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ZEROS OF RANDOM ANALYTIC FUNCTIONS

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Contents

Abstract

In this project we deal with random analytic functions. Here we specifically use Gaussian analytic functions. Without technicalities, a GAF f (for short) is a random holomorphic function on a region of $\mathbb{C}$ such that $(f(z_1), ..., f(z_n))$ is a random vector with normal distribution. One way to generate them is using linear combinations of holomorphic functions whose coefficients are Gaussian random variables in $\mathbb{C}$ (or in $\mathbb{R}$ in special cases). For finding the zero set of a GAF we work on four isometric - invariant Hilbert spaces of analytic functions: the Fock space in $\mathbb{C}$, the finite space of polynomials in $\mathbb{S}^2$, the weighted Bergman space in $\mathbb{D}$ and the Paley - Wiener space. The first intensity determines the average of the distribution of the zero set of a GAF, and the Edelman - Kostlan formula gives an explicit expression of it. A result of uniqueness, called Calabi's Rigidity, concludes that the first intensity determines the distribution of the zero set of a GAF. At the end, some examples made in C++ and `gnuplot` clarify the theory in these Hilbert spaces.

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Introduction

Most of the physical phenomena that we witness every day can be modeled by random processes. One of them could be the arrival of people in a queue of a supermarket or the distribution of trees in a forest. This effort to model phenomenons using probability theory also includes the physic of quantum particles. In the last century, one of the most relevant and challenging problems was creating point processes to simulate the distribution of fermions, particles characterized by its repulsion. The Poisson process was one of the candidates, but it was discarded because it does not have any repulsion behavior. Then, it was mandatory to generate a random point process that had an anti-clumping behavior, that it was indifferent where to study this process and, obviously, that it simulates the distribution of the particles described before. For this, random analytic functions were considered in determined spaces of functions. The advantage after assuming all the properties above and some concepts explained in this project was that the random process is formed by the points where a random analytic function f is zero. This last type of functions can be, for example, of the form:

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0,$$

where $(a_n)_n$ are random variables, for all $n \in \mathbb{N} \cup \{\infty\}$.
In Chapter 1 we see that the random variables or vectors to consider are Gaussian ones in $\mathbb{C}$, and these are useful to define the functions that are the keystone of this project, called Gaussian Analytic Functions (or GAF, for short). At the end of this chapter, we give the general expression of the covariance kernel in a Hilbert space.
In Chapter 2 we introduce the main Hilbert spaces to study the zero sets of a GAF, and those are the Fock space in $\mathbb{C}$, the space of polynomials of finite degree in $\mathbb{S}^2$, the weighted Bergman space in $\mathbb{D}$ and the space of Paley - Wiener. We prove that these spaces have interesting isometric properties with regard to Möbius transformations.
In Chapter 3 we deal with the first intensity for a point process, and the Edelman - Kostlan formula is one of the most important results of this project, because it allows to us to understand how the zero set of a GAF is distributed. Finally, a uniqueness theorem called Calabi's Rigidity establishes that the first intensity determines the distribution of the zero set of a GAF.
In Chapter 4 we put theory into practice. C++ coding has been used here, and graphics made with `gnuplot` clarify the explanations. Here, we consider finite Hilbert spaces and we compute the zero set and the first intensity of a GAF in these spaces.
In the Annex, a brief theory section precedes the C++ programs used in the last chapter.

In the execution of this project, *Zeros of Gaussian Analytic Functions and Determinantal Point Processes* by John Ben Hough, Manjunath Krishnapur, Yuval Peres and Bálint Virág [2] has been the step-by-step book that has guided me through these pages, and the majority of the results and definitions has been extracted from there. Other few results and procedures in Chapter 3 are from the magnificent paper *Zeroes of Gaussian Analytic Functions with translation - invariant distribution*, by Naomi Feldheim [8]. Also the book *Anàlisi complexa*, by Joaquim Bruna and Julià Cufí [4] has been an excellent and helpful source. For the computing section I used C++ and the graphic program `gnuplot`.

Chapter 1

Gaussian analytic functions

1.1 Complex Gaussian distribution

Before introducing this section, we should establish some terminology. Let be X a random variable. We say that X follows a *normal distribution* if its density function is

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}},$$

where $\mu \in \mathbb{R}$ and $\sigma \in (0, +\infty)$. We denote $X \sim N_{\mathbb{R}}\left(\mu, \sigma^2\right)$ when the random variable X follows a normal distribution. In addition, μ represents the mean of the random variable and σ^2 its variance.

Definition 1.1. *A* standard complex Gaussian *is a complex-valued random variable with density function*

$$f(z) = \frac{1}{\pi} e^{-|z|^2}$$

with regard to the Lebesgue measure on the complex plane and z is a complex value.

