Quantum percolation in complex networks

Author: Andrés Laverde Marín.
Facultat de Física, Universitat de Barcelona, Diagonal 645, 08028 Barcelona, Spain.[*]

Advisor: Marián Boguñá

Abstract: Complex quantum networks will be essential in the future either for the distribution of quantum information (telecommunications) or for studying complex quantical linked systems among others. Complex networks have a wide variety of properties that will help us understand how complex quantum networks behave. Here we will study how this complex networks perform using local quantum transformations at the nodes. We will focus specifically on the Internet network and we will study its behaviour from a current and quantum perspective.

I. INTRODUCTION

A large part of the current systems, such as the Internet network, neural network or social networks amongst many others, can be modeled with graph theory, where the system is described by a set of N nodes and L edges, each one of these links one node to another. Within complex networks we can find both symmetric and antisymmetric networks, i.e. a square network is a totally random network and a scientist connections lattice is totally antisymmetric. Due to the great variety of systems that can be described through a complex network, a line of research has been generated, which includes from mathematical to theoretical physical models in order to reveal the principles that govern these lattices. Knowing the properties of these networks allows us to know how the system will behave; for instance, if we study the internet network we can see how it will behave in case of a global internet repeaters failure.

In the future, we hope to develop communication systems based on quantum physics, such as the quantum internet[1]. Quantum networks offer new possibilities and phenomena, which we can compare with their classical equivalent. One of the new advantages is that complex quantum network offers us the possibility of sending encrypted messages securely, without anyone being able to intercept the message and leaving a trace.

The objective of this study is the analysis of a same network in a classic and quantum way. To compare them, we can understand what advantages or disadvantages these new complex quantum networks bring us over the already known complex networks. We will focus mainly on the Internet network, although, it does not imply that this comparative study can not be extended to more than one network.

[*]Electronic address: andreslaverdemarin@gmail.com

II. BOND PERCOLATION THEORY

The objective of modeling complex networks is to understand the systems they represent without having to know its exact structure, i.e. only by knowing their statistical properties we can understand their behavior. Once measured and quantified these properties, we must transform these results into conclusions on how the system can work. The phenomenon of percolation describes, in a simple way, the criticality of some complex systems, such as physical, biological and social phenomena. Canonical studies of this systems are within statistical mechanics and are fundamentally associated with phase transitions[3].

Usually the percolation process is understood as the movement of a fluid in a porous material. This definition can help us to understand the process of percolation. We can imagine this process as a fluid that can flow between two nodes with a probability $p \in [0, 1]$. When the probability is $p = 1$, we will have all the edges connected, therefore, the liquid can flow through the whole network. When the probability is $p = 0$ we will have all the edges disconnected and the liquid will not be able to flow between nodes as there will be no edges connecting it. Fig.(1).

In our case, we study real networks, with a number of fixed and connected nodes in a concrete way. To understand how the network behaves we can imagine that each one of the edges has a probability p of being open and a probability $1-p$ of being closed, this way of understanding the network is knowning as bond percolation [3]. The percolation threshold p_c is a concentration p at which a macroscopic connected component (or giant component (GC)) emerges [3], see Fig.(1).

Below p_c, the network is composed of a large number of small disconnected components with a size distribution approaching a power law as p tends to p_c. Above p_c, the giant component grows and becomes equal to the entire system when $p = 1$. For finite systems the giant component is defined as the largest connected component of the system.

FIG. 1: Square lattice. Each edge is represented with white squares and black squares. The white one is disconected and black one is connected.We observe the emergence of GC with $p_c = 0.59$, with $p < p_c$ we only see disconnected islands.

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III. MODELLING THE NETWORK AS A QUANTUM NETWORK

We can consider this modelling, as in Fig.(2), where each edge correspond to pairs of pure entanglement state, mathematically we can express it as $|\Psi\rangle^{\otimes 2}$, where the wave function is defined as $|\Psi\rangle = \sqrt{\lambda_0}\,|00\rangle + \sqrt{\lambda_1}\,|11\rangle$, representing two qubit states, where $\sqrt{\lambda_0} \geq \sqrt{\lambda_1} \geq 0$ are the Schmidt coefficients. Each of these partially entangled states can be converted into a singlet (maximum entangled state) with the singlet conversion probability $p = min[1, 2(1-\lambda_0)]$ [5] . With this transformation we guarantee that as we are in a maximally linked state, no information is lost between states [5].However, if the two nodes share two copies of $|\Psi\rangle$, the probability that at least one of the two states is converted into a singlet state is $p_2 = 2p - p^2$, where p is the singlet conversion probability, previously defined.

