The role of the Sommerfeld enhancement in dark matter physics

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Abstract: The dark matter annihilation cross section could suffer a boost due to the Sommerfeld enhancement at low energy. The importance of this process holds in the quantification of the effect on the cross section that an interactive potential produces before particle annihilation. In order to provide insight into the Sommerfeld enhancement we analytically compute it for the potential well and the attractive Coulomb potential. Finally, within the use of a numerical program we compute the Sommerfeld enhancement for the Yukawa potential since it is not analytically solvable.

I. INTRODUCTION

Although the theory of the Sommerfeld enhancement was developed almost a century ago by Arnold Sommerfeld [1] its current impact in modern physics is remarkable since recent investigations suggests that dark matter's cross section could be affected by the Sommerfeld enhancement giving rise to a paradigm shift in the interpretation of cosmic rays data produced by the annihilation of dark matter particles [2] - from now on referred as DM particles for sake of abbreviation. The cosmic ray detector PAMELA has indicated a sharp increase in the positron fraction from 10-100 GeV compared to what is expected from high-energy cosmic rays interacting with the interstellar medium [3]. This could be explained with dark matter annihilation $\chi\,\bar{\chi}$ into $e^+\, e^-$ (see FIG. 1 left); nevertheless, this requires a cross section larger than the typical coming from DM models [4]. Models including a long range attractive force as the Yukawa potential between DM particles can enhance the annihilation cross sections to the observed levels, a phenomenon referred to as the Sommerfeld enhancement. The presence of this force can enhance the bare annihilation cross sections to the observed levels (see FIG. 1 right).

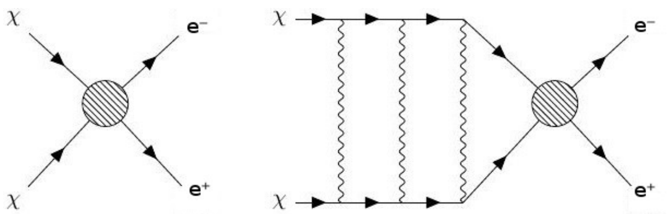

FIG. 1: Left: bare annihilation cross section. Right: dressed annihilation cross section; corrections by light mediator particle with Yukawa potential.

First of all, in order to gain more insight about this effect we start with its classical analogy. Let's consider a point-like particle moving towards a massive object of radius R. *Neglecting gravity*, the bare cross section for the particle to hit the object is $\sigma^0 = \pi R^2$. *Considering gravity*, the dressed cross section is actually $\sigma = \pi b^2$ where b is the distance of closest approach of the orbit which can be larger than R (see FIG. 2). It is also commonly seen b being referred as the impact parameter. The relation between the bare and dressed cross sections can be obtained if the conservation of energy and angular momentum are imposed:

$$\sigma = \sigma^0 \left(1 + \frac{v_{esc}^2}{v^2}\right) \tag{1}$$

where v is the velocity of the particle at infinity and $v_{esc}^2 = 2GM/R$ is the escape velocity. We have to keep in mind that $\sigma^0 = \sigma$ at $G = 0$.

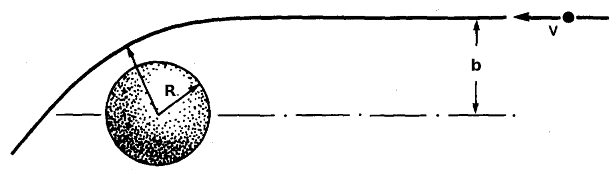

FIG. 2: Representation of how the impact parameter changes due to gravity.

The annihilation cross section of two particles in the non-relativistic regime can be rigorously determined in Quantum Field Theory which allows the computation of the Sommerfeld enhancement factor [5]. Here we give an intuitive approach to get the Sommerfeld enhancement.