To define the latter density function we may write the random variable Z as $X + iY$, where X and Y are independent identically distributed (i.i.d. for short) random variables with $N_{\mathbb{R}}\left(0, \frac{1}{2}\right)$ distribution. Indeed, considering the random vector (X, Y) we have that

$$f_X(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$$

and

$$f_Y(y) = \frac{1}{\sqrt{\pi}} e^{-y^2}.$$

Using the fact that X and Y are independent, we obtain:

$$f_{(X,Y)}(x, y) = f_X(x) f_Y(y) = \frac{1}{\pi} e^{-(x^2+y^2)}.$$

Thus, if $z = x + iy$, we have that $|z|^2 = x^2 + y^2$, and the density function is

$$f_Z(z) = \frac{1}{\pi} e^{-|z|^2}.$$

Definition 1.2. *Let Z_k be an i.i.d. standard complex Gaussian random variables, where $1 \leq k \leq n$. Then $Z := (Z_1, ..., Z_n)^t$ is a* standard complex Gaussian random vector. *If M is a complex matrix of dimension $m \times n$, then $W = MZ + \mu$ is an m-dimensional complex Gaussian vector with mean μ, which is an $m \times 1$ vector, and covariance $\Sigma = MM^*$, which is an $m \times n$ matrix and M^* is the conjugate transpose of M. We denote the distribution of W as $N^m_{\mathbb{C}}(\mu, \Sigma)$.*

By the last definition, the density function of a standard complex Gaussian vector Z is:

$$f_Z(z) = f_{(Z_1,...,Z_n)}(z_1, ..., z_n) = \prod_{k=1}^{n} f_{Z_k}(z_k) = \frac{1}{\pi^n} \prod_{k=1}^{n} e^{-\overline{z_k} z_k} = \frac{1}{\pi^n} e^{-\|z\|^2},$$

where $z = (z_1, ..., z_n)^t$ is a complex vector of n components. Recall that the second equality of the formula above is a consequence of the independence of the random components of the vector Z.
Following the book [11] (page 77, chapter 3), if Z follows a $N^n_{\mathbb{C}}(\mu, \Sigma)$ distribution, where

$$\mu := \mathbb{E}[Z]$$

and

$$\Sigma := \mathbb{E}\left[(Z - \mu)(\overline{Z} - \overline{\mu})^t\right],$$

then Z has density function with regard to the Lebesgue measure on $\mathbb{C}$:

$$f_Z(z) = \frac{1}{\pi^n} \frac{1}{\det(\Sigma)} e^{-(\overline{Z}-\overline{\mu})^t \Sigma^{-1}(Z-\mu)},$$

remarking that z is an n-dimensional complex vector, Σ is regular, $(\overline{Z} - \overline{\mu})^t \Sigma^{-1}(Z - \mu)$ is real and f_Z is a real-valued function.

The following results can be applied to complex Gaussian random variables or vectors, but we will prove them in the real case, because, as we saw before, a complex Gaussian random variable is a two-component random vector, and those components are the real and imaginary part of the complex random variable. Moreover, each of them has a real Gaussian distribution and they are independent. To get the main results and proofs in multidimensional normal distribution, see pages from 126 to 130 in [14].

Proposition 1.3. *Let Z_n be a random variable that has $N_{\mathbb{R}}(\mu_n, \Sigma_n)$ distribution. Then, Z_n converges in distribution to a random variable Z with $N_{\mathbb{R}}(\mu, \Sigma)$ distribution if and only if the sequences $(\mu_n)_n$ and $(\Sigma_n)_n$ converge respectively to μ and Σ. In other words, weak limits of real Gaussians are real Gaussians.*

Proof. Each element of $(Z_n)_n$ has $N_{\mathbb{R}}(\mu_n, \Sigma_n)$ distribution. Then, the characteristic function of Z_n is, for all s of $\mathbb{R}$:

$$\varphi_{Z_n}(s) = e^{is\mu_n - \frac{1}{2}s^2\Sigma_n}.$$

If we suppose that $(\mu_n)_n$ and $(\Sigma_n)_n$ converge respectively to μ and Σ, we get:

$$\varphi_{Z_n}(s) = e^{is\mu_n - \frac{1}{2}s^2\Sigma_n} \overset{n \to +\infty}{\longrightarrow} \varphi_Z(s) = e^{is\mu - \frac{1}{2}s^2\Sigma}.$$

The characteristic function is continuous at the point 0. By Paul-Lévy's theorem, the sequence $(Z_n)_n$ converges in distribution to Z and this last random variable has $N_{\mathbb{R}}(\mu, \Sigma)$ distribution.
For the converse, if the sequence of random variables $(Z_n)_n$ converges in distribution to the random variable Z, and this last one has $N_{\mathbb{R}}(\mu, \Sigma)$ distribution, then their characteristic functions converge; that is:

$$\varphi_{Z_n}(s) = e^{is\mu_n - \frac{1}{2}s^2\Sigma_n} \overset{n\to+\infty}{\longrightarrow} \varphi_Z(s) = e^{is\mu - \frac{1}{2}s^2\Sigma},$$

for all real s. Therefore the sequences $(\mu_n)_n$ and $(\Sigma_n)_n$ respectively converge to μ and Σ.