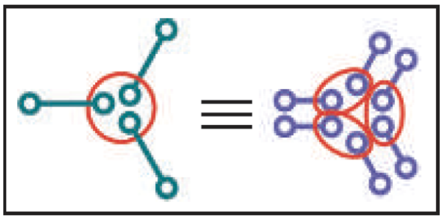

FIG. 2: Equivalence between a classic and a quantum node. On the left we see a node with three classic connections, where there are three neighbours connected with a simple edge. On the right we can see these same neighbours connected quantum, i.e. with two links per neighbor. Figure from [4]

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To apply the classic percolation in a quantum complex network, we need to consider that the network now has the same number of nodes, but the edges that join the nodes are doubled, see Fig.(2), where each edge corresponds to the partially entanglement state $|\Psi\rangle$. A better way to understand this, is to perceive it with the same number of nodes and edges, but now the probability that a link is open is no longer p but p_2 as the links are two copies of $|\Psi\rangle$.We can see this in Fig.(2). The process of applying this transformation to each edge of the classical network, whether optimally or not, is known as classical entanglement percolation [4].

So far we have only made a quantum transformation allowing the network to be maximally entangled. But we can obtain new characteristics if we apply other local quantum transformations that transform the geometry of the network, it must be taken into account that this change in geometry involves a change in the percolation threshold. [4]. To understand this new quantum property first we need to comprehend the entaglement swapping. Let us now consider a node (c) of our network, in which we make a Bell measurement in two qubits that belong to a $|\Psi\rangle$ state and each edge is shared with a different node (a and b). Once this operation is done, the nodes (a and b) of the central node (c) are untangled and a tangle is created between nodes a and b [7] , see Fig.(3). Thanks to this type of quantum transformation we can converte a honeycomb lattice into a triangular lattice, ensuring that no information is lost [4]. In this type of transformation we can see that the percolation threshold is lower for the triangular lattice than for the honeycomb lattice [3], which represents an improvement over the previous lattice.

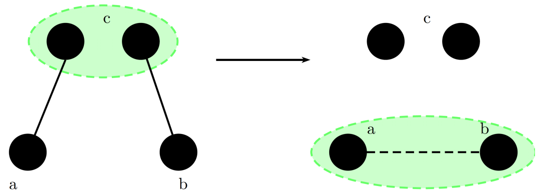

FIG. 3: Entangelment swapping

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IV. INTERNET NETWORK

Many times we believe that the WWW and the Internet network are the same, which is not true. They are very different networks from each other. In a general way, we can define the Internet network as a worldwide network of computers interconnected by cables, this allows an exchange of information between them. The vertices of this network are:

- The computers of the users.
- The servers (computers or programs providing a network service).
- The routers that arrange traffic across the Internet.

Connections are undirected, and traffic (including its direction) changes constantly [10]. In table I we summarize the main characteristics of the network, which will be studied below.

	N	L	$\langle \overline{c} \rangle$	$\langle k \rangle$	Vertex type	Format
Internet AS	23752	65048	0.20	4.92	AS	Undirect

TABLE I: Main characteristics of the internet network, Number of nodes N, number of edges L, Clustering coefficient $\langle \overline{c} \rangle$, Average degree $\langle k \rangle$, Vertex type, Network format.

To understand the topology of this network, we can start by studying its cumulative degree distribution Eq. (1), which is the probability that a randomly selected node has k or more edges, according to the behavior of $P_c(k)$ that we can see in the Fig.(5). The internet network has a scale-free topology, which means, it is a network with nodes with a large number of neighbours and nodes with very few neighbouneirs. In Fig. (4) we can see this since the nodes in the middle have many more connections than those at the ends. In this network representation, the size of the nodes is proportional to the logarithm of their degrees. We can see that there are very large nodes (they have many connections) in the center and very small nodes (they have few connections) at the ends, which is consistent with the representation of $P_c(K)$ that we found in the simulations made to obtain the characteristics of the Internet network, Fig.(5) and Fig.(6).