First, let's consider two non-relativistic particles moving towards an annihilation zone. For sake of simplicity we will consider this zone to be point-like; thus, the annihilation potential can be written as $U_{ann} = U_0 \delta^3(\vec{r})$. On the other hand, we can also have an interacting potential that affect both particles before their annihilation; this interacting Hamiltonian will be written as $H_V = V(r)$. With this, we can define **the Sommerfeld enhancement factor** S as the ratio of the annihilation cross section with and without the potential $V(r)$:

$$S = \frac{\sigma}{\sigma^0} \tag{2}$$

where the cross section of the system to be proportional to the squared of the system wavefunction. Thus:

$$\sigma = |\langle i|U_{ann}|f\rangle|^2 = \\ = \left| \int dr^3 \psi(r) U_{ann}(r) \psi_F(r) \right|^2 \tag{3}$$

where we identify $|i\rangle$ as the initial state of the system $\chi\,\bar{\chi}$ and $|f\rangle$ as the final state of the $e^+\ e^-$ system. Moreover, in the case of local DM DM annihilation, i.e. assuming a zero range annihilation amplitude between the two DM particles, the non-perturbative correction reads:

$$\sigma = |\psi(0) U_{ann}(0) \psi_F(0)|^2 \tag{4}$$

The same procedure can be done in order to compute σ^0. Thus, the Sommerfeld enhancement S can be directly estimated as:

$$S = \frac{|\psi(0)|^2}{|\psi^0(0)|^2} \tag{5}$$

where ψ is the wavefunction under the effect of $V(r)$ before annihilating (see FIG. 1 left) and ψ^0 is the bare wavefunction without the interacting potential (see FIG. 1 right), namely, the wavefunction ψ in the limit where $V \to 0$.

Here we work out in detail the relative wavefunction which plays a key role when computing the Sommerfeld enhancement [6]. When considering a two-particle system the Schrödinger equation for the total wavefunction $\psi(r) = \psi_1(r_1)\psi_2(r_2)$, where $r \equiv |r_2 - r_1|$, can be reduced into:

$$\left[-\frac{\nabla^2_{rel}}{2\mu} - \frac{\nabla^2_{tot}}{2M} \right] \psi = \left(E_{lab} - V(r) \right) \psi \\ -\frac{\nabla^2_{rel}}{2\mu} \psi = \left(E_{CM} - V(r) \right) \psi \tag{6}$$

where it is needed to define the total mass $M \equiv m_1 + m_2$ and the reduced mass $\mu \equiv m_1 m_2/(m_1 + m_2)$. On the other hand, we have also introduced the operators: $\nabla_{tot} = \nabla_1 + \nabla_2$, $\nabla_{rel}/\mu = \nabla_1/m_1 - \nabla_2/m_2$. When considering a system of two identical DM particles we get $m_1 = m_2 \equiv m_\chi$; thus, $M = 2m_\chi$ and $\mu = m_\chi/2$. Moreover, for a central potential the Schrödinger equation decouples into separate equations for each partial wave. If we propose as solution to Eq. (6):

$$\psi = \sum_{l=0}^{\infty} \frac{i^l e^{i\delta_l}(2l+1)}{k} P_l(\cos(\theta)) R_l(r) \tag{7}$$

we obtain the radial Schrödinger equation for the two-particle system which reads:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dR_l}{dr} \right) = \left[2\mu \left[V(r) - E \right] + \frac{l(l+1)}{r^2} \right] R_l \tag{8}$$

where we may remember that R_l is the radial wavefunction and P_l are the Legendre polynomial. We look for a solution to the Schrödinger equation satisfying the usual asymptotic condition at large separation [6]:

$$\psi_{r\to\infty} \to e^{ikz} + \frac{e^{ikr}}{r} f(\theta) \\ = \sum_{l=0}^{\infty} (2l+1) P_l(\cos\theta) \left[\frac{e^{ikr}(1 + 2ikf_l) - e^{-ikr}(-1)^l}{2ikr} \right] \tag{9}$$

Now, it is quite useful to define $S_l \equiv 1 + 2ikf_l$ where it could be proven that $|S_l| = 1$. Thus, we could redefine it as $S_l \equiv e^{2i\delta_l}$ leading to:

$$= \sum_{l=0}^{\infty} (2l+1) P_l(\cos\theta) \left[\frac{e^{ikr+2i\delta_l} - e^{-ikr+l\pi}}{2ikr} \right] \\ = \sum_{l=0}^{\infty} \frac{i^l e^{i\delta_l}(2l+1)}{k} P_l(\cos\theta) \frac{\sin(kr - l\pi/2 + \delta_l)}{r} \tag{10}$$

where $f = \sum_{l=0}^{\infty} (2l+1) f_l P_l(\cos\theta)$ is the scattering amplitude. Moreover, for a central potential the Schrödinger equation decouples into separate equations for each partial wave. The solution to Eq. (8) behave as free-waves at large r:

$$R_{l\,r\to\infty} \to \sin(kr - l\pi/2 + \delta_l)/r \tag{11}$$

where we have imposed normalisation at infinity. The constants A and δ_l are fixed by two boundary conditions. One is $R_l(0) = A$, needed in order to $\psi(0)$ be finite. The second one comes from the asymptotic condition - Eq. (11).