□

Thus we have the following result with complex Gaussian.

Proposition 1.4. *Let Z_n be a random variable that has $N_{\mathbb{C}}(\mu_n, \Sigma_n)$ distribution. Then, Z_n converges in distribution to a random variable Z with $N_{\mathbb{C}}(\mu, \Sigma)$ distribution if and only if the sequences $(\mu_n)_n$ and $(\Sigma_n)_n$ converge respectively to μ and Σ. In other words, weak limits of complex Gaussians are complex Gaussians.*

Proposition 1.5. *Let Z be an n-dimensional random vector that has $N_{\mathbb{R}}(\mu, \Sigma)$ distribution. If A is a matrix of order $m \times n$, then the random vector AZ has $N_{\mathbb{R}}(A\mu, A\Sigma A^t)$ distribution.*

Proof. For all m-dimensional real vector s we have:

$$\varphi_{AZ}(s) = \varphi_Z(A^t s) = e^{is^t A\mu - \frac{1}{2}(A^t s)^t \Sigma (A^t s)} = e^{is^t(A\mu) - \frac{1}{2}s^t(A\Sigma A^t)s}.$$

Then AZ has $N_{\mathbb{R}}(A\mu, A\Sigma A^t)$ distribution, because it is the characteristic function of a random vector that has $N_{\mathbb{R}}(A\mu, A\Sigma A^t)$ distribution. □

The last proposition assures us that if Z is an n-dimensional random vector such that all linear combination of its components has normal distribution, then Z has a multidimensional normal distribution.
With this, we have the analogous result for complex random vectors with the same remark.

Proposition 1.6. *Let Z be an n-dimensional random vector that has $N_{\mathbb{C}}(\mu, \Sigma)$ distribution. If A is a matrix of order $m \times n$, then the random vector AZ has $N_{\mathbb{C}}(A\mu, A\Sigma A^*)$ distribution.*

Proposition 1.7. *The mean and the covariance of a real Gaussian random vector determines its distribution.*

Proof. Let Y be n - dimensional random vector such that $Y = AX + \mu$, where A is a matrix of order n, μ an n - dimensional vector and X a random vector that has $N_{\mathbb{R}}(0, Id)$ distribution. We want to see that Y has a normal distribution with mean μ and covariance matrix $\Sigma = AA^t$. Using characteristic functions, we have, for any n - dimensional real vector s:

$$\varphi_Y(s) = \varphi_{AX+\mu}(s) = \mathbb{E}\left[e^{is^t(AX+\mu)}\right] = e^{is^t\mu}\varphi_{AX}(s) \overset{(*)}{=} e^{is^t\mu}e^{-\frac{1}{2}s^tAIdA^ts} = e^{is^t\mu-\frac{1}{2}s^t\Sigma s},$$

where in $(*)$ we use Proposition 1.5. Then Y has a $N_{\mathbb{R}}(\mu, \Sigma)$ distribution. □

Thus, by this proposition we have:

Proposition 1.8. *The mean and the covariance of a complex Gaussian random vector determines its distribution.*

1.2 Gaussian analytic functions

To introduce and prove all the statements of this section we must consider the space of analytic functions on a region $\Omega \subset \mathbb{C}$ with the topology of uniform convergence on compact sets. Let be $\mathcal{A}(\Omega)$ the space before described. Now, given f, g of $\mathcal{C}(\Omega)$ (the space of continuous functions in Ω) and following [7], we can define the norm:

$$d_K(f,g) :=\| f - g \|_{L^\infty(K)}:= \sup_{z\in K} |f(z) - g(z)|,$$

and the metric:

$$\rho(f,g) := \sum_{n=1}^{+\infty} \frac{1}{2^n} \frac{d_{K_n}(f,g)}{1 + d_{K_n}(f,g)},$$

where K and K_n are compact sets of Ω. In addition, it exists a sequence of compact sets $(K_n)_n$ of Ω such that $\Omega = \bigcup_{n=1}^{+\infty} K_n$ and $K_n \subset \overset{\circ}{K}_{n+1} \subset K_{n+1}$.
We have the next:

Theorem 1.9. *The space $(\mathcal{A}(\Omega), \rho)$ is a separable, complete and metric space.*

For the proof, Corollary 2.3, page 152, in [7] gives us that $(\mathcal{A}(\Omega), \rho)$ is a complete metric space since $(\mathcal{C}(\Omega), \rho)$ also is (for the last statement see Proposition 1.12. in [7], page 145). Separability is accomplished in disks or annulus due to the uniform convergence on compact sets of Ω of the Taylor series and Laurent ones, respectively. For other spaces, Runge's theorem is required (see page 198 in [7]).