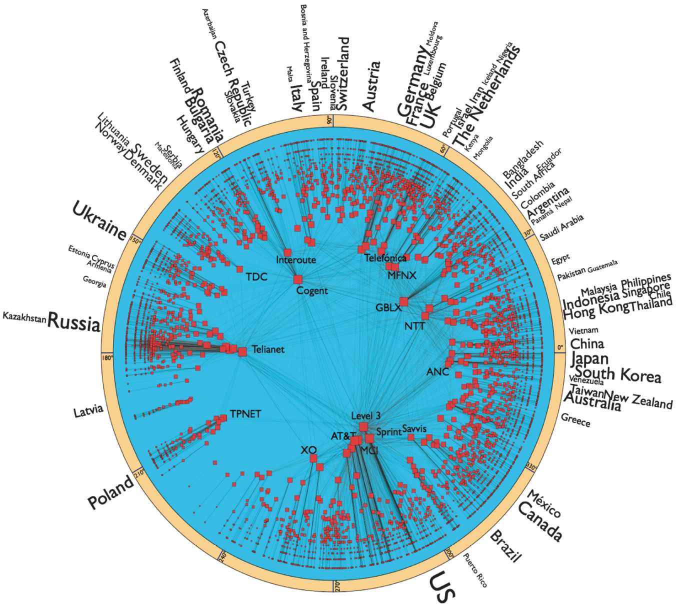

FIG. 4: Hyperbolic representation of the network we have used for this document, Figure from [8].

$$P_c(k) = \sum_{k \leq k'} P(k') \qquad (1)$$

The next thing we will focus on understand is the topology of the network is the average nearest neighbours degree Eq. (2) depicting a measurement of the tendecy of nodes to connect to peers in terms of the degree. On the right of Fig.(5) we can see $k_{nn}(k)$ which represents the average nearest neighbours degree of the internet network, which decreases in function of the degrees. This presents a disassortative pattern of degree, i.e the internet network connects in such a way that small degree nodes tend to connect with large degree nodes, we can see this in Fig.(4), we see that the smaller nodes (the ones at the end) are not connected to each other but they are connected to the large nodes (central nodes). This is what is expected in a scale-free network if we want to add new nodes these will tend to be added to nodes that already have many connections, i.e. new nodes will not be added randomly (if they did, we expect a distribution in the $P(k)$). For instance, if you just bought a new computer and need internet, you do not choose at random company to hire the service from, but you are informed of which company offers the best service. If all the people who hire internet do it in this way, we will see that the network will evolve in such a way that the best company will have more customers. Therefore the degree of the edges are increased for this company.

$$k_{nn}(k) = \sum_{k'} k' P(k'|k) = \frac{1}{N_k} \sum_{i \in k} \frac{1}{k} \sum_{j} a_{i,j} k_j \qquad (2)$$

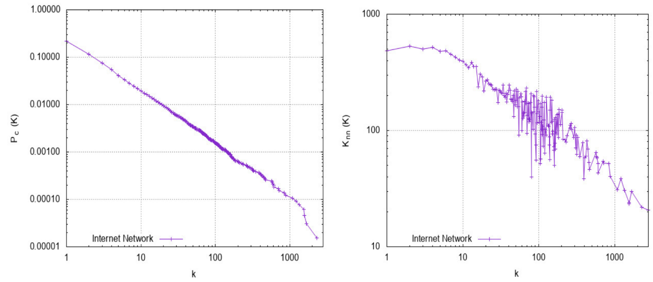

FIG. 5: The graphic on the left shows the comulative degree distribution $P_c(k)$ and the graphic on the right shows the average nearest neighbours degree $k_{nn}(k)$

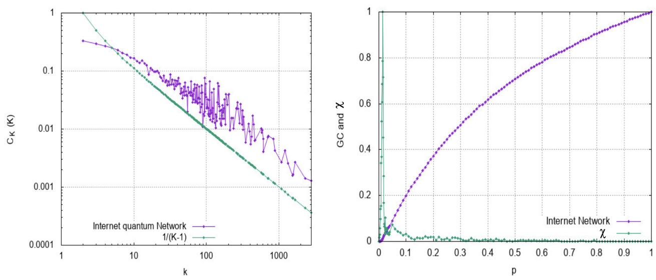

FIG. 6: The graphic on the left shows Clustering coefficient $C_k(k)$ and the graphic on the right shows the GC and χ as a funtions of p.