Since the angular part factorises and it does not depend on the potential - we may remember we are considering central potentials - the Sommerfeld enhancement reduces to:

$$S_l = \frac{|R_l(0)|^2}{|R_l^0(0)|^2} \tag{12}$$

We may remember that since we are only interested in studying systems which $l = 0$ we arrive to the equation:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dR_0}{dr} \right) = 2\mu \left[V(r) - E \right] R_0 \tag{13}$$

which needs to fulfil the boundary conditions:

$$R_0(0) = A \ , \quad R_0(r \to \infty) \simeq \sin(kr + \delta_0)/r \tag{14}$$

In this case, $S_0 = A^2$, being $R_0^0(r) = \sin(kr)/r$. Equivalently, without potential we set the normalisation to be $A = 1$ and $\delta_0 = 0$.

As a sum, in order to get the Sommerfeld enhancement factor S_0 we need to solve the radial Schrödinger equation for $R_0(r)$ - which is Eq. (13) - which needs to accomplish the boundary conditions given by Eq. (14).

II. SOLVING SCHRÖDINGER EQUATION FOR SCATTERING STATES

This section of the report consists on computing the Sommerfeld enhancement for some potentials: the potential well, the attractive Coulomb potential and the Yukawa potential. Since the radial Schrödinger equation given by the Yukawa potential can not be analytically solved a simple program in *Python* has been written in order to compute the Sommerfeld enhancement.

A. The potential well

As said in the introduction, what we need to solve is the radial Schrödinger equation for the given potential which in this case is:

$$V = \begin{cases} -V_0 & \text{for } 0 \leq r < R \\ 0 & \text{for } r > R \end{cases} \tag{15}$$

where $V_0 > 0$. Here, we are only interested in solving the Schrödinger equation for $E > 0$ states - continuum spectrum - in order to compare with the charge-less particle case and then to get the Sommerfeld enhancement between both cases. From now on the subindex $R_0(r)$ will be omitted for sake of simplicity. For this potential the radial part of Schrödinger equation reads:

$$\begin{aligned} \frac{d^2 u_{in}(r)}{dr^2} - k_{in}^2 u_{in}(r) = 0, \quad \text{for } 0 \leq r < R \\ \frac{d^2 u_{out}(r)}{dr^2} - k^2 u_{out}(r) = 0, \quad \text{for } r > R \end{aligned} \tag{16}$$

where we have defined $u(r) \equiv rR(r)$, $k^2 \equiv 2\mu E$ and $k_{in}^2 \equiv 2\mu(E+V_0)$. Since Eq. (16) is the harmonic oscillator equation the more general solution can be expressed as:

$$\begin{aligned} u_{in}(r) = A \sin(k_{in} r) + B \cos(k_{in} r), \quad \text{for } 0 \leq r < R \\ u_{out}(r) = C \sin(\ kr\ ) + D \cos(\ kr\ ), \quad \text{for } r > R \end{aligned} \tag{17}$$

Since we know from Eq. (14) that $u_{in}(0) = 0$ it is fulfilled that $B = 0$ and also that $C = 1$ since we want to normalise u_{out} at infinity. Moreover, the constant D can be reabsorbed as a phase which leads to:

$$\begin{aligned} u_{in}(r) = A \sin(k_{in} r), \quad \text{for } 0 \leq r < R \\ u_{out}(r) = \sin(kr + \delta_0), \quad \text{for } r > R \end{aligned} \tag{18}$$

Finally, we find A by imposing continuity in the wavefunction and its derivative in $r = R$:

$$\begin{aligned} A \sin(k_{in} R) = \sin(kR + \delta_0) \\ A\, k_{in} \cos(k_{in} R) = k \cos(kR + \delta_0) \end{aligned} \tag{19}$$

where we can find the constrain:

$$A = \pm \frac{1}{\sqrt{\sin^2(k_{in}R) + \frac{k_{in}^2}{k^2}\cos^2(k_{in}R)}} \tag{20}$$