Definition 1.10. *Let f be a random variable on a probability space taking values in the space of analytic functions on a region Ω of $\mathbb{C}$. We say that f is a* Gaussian analytic function *(or GAF, for short) on Ω if the random vector $(f(z_1), ..., f(z_n))$ has a mean zero complex Gaussian distribution for every $n \geq 1$ and every $z_1, ..., z_n$ of Ω. In addition, the components of the last random vector are not necessarily independent.*

As a remark, for any $z_1, ..., z_n$ of Ω for all $n \geq 1$, the random vector $(f(z_1), ..., f(z_n))$ has $N^n_{\mathbb{C}}(0, \Sigma)$ distribution, where Σ is the covariance matrix $K(z_i, z_j)$ for all $i, j \leq n$. Then, by Proposition 1.8, the covariance kernel K determines the distribution of f.
An interesting question to formulate is how can we generate Gaussian analytic functions. The next result addresses this problem.

Proposition 1.11. *Let $(f_n)_n$ be a sequence of analytic functions in Ω and let $(\zeta_n)_n$ be i.i.d. standard complex Gaussian random variables, that is, each ζ_n follows a $N_{\mathbb{C}}(0,1)$ distribution. If $\sum_{n=1}^{+\infty} |f_n(z)|^2$ converges uniformly on compact sets on Ω, then $f(z) = \sum_{n=1}^{+\infty} \zeta_n f_n(z)$ converges uniformly almost surely on compact sets on Ω and f defines a GAF. Furthermore, f has covariance kernel $\mathcal{K}_f(z, w) = \sum_{n=1}^{+\infty} f_n(z)\overline{f_n(w)}$.*

Lemma 1.12. (Kolmogorov's inequality) *Under the same hypothesis than the last proposition, and assuming that K is a compact subset of Ω and the random variable $X_n = \sum_{k=1}^{n} \zeta_k f_k$ is defined in the Lebesgue space $L^2(K)$, it holds the inequality:*

$$\mathbb{P}\left(\sup_{1\leq j\leq n} \| X_j \|_{L^2(K)} \geq \varepsilon\right) \leq \frac{1}{\varepsilon^2} \sum_{j=1}^{n} \| f_j \|^2_{L^2(K)},$$

for a given $\varepsilon > 0$.

Proof. (see proof of Lemma 2.2.3. in [2], pages 16 and 17) We define, for all $k < n$, the set $Z = \{\zeta_j \;:\; j \leq k\}$. It holds that:

$$\mathbb{E}\left[\| X_n \|^2_{L^2(K)} \mid Z\right] = \| X_k \|^2_{L^2(K)} + \sum_{j=k+1}^{n} \| f_j \|^2_{L^2(K)} .$$

Indeed,

$$\mathbb{E}\left[\| X_n \|^2_{L^2(K)} \mid Z\right] \overset{(*)}{=} \mathbb{E}\left[\sum_{j=1}^{n} \| \zeta_j f_j \|^2_{L^2(K)} \mid Z\right] \tag{1.1}$$

$$= \mathbb{E}\left[\sum_{j=1}^{k} \| \zeta_j f_j \|^2_{L^2(K)} + \sum_{j=k+1}^{n} \| \zeta_j f_j \|^2_{L^2(K)} \mid Z\right] \tag{1.2}$$

$$= \mathbb{E}\left[\sum_{j=1}^{k} \| \zeta_j f_j \|^2_{L^2(K)} \mid Z\right] + \mathbb{E}\left[\sum_{j=k+1}^{n} \| \zeta_j f_j \|^2_{L^2(K)} \mid Z\right] \tag{1.3}$$

$$= \sum_{j=1}^{k} \| \zeta_j f_j \|^2_{L^2(K)} + \sum_{j=k+1}^{n} \left(\| f_j \|^2_{L^2(K)} \underbrace{\mathbb{E}\left[|\zeta_j|^2\right]}_{1}\right) \tag{1.4}$$

$$= \sum_{j=1}^{k} \| \zeta_j f_j \|^2_{L^2(K)} + \sum_{j=k+1}^{n} \| f_j \|^2_{L^2(K)} \tag{1.5}$$

$$= \| X_k \|^2_{L^2(K)} + \sum_{j=k+1}^{n} \| f_j \|^2_{L^2(K)} . \tag{1.6}$$

In $(*)$ we use the independence of $(\zeta_n)_n$. The first sum of (1.4) is due to the next property: if Y is an integrable random variable, then $\mathbb{E}\left[Y|Y\right] = Y$. The second sum of the same line is given because the sequence $(\zeta_j)_{j>k}$ is independent of the sequence $(\zeta_j)_{j\leq k}$. Now, defining the stopping time $\tau = \inf\{n \in \mathbb{N} \, : \| X_n \|_{L^2(K)} > \varepsilon\}$, we have that:

