} \log \left[\frac{e^{a\tau^*} - 1}{e^{a\tau^*} + 1} \cdot \frac{e^{a\tau_1} + 1}{e^{a\tau_1} - 1} \right] \right). \end{aligned} \tag{32}$$

The complete expression for L in the mild variant is given piecewise by

$$L(\tau_1) = \begin{cases} L^{tough}(\tau_1) & \text{if} \quad \tau_1 \leq \tau^{lim} \\ L^{mild}(\tau_1) & \text{if} \quad \tau^{lim} < \tau_1 \leq \tau^* \\ \gamma h & \text{if} \quad \tau^* < \tau_1 \end{cases}$$

V. CONCLUSIONS

Finally, we arrive at the expression of the proper length of the thread in both variants. Results in terms of τ_1 are plotted in figure 5. We observe that proper length is not constant when acceleration takes place.

Surprisingly, in the tough variant there is a measuring time limit beyond which proper length of the thread becomes infinite, whereas in the mild version it keeps always under a finite value γh. This would not have happened if we had taken τ_2 as a parameter instead τ_1. Tough or mild, it becomes clear that the thread is stretched in both variants, and thus it will break.

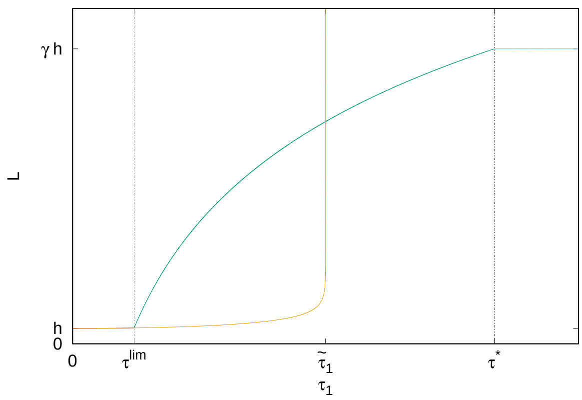

FIG. 5: Length in terms of τ_1 in the tough variant (yellow) and the mild one. They both coincide between 0 and τ^{lim}.

Acknowledgments

I would like to thank my advisor, Josep Llosa, for investing such a great amount of time taking care of this project, and for his passion for physics. To my whole family, especially my parents and brother for their patience, confidence and generosity. To my grandfather and my grandma, with all my affection. To my friends and fellows, especially you two Jaime and Víctor... and to you Cristina, my partner, for sustaining me throughout my university experience providing me with care and love.

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