Thus, the squared module of the wave functions at origin is:

$$\boxed{|R_0^{well}(0)|^2 = \frac{1}{\frac{k^2}{k_{in}^2}\sin^2(k_{in}R) + \cos^2(k_{in}R)} = S_0^{well}} \tag{21}$$

If we consider there is no potential, which is $V_0 = 0$, then we have $k_{in} = k$ which leads to $S_0 = 1$ as expected. In order to have a better understanding of the behaviour of the Sommerfeld enhancement we can plot it as function of $\rho_{in} \equiv k_{in} R$. Let's consider $\frac{\rho^2}{\rho_{in}^2} = \frac{1}{5}$:

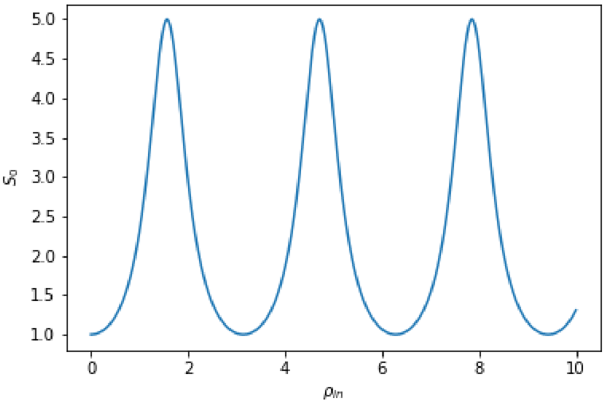

FIG. 3: Sommerfeld enhancement factor for the potential well.

Finally, Let's now see how the Sommerfeld enhancement behaves for some limits:

1. We always get $\frac{k^2}{k_{in}^2} < 1$ which leads to $S_0^{well} \geq 1$.
2. When $\rho_{in} = \frac{1}{2}\pi + n\pi$ for $n = 0, 1, 2\ldots$ there are maximums of value: $S_0 = 1 + \frac{V_0}{E}$. On the other hand, when $\rho_{in} = n\pi$ it is accomplished that $S_0 = 1$ for any value of V_0.

B. The attractive Coulomb potential

In this section we will compute the Sommerfeld enhancement factor for the attractive Coulomb potential which is $V(r) = -\frac{\alpha}{r}$. Now, we admit solutions for any value of E. For this potential the radial Schrödinger equation when $l = 0$ can be rewrited as:

$$\frac{d^2 u(\rho)}{d\rho^2} + \left(1 + \frac{2\eta}{\rho}\right) u(\rho) = 0 \tag{22}$$

where we have defined:

$$u(r) \equiv rR(r) \ , \quad \rho \equiv kr \ , \quad \eta \equiv \frac{\mu\alpha}{k} = \frac{\alpha}{\beta} \tag{23}$$

If we solve Eq. (22) we could prove that the Sommerfeld enhancement factor for the attractive Coulomb potential is:

$$\boxed{S_0^{Coulomb} = \frac{2\pi\eta}{1 - \exp\left(-2\pi\eta\right)}} \tag{24}$$

Now we will represent the Sommerfeld enhancement factor as function of β in order to gain more insight:

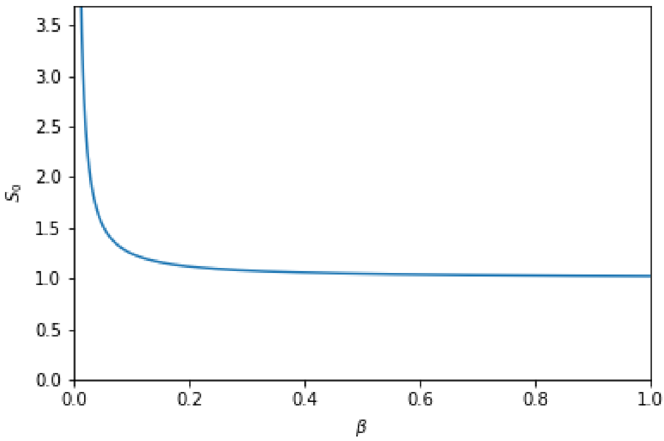

FIG. 4: Sommerfeld enhancement factor for the attractive Coulomb potential as function of the velocity.