$$\mathbb{E}\left[\| X_n \|^2_{L^2(K)}\right] \geq \sum_{k=1}^{n} \mathbb{E}\left[\| X_n \|^2_{L^2(K)} \mathbb{1}_{\{\tau=k\}}\right] = \sum_{k=1}^{n} \mathbb{E}\left[\mathbb{E}\left[\| X_n \|^2_{L^2(K)} \mathbb{1}_{\{\tau=k\}} \mid Z\right]\right] \tag{1.7}$$

$$= \sum_{k=1}^{n} \mathbb{E}\left[\left(\| X_k \|^2_{L^2(K)} + \sum_{j=k+1}^{n} \| f_j \|^2_{L^2(K)}\right) \mathbb{1}_{\{\tau=k\}}\right] \tag{1.8}$$

$$= \sum_{k=1}^{n} \left(\mathbb{E}\left[\| X_k \|^2_{L^2(K)} \mathbb{1}_{\{\tau=k\}}\right] + \underbrace{\mathbb{E}\left[\sum_{j=k+1}^{n} \| f_j \|^2_{L^2(K)} \mathbb{1}_{\{\tau=k\}}\right]}_{\geq 0}\right) \tag{1.9}$$

$$\geq \sum_{k=1}^{n} \mathbb{E}\left[\| X_k \|^2_{L^2(K)} \mathbb{1}_{\{\tau=k\}}\right] \geq \sum_{k=1}^{n} \varepsilon^2 \mathbb{E}\left[\mathbb{1}_{\{\tau=k\}}\right] = \varepsilon^2 \sum_{k=1}^{n} \mathbb{P}\left(\{\tau = k\}\right) \tag{1.10}$$

$$= \varepsilon^2 \mathbb{P}\left(\biguplus_{k=1}^{n} \{\tau = k\}\right) = \varepsilon^2 \mathbb{P}\left(\tau \leq n\right). \tag{1.11}$$

The equality of line (1.7) maintains because, given integrable random variables X, Y, it satisfies the equation $\mathbb{E}\left[\mathbb{E}\left[X|Y\right]\right] = \mathbb{E}\left[X\right]$. Thus,

$$\mathbb{P}\left(\sup_{1\leq j\leq n} \| X_j \|_{L^2(K)} \geq \varepsilon\right) \leq \frac{1}{\varepsilon^2} \sum_{j=1}^{n} \| f_j \|^2_{L^2(K)} .$$

□

Proof of Proposition 1.11 (see proof of Lemma 2.2.3. in [2], pages from 16 to 18). First of all, we must prove that X_n is a Cauchy sequence in $L^2(K)$ almost surely. For this, we can see that, for a given natural n_0 such that $n, m \geq n_0$:

$$\mathbb{P}\left[\sup_{n,m\geq n_0} \| X_m - X_n \|_{L^2(K)} \geq 2\varepsilon\right] \overset{n_0 \to +\infty}{\longrightarrow} 0.$$

Indeed, take the sequence $(X_{n_0+n} - X_{n_0})_n$. We have

$$\begin{aligned}
\mathbb{P}\left[\sup_{n,m\geq n_0} \| X_m - X_n \|_{L^2(K)} \geq 2\varepsilon\right] &\leq \mathbb{P}\left[\sup_{n\geq 1} \| X_{n_0+n} - X_{n_0} \|_{L^2(K)} \geq \varepsilon\right] \\
&= \lim_{k\to\infty} \mathbb{P}\left[\sup_{1\leq n\leq k} \| X_{n_0+n} - X_{n_0} \|_{L^2(K)} \geq \varepsilon\right] \\
&\overset{(*)}{\leq} \lim_{k\to\infty} \frac{1}{\varepsilon^2} \sum_{j=n_0+1}^{n_0+k} \| f_j \|^2_{L^2(K)} = \frac{1}{\varepsilon^2} \sum_{j=n_0+1}^{+\infty} \| f_j \|^2_{L^2(K)},
\end{aligned}$$

which the last expression tends to zero when n_0 tends to infinite and the last lemma is used in $(*)$. Therefore it exists a natural number n_0 such that for all $n \geq n_0$, the sequence $(X_{n_0+n} - X_{n_0})_n$ satisfies

$$\mathbb{P}\left[\| X_{n_0+n} - X_{n_0} \|_{L^2(K)} \leq \varepsilon\right] = 1,$$

and X_n is a Cauchy sequence in $L^2(K)$. An important remark is that K is not fixed, it can be any compact set of Ω. Indeed, by an exhaustive sequence of compact sets of Ω, the compact K_n contains the sequence of compact sets $(K_i)_{i<n}$ for all natural n. If we apply the result we just proved to K_n, the property remains true with any compact of the sequence $(K_i)_{i<n}$ for all natural n.