To conclude this presentation of the topology of the Internet network, we need to explain three others properties that characterize the network:

- Grade-dependent grouping coefficient, $C_k(K)$.
- Variance, χ.
- Percolation thresholds, p_c.

The first, the grade-dependent grouping coefficient or grouping spectrum defined by the Eq. (3), is the average of the local grouping coefficient over the degree classes. It gives us an idea of how much the network is grouped. In Fig.(6), we see two curves represented, the green one refers to $\frac{1}{k-1}$ and the purple one represents $C_k(K)$. The first thing we can observe is that this coefficient is much bigger than the one we would obtain in a random network. This means that the obtained values cannot be explained with the model of a random network. If we compare $C_k(K)$ and $\frac{1}{k-1}$, we see that most points of $C_k(K)$ are above the value $\frac{1}{k-1}$, which means that it is a network with strong clustering. Is very interesting to see that although it is a very large network (it a large number of nodes) with a very high clustering, it has a very low average path length [11].

For the second property we can look at Fig.(6). On the right we see how the giant component of the network varies in function of p. The variance, χ, is defined accordingly to the Eq.(4). It is proportional to the probability that concentration p, as spanning cluster ($GC \neq 0$), starts to appear and according to this, we can find p_c which corresponds to the maximum point of χ is found. This is the same point where the GC becomes zero.

According to this, for the classical internet network we found a third property; a value of $p_c = 0,014$. Which shows us that it is a highly connected network. With a very small value of p the GC becomes larger than zero, therefore we begin to have a connected network.

$$\bar{c}(k) = \frac{1}{N_k}\sum_{i\in k} c_i = \frac{2}{k(k-1)P(k)}\frac{1}{N}\sum_{i\in k} T_i \tag{3}$$

$$\chi = \frac{\langle GC^2\rangle - \langle GC\rangle^2}{\langle GC\rangle} \tag{4}$$

V. INTERNET QUANTUM COMPLEX NETWORK

The Internet network we have described is the network we know today. Now we are going to study how it behaves and what properties it acquires when we apply the quantum transformations described in section III. First we must consider that every classic internet edge becomes a quantum double edge, $|\Psi\rangle^{\otimes 2}$, see Fig.(7) so that the probability we keep in mind to apply bond percolation is p_2. If we only take into account this first local change in the original network we can see that all the properties of the network are maintained , except p_c, we see that for the internet network, if we take into account this first transformation, the threshold probability is $p_c = 0,006$ to the left of the one we had found for the classical internet network, therefore this network percolates before, see Fig.(8) and Fig.(9), in this figure you can see how the $C_k(K)$, the $P_c(K)$ and the $K_{nn}(k)$ of the classical network and the new quantum network with multilinks are the same. This first transformation is necessary because we cannot compare a classical network with a quantum network, therefore what we transform the internet network into one with new quantum edges, but keeping its topology. In this way we will be able to see the effect that the swapping change has on the network.

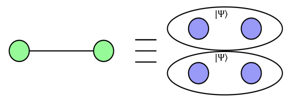

FIG. 7: Equivalence between a classic edge and a quantum edge

Finally we apply the swapping transformation in the network, see Fig.(3). When we apply this second transformation, the topology of the network changes because the nodes that were not connected before now can be connected and the nodes that were connected before can remain connected. When we connect these two neighbours there may be still a border between the selected neighbours and the central node. This new change in the topology can be seen in Fig.(9). As you can see both $C_k(k)$, $K_{nn}(k)$ and $P_c(k)$ have different behaviors but the general characteristics of the network are maintained, it is still a network with a strong clustering and where small range nodes still tend to connect with large range nodes. What we have achieved with this transformation is that the threshold of percolation is smaller than the one that the network had with the new quantum edges, $p_c = 0,004$, see Fig.(8), maintaining the general characteristics of the classical network. Therefore the new construction implies that the network with swapping transformation percolates before the network with quantum edges.