Finally, as we did with the potential well, we will study some interesting limits:

1. When $v \to \infty$ we have $S_0 \to 1$. This is because of at high velocities the interaction with the potential becomes negligible. Moreover, since we work in non-relativistic theories it is not accomplished that for $v \to c$ we have $S_0 \not\to 1$.
2. When $v \to 0$, the Sommerfeld enhancement $S_0 \simeq \pi\alpha/\beta$: this is why the Sommerfeld enhancement is often referred as a $1/v$ enhancement.

C. The Yukawa potential

In this last section we will study a particle interacting through a Yukawa potential. We consider a dark matter particle of mass m_χ whose velocity is β and interacts with $V(r) = -\frac{\alpha}{r}e^{-m_V r}$, that is, an attractive Yukawa potential mediated by a boson of mass m_V. Following the recipe given by R. Iengo in Ref. [7] a numerical code has been written.

Using the same definitions than in Eq. (23) and defining $\zeta \equiv \frac{m_V}{\alpha m_\chi}$ we notice that the radial Schrödinger equation for $l = 0$ transforms into:

$$\frac{d^2u(\rho)}{d\rho^2} + \left(1 + \frac{2\eta e^{-\zeta\rho}}{\rho}\right) u(\rho) = 0 \tag{25}$$

which is essentially the same than Eq. (22) when $\zeta = 0$. In other words, the Yukawa potential is the generalisation of the Coulomb potential where $\zeta \neq 0$. On the other hand, if we consider $R(\rho)$ the equation above can be expressed as:

$$\frac{d^2R(\rho)}{d\rho^2} + \frac{2}{\rho}\frac{dR(\rho)}{d\rho} + \left(1 + \frac{2\eta e^{-\zeta\rho}}{\rho}\right) R(\rho) = 0 \tag{26}$$

Since the Schrödinger equation for the Yukawa potential is not analytically solvable we need to solve it numerically. Since we will use a numerical method that needs both initial conditions - value of the wavefunction and its derivative at the origin - we need to rewrite both conditions of Eq. (14). First of all, since we need to determine A in order to compute the Sommerfeld enhancement we cannot use it as an initial condition. From differential equations' theory we know that if $f(x)$ is solution of a differential equation, this function multiplied by a constant will also be solution of this differential equation. Thus, we can multiply $R(\rho)$, which is the solution of the Schrödinger equation, by $1/A$. Thus; $\Phi(\rho) = R(\rho)/A$ will also be solution of Eq. (26) which will be rewrited as:

$$\Phi''(\rho) + \frac{2}{\rho}\Phi'(\rho) + \left(1 + \frac{2\eta e^{-\zeta\rho}}{\rho}\right)\Phi(\rho) = 0 \tag{27}$$

and its solution will accomplish $\Phi(0) = 1$. Now, we will use Frobenius' method in order to transform the second boundary condition into a condition at the origin for the derivative. Let's consider a second order function for Φ that accomplish $\Phi(0) = 1$. Thus:

$$\Phi(\rho) = 1 + \Phi'(0)\rho + \Phi''(0)\rho^2 \tag{28}$$

When introducing this solution into Eq. (27) we find a condition for the derivative which is $\Phi'(0) = -\eta$.

On the other hand, if we solve Eq. (27) in the limit where $\rho \to \infty$ we arrive to the asymptotic solution:

$$\rho\Phi(\rho)_{\rho\to\infty} \to \frac{1}{A}\sin\left(\rho + \delta_0\right) \tag{29}$$

Finally, we may notice that the way of determining the Sommerfeld enhancement has been affected due to the redefinition of variable. Again, the Sommerfeld enhancement factor reads:

$$\boxed{S_0^{Yukawa} = A^2} \tag{30}$$

As a sum, in order to compute the Sommerfeld enhancement we only need to determine the value of A which is given by the asymptotic behaviour of the wavefunction. For achieving that it will be needed to solve numerically Eq. (27) whose initial conditions are:

$$\Phi(0) = 1 \ , \quad \Phi'(0) = -\eta \tag{31}$$

One way of computing A is defining $F(\rho) \equiv \rho\Phi(\rho)$; thus, if we compute $F(\rho)^2 + F(\rho - \pi/2)^2$ for large values of ρ we get:

$$\begin{aligned} F(\rho)^2 + F(\rho - \pi/2)^2_{\rho\to\infty} &\simeq \\ &\simeq A^2\left[\sin^2\left(\rho + \delta_0\right) + \cos^2\left(\rho + \delta_0\right)\right] \\ &= A^2 \end{aligned} \tag{32}$$