For the uniform convergence on compact sets of Ω, we have to consider the disk $\mathbb{D}(z_0, 4R)$ on Ω. Since the sequence $(f_n)_n$ is analytic, X_n is also analytic on Ω for all natural n. Therefore, by Cauchy's integral formula:

$$X_n(z) = \frac{1}{2\pi i} \int_{|w-z_0|=r} \frac{X_n(w)}{w-z} dw,$$

where $w = z_0 + re^{i\theta}$, $|z - z_0| < r$, $0 \leq \theta \leq 2\pi$ and $2R < r < 3R$. For a given z of the disk $\mathbb{D}(z_0, R)$, consider the annulus $A = A(z_0; 2R, 3R)$. Then

$$\begin{aligned} X_n(z) &= \frac{1}{2\pi i} \int_{|w-z_0|=r} \frac{X_n(w)}{w-z} dw = \frac{1}{2\pi i R} \int_{2R}^{3R} \int_{|w-z_0|=r} \frac{X_n(w)}{w-z} dw dr \\ &= \frac{1}{2\pi i R} \int_{2R}^{3R} \int_0^{2\pi} \frac{X_n(z_0 + re^{i\theta})}{z_0 - z + re^{i\theta}} ire^{i\theta} d\theta dr = \frac{1}{2\pi} \int_A X_n(w)\phi_z(w) dm(w), \end{aligned}$$

where $dm(w) = r d\theta dr$ and

$$\phi_z(w) = \frac{e^{i\theta}}{R(w-z)}.$$

Observe that the collection $\{\phi_z\}_z$ is uniformly bounded in the Lebesgue space $L^2(A)$. Indeed:

$$\begin{aligned} \| \phi_z(w) \|_{L^2(A)}^2 &= \int_A \frac{1}{R^2} \frac{1}{|w-z|^2} dm(w) = \frac{1}{R^2} \int_{2R}^{3R} \int_0^{2\pi} \frac{r}{|w-z|^2} d\theta dr \\ &\leq \frac{1}{R^2} \int_{2R}^{3R} \int_0^{2\pi} \frac{r}{R^2} d\theta dr = \frac{2\pi}{R^4} \int_{2R}^{3R} r dr = \frac{2\pi}{R^4} \frac{5R^2}{2} = \frac{5\pi}{R^2}, \end{aligned}$$

where the inequality above is given by $|z_0 - z + re^{i\theta}|^2 \geq R^2$, recalling the fact that z belongs to $\mathbb{D}(z_0, R)$ and $2R < r < 3R$. Thus

$$\| \phi_z(w) \|_{L^2(A)} \leq \frac{\sqrt{5\pi}}{R}.$$

Now, let K be the disk $\mathbb{D}(z_0, 4R)$. As we saw before, X_n is a Cauchy sequence in $L^2(K)$. Then it exists a random variable X of $L^2(K)$ such that $\| X - X_n \|_{L^2(K)}$ tends to zero when n tends to infinite and

$$\frac{1}{2\pi} \int_A X_n(w)\phi_z(w) dm(w) \overset{n\to+\infty}{\longrightarrow} \frac{1}{2\pi} \int_A X(w)\phi_z(w) dm(w)$$

uniformly for all z of $\mathbb{D}(z_0, R)$. Indeed, for all $\varepsilon > 0$ there exists a natural n_0 such that for all $n, m \geq n_0$ it holds $\| X_n - X_m \|_{L^2(A)} < \varepsilon$. Then, we can use the last n_0 to state that:

$$\sup_{z \in \mathbb{D}(z_0,R)} |X_n(z) - X_m(z)| \leq \| X_n - X_m \|_{L^2(A)} \| \phi_z \|_{L^2(A)} \leq \varepsilon \frac{\sqrt{5\pi}}{R}.$$

Since ε is arbitrary, we can tend ε to zero and the uniform convergence on $\mathbb{D}(z_0, R)$ is satisfied.
Therefore X_n converges uniformly to X on compact sets of Ω almost surely, and X is an analytic function on Ω by Weierstrass' theorem.
By hypothesis, $(\zeta_n)_n$ are complex Gaussian with mean zero, therefore X_n is also a complex Gaussian with mean zero, by Proposition 1.6, and defines a GAF by definition. Since limits of complex Gaussians are complex Gaussians, by Proposition 1.4, X is also a GAF.
For the covariance formula, we have:

$$\begin{aligned}
\mathcal{K}_f(z,w) &= \mathbb{E}\left[f(z)\overline{f(w)}\right] = \mathbb{E}\left[\left(\sum_{n=1}^{+\infty} \zeta_n f_n(z)\right)\left(\sum_{n=1}^{+\infty} \overline{\zeta_n f_n(w)}\right)\right] \\
&= \mathbb{E}\left[\sum_{n=1}^{+\infty}\sum_{k=1}^{n} \zeta_k f_k(z)\overline{\zeta_{n-k} f_{n-k}(w)}\right] = \sum_{n=1}^{+\infty}\sum_{k=1}^{n} \mathbb{E}\left[\zeta_k f_k(z)\overline{\zeta_{n-k} f_{n-k}(w)}\right] \\
&= \sum_{n=1}^{+\infty}\sum_{k=1}^{n} f_k(z)\overline{f_{n-k}(w)}\mathbb{E}\left[\zeta_k\overline{\zeta_{n-k}}\right] \overset{(*)}{=} \sum_{m=1}^{+\infty} f_m(z)\overline{f_m(w)},
\end{aligned}$$