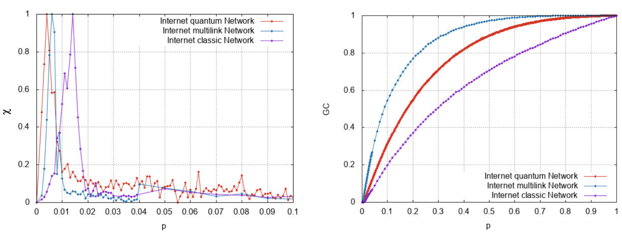

FIG. 8: The graph on the right shows $\chi(p)$, the graph on the left shows $GC(P)$, for the classic internet network (purple), the quantum network with multiple edges (blue) and the quantum network applying swapping (red).

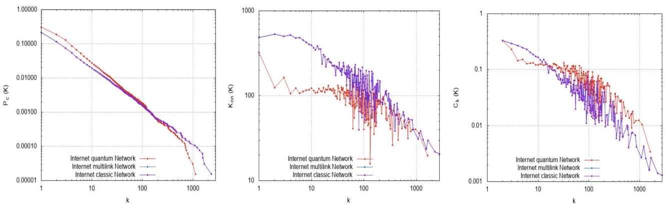

FIG. 9: The graphic on the left shows the comulative degree distribution $P_c(k)$ and the graphic in the middle shows the coeficient grouping $C_k(k)$ and on the right shows the average nearest neighbours degree $k_{nn}(k)$, for the classic internet network (purple), the quantum network with multiple edges (blue) and the quantum network applying swapping (red).

VI. CONCLUSIONS

We have studied the percolation of entanglement in the Internet network, which we can characterize by its statistical properties without knowing its exact structure and which shows that with purely local quantum transformations, we can lower the threshold of percolation of the network, p_c, without excessively changing its global characteristics. When we mix the bond percolation of complex networks with the local transformations of quantum mechanics, we see that networks undergo non-trivial changes, opening new paths to explore how classical networks can be improved by applying these transformations.

Although this has already been demonstrated for synthetic networks in different studies, we have seen that this theory is also valid for a real network such as the Internet. If the future of telecommunications is quantum, we think it's a good starting point to know that quantum networks can substantially improve information delivery without drastically changing the structure we have today.

Acknowledgments

I would like to thank Dr. Marián Boguñá especially for his work, for showing me the fantastic world of complex networks and the time he took to help me understand it. I also thank my mother, my grandparents and my aunt Lucy for giving me the opportunity to get here and for the unconditional support they have offered me over the years. Finally, I would also like to thank Neus, Luisa, Xavier and Núria, because without them, this work would not have been possible.

[1] Kimble, H.J. The quantum Internet, *Nature Physics* **453**, 1023-1030 (2008)

[2] Perseguers, S. and Lewenstein, M. and Acn, A. and Cirac, J. I. Quantum random networks, *Nature Physics* **6**, 539-543 (2010)

[3] D. Staufer, *Introduction to Percolation Theory* (Taylor & Fran cis, London, 1985).

[4] Acn, Antonio and Cirac, J. Ignacio and Lewenstein, Maciej. Entanglement percolation in quantum networks, *Nature Physics* **3**, 256 EP (2007)

[5] Vidal, Guifré, Entanglement of Pure States for a Single Copy, *Physical Review Letters* **83**, 1046-1049 (1999)

[6] Shimony, Abner, Bell's Theorem, *The Stanford Encyclopedia of Philosophy* `https://plato.stanford.edu/archives/fall2017/entries/bell-theorem/`(2017)

[7] Cuquet, Martí and Calsamiglia, John, Entanglement Percolation in Quantum Complex Networks, *Physical Review Letters* **103**, 240503, 1-4 (2009)

[8] Marián Boguñá, Fragkiskos Papadopoulos, Dmitri Krioukov, Sustaining the Internet with hyperbolic mapping, *Nature Communications* **1**, 62 (2010)

[9] M Ángeles Serrano, Marián Boguñá, Clustering in complex networks. I. General formalism, *Physical Review E*, **74**, 056114 (2006)

[10] S. N. Dorogovtsev and J. F. F. Mendes, Evolution of Networks: From Biological Nets to the Internet and WWW, *Oxford University Press* (2003)

[11] M Ángeles Serrano, Marián Boguñá, Albert Diaz-Guilera, Modeling the Internet, *The European Physical Journal B-Condensed Matter and Complex Systems* **50**, 249-254 (2006)