If we consider the same conditions than in Ref. [8] which are $\alpha = \frac{1}{30}$ and the mass of the shared particle $m_V = 90$ GeV we can plot $F(\rho)^2 + F(\rho - \pi/2)^2$ in order to see how it behaves. If we consider the mass of the DM particles to be $m_\chi = 10$ TeV we obtain:

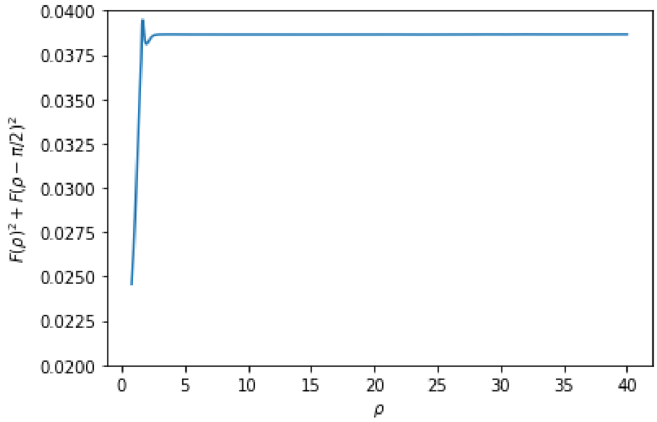

FIG. 5: Representation of $F(\rho)^2 + F(\rho - \pi/2)^2$. As stated before, its value becomes constant for $\rho \to \infty$.

Once this is known we can plot the Sommerfeld enhancement factor as function of the mass of the DM particles at different velocities. As before, we consider the values used in Ref. [8]:

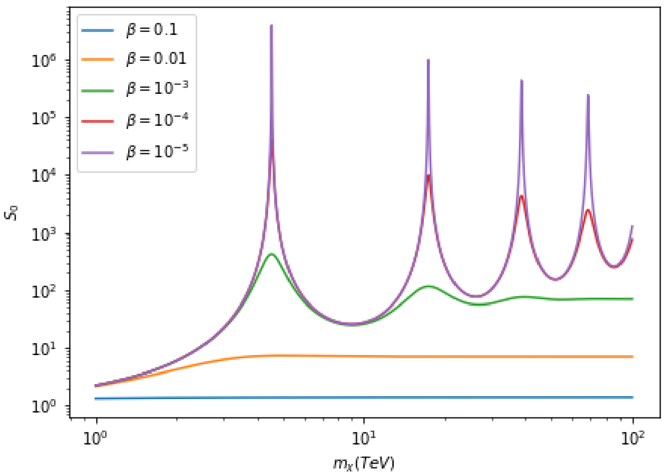

FIG. 6: Sommerfeld enhancement factor for the Yukawa potential as function of the mass of the DM particles.

Here we can see how there is a resonant pattern, that is, for some values of the mass of the DM particles the Sommerfeld enhancement is extremely large ($10^5 - 10^6$). On the other hand, we can also see how it is more important for small velocities as happened in the Coulomb potential since for large velocities we have $S_0 \simeq 1$. The results obtained are in agreement with the ones presented in the literature [8].

III. CONCLUSIONS

As said in the introduction, although its long life, the Sommerfeld enhancement could suppose a milestone in the understanding of dark matter physics since it could explain the increase in the positron fraction compared to what is expected from high-energy cosmic rays interacting with the interstellar medium.

First, we gained some intuition by determining the Sommerfeld enhancement for the potential well. Then, we studied a more realistic potential which was the electromagnetic one. There we achieved seeing how velocity could affect the Sommerfeld enhancement and why it is widely called as the $1/v$ enhancement. And finally, after writing a numerical code in *Python* we have proven that the Sommerfeld enhancement could boost the annihilation cross section of dark matter. This effect presented a clear dependence on DM particles' mass giving rise to a resonant patter which could give some indirect information on dark matter's mass.

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