where in $(*)$ it holds the following. If $k = n-k$, then $\mathbb{E}\left[\zeta_k\overline{\zeta_k}\right] = \mathbb{E}\left[|\zeta_k|^2\right] = 1$. Otherwise, if $k \neq n-k$, then, by the independence of $(\zeta_n)_n$ and remembering that $\zeta_n \sim N_{\mathbb{C}}(0,1)$, we have that $\mathbb{E}\left[\zeta_k\overline{\zeta_{n-k}}\right] = \mathbb{E}\left[\zeta_k\right]\mathbb{E}\left[\overline{\zeta_{n-k}}\right] = 0$. Thus

$$\mathcal{K}_f(z,w) = \sum_{m=1}^{+\infty} f_m(z)\overline{f_m(w)}.$$

□

We notice that in Proposition 1.11 there is no restriction about which space we must determine a GAF. Although, in this project we will calculate them in Hilbert spaces, concretely in Hilbert spaces of analytic functions on a region Ω. We denote this space as $\mathcal{H}$.
We have the next result:

Proposition 1.13. *Consider the space $\mathcal{H}$ as described before. Let $(f_n)_n$ be an orthonormal basis of $\mathcal{H}$ and assume that for all z of Ω, there exists a positive constant C_z such that $|f(z)| \leq C_z \| f \|_{\mathcal{H}}$ for all function f of $\mathcal{H}$. Moreover, assume that the norm is continuous at z. Then $\sum_{n=1}^{+\infty} |f_n(z)|^2$ converges uniformly on compact sets of Ω.*

Proof. For all z of Ω we consider the punctual evaluation operator at z, defined as:

$$\begin{aligned}
E_z : \mathcal{A}(\Omega) &\longrightarrow \Omega \\
f &\longmapsto E_z(f) = f(z)
\end{aligned}$$

This operator is linear and continuous. Indeed, for any f and g of $\mathcal{H}$ we have

$$E_z(f+g) = (f+g)(z) = f(z) + g(z) = E_z(f) + E_z(g),$$

and for a complex scalar λ we get

$$E_z(\lambda f) = (\lambda f)(z) = \lambda f(z) = \lambda E_z(f).$$

The continuity is straightforward because $|f(z)| \leq C_z \parallel f \parallel_{\mathcal{H}}$. Indeed:

$$|E_z(f)| = |f(z)| \leq C_z \parallel f \parallel_{\mathcal{H}} .$$

Thus $\parallel E_z \parallel_{\mathcal{H}} \leq C_z$ and the operator is continuous.
For the convergence result, we have to see that:

$$\parallel E_z \parallel_{\mathcal{H}}^2 = \sum_{n=1}^{+\infty} |f_n(z)|^2.$$

Since $(f_n)_n$ is an orthonormal basis, by Fischer - Riesz theorem:

$$f = \sum_{n=1}^{+\infty} (f, f_n)_{\mathcal{H}} f_n,$$

and Parseval's equality assures that

$$\parallel f \parallel_{\mathcal{H}}^2 = \sum_{n=1}^{+\infty} |(f, f_n)_{\mathcal{H}}|^2.$$

We have that:

$$\begin{aligned} |E_z(f)|^2 = |f(z)|^2 = \left| \sum_{n=1}^{+\infty} (f, f_n)_{\mathcal{H}} f_n(z) \right|^2 &\leq \sum_{n=1}^{+\infty} |(f, f_n)_{\mathcal{H}}|^2 \sum_{n=1}^{+\infty} |f_n(z)|^2 \\ &= \parallel f \parallel_{\mathcal{H}}^2 \sum_{n=1}^{+\infty} |f_n(z)|^2. \end{aligned}$$

Then

$$|E_z(f)| \leq \parallel f \parallel_{\mathcal{H}} \left(\sum_{n=1}^{+\infty} |f_n(z)|^2 \right)^{\frac{1}{2}}.$$

Therefore:

$$\parallel E_z \parallel_{\mathcal{H}}^2 \leq \sum_{n=1}^{+\infty} |f_n(z)|^2.$$

For the other inequality, we consider the function $g(\cdot) = \sum_{n=1}^{+\infty} \overline{f_n(z)} f_n(\cdot)$. We have that

$$\parallel g \parallel_{\mathcal{H}}^2 = \sum_{n=1}^{+\infty} |f_n(z)|^2 \text{ and } g(z) = \sum_{n=1}^{+\infty} |f_n(z)|^2.$$

Knowing the fact that $\| E_z \|_{\mathcal{H}}$ is the minimum constant value that holds:

$$|g(z)| \leq \| E_z \|_{\mathcal{H}} \| g \|_{\mathcal{H}},$$

thus:

$$\sum_{n=1}^{+\infty} |f_n(z)|^2 \leq \| E_z \|_{\mathcal{H}} \left(\sum_{n=1}^{+\infty} |f_n(z)|^2 \right)^{\frac{1}{2}},$$

and we arrive to the other inequality:

$$\sum_{n=1}^{+\infty} |f_n(z)|^2 \leq \| E_z \|_{\mathcal{H}}^2 .$$

□

Since E_z is a bounded linear functional then, by Riesz's representation theorem, it exists a unique function K_z of $\mathcal{H}$ for all z of Ω such that

$$f(z) = (f, K_z)_{\mathcal{H}},$$

for all f of $\mathcal{H}$. The function K_z is the reproducing kernel of $\mathcal{H}$ at z.
Following [13] (page 375), if we start with a Hilbert space $\mathcal{H}$ of analytic functions on a region Ω with reproducing kernel K_z for all z of Ω, we can compute a GAF as

$$f(z) = \sum_{n=1}^{+\infty} \zeta_n f_n(z),$$

where the sequence $(f_n)_n$ is an orthonormal basis of $\mathcal{H}$ and the elements of the sequence $(\zeta_n)_n$ are random variables with $N_{\mathbb{C}}(0,1)$ distribution. Furthermore, the GAF has covariance kernel

$$\mathcal{K}_f(z,w) = \sum_{n=1}^{+\infty} f_n(z)\overline{f_n(w)}.$$

This construction does not depend of the chosen basis of $\mathcal{H}$. For this we will prove that $K_z = \mathcal{K}_f$ or, similarly, that $(f, K_z)_{\mathcal{H}} = (f, \mathcal{K}_f)_{\mathcal{H}}$, for any function f of $\mathcal{H}$. We know that $(f, K_z)_{\mathcal{H}} = f(z)$ for z of Ω. On the other hand, for any w of Ω, we have that:

$$(f, \mathcal{K}_f(z,w))_{\mathcal{H}} = \left(f, \sum_{n=1}^{+\infty} f_n(z)\overline{f_n(w)} \right)_{\mathcal{H}} = \sum_{n=1}^{+\infty} f_n(z)(f, f_n)_{\mathcal{H}} = f(z),$$

where in the last equation we use the Fischer - Riesz theorem by the fact that $(f_n)_n$ is an orthonormal basis. Therefore we got the equality we wanted.
Now, as a curiosity, there is an interesting theorem in the theory of reproducing kernels on Hilbert spaces. We proved before that for a given Hilbert space we can determine the reproducing kernel, but the converse is also true, as is explained in the paper [1] by Aronszajn:

Theorem 1.14. (Moore - Aronszajn's theorem) *Let K be a hermitian, positive definite covariance kernel on a region Ω. If for a given complex values z and w in Ω, $K(z,w)$ is holomorphic in z and anti-holomorphic in w and $K(z,w)$ is bounded on compact sets of Ω, then there is a unique Hilbert space $\mathcal{H}$ of holomorphic functions in Ω such that K is its reproducing kernel.*

Chapter 2

Isometry-invariant zero sets

In this chapter we will use the concepts explained before in four Hilbert spaces of functions on different domains, and those domains are the complex plane, the Riemann sphere and the hyperbolic plane. For each one of them we will concrete the collection of Hilbert spaces given by a parameter, prove that the chosen basis is an orthonormal one, compute the covariance kernel and see that each GAF is invariant under suitable transformations.
For results about isometry-invariant zero sets consult Section 2.3. in Chapter 2 of [2].

2.1 The complex plane $\mathbb{C}$

Fixing a real parameter $L > 0$, we consider the Fock space, defined as

$$\mathcal{F}_L := \left\{ f \in \mathcal{A}(\mathbb{C}) \ : \| \ f \ \|^2_{\mathcal{F}_L} = \frac{L}{\pi} \int_{\mathbb{C}} |f(z)|^2 \, e^{-L|z|^2} dm(z) < +\infty \right\}.$$

Let us denote:

$$e_n = \frac{\sqrt{L^n}}{\sqrt{n!}} z^n.$$

Then we have that $(e_n)_{n=0}^{+\infty}$ is an orthonormal basis of $\mathcal{F}_L$. We must see that $(e_n, e_n)_{\mathcal{F}_L} = 1$ and $(e_n, e_m)_{\mathcal{F}_L} = 0$, for all $n \neq m$.
On one hand we have:

$$\begin{aligned}(e_n, e_n)_{\mathcal{F}_L} &= \frac{L}{\pi} \int_{\mathbb{C}} \frac{L^n |z|^{2n}}{n!} e^{-L|z|^2} dm(z) = 2 \frac{L^{n+1}}{n!} \int_0^{+\infty} r^{2n+1} e^{-Lr^2} dr \\ &= \frac{L^{n+1}}{n!} \int_0^{+\infty} r^{2n} e^{-Lr^2} 2r dr \overset{(*)}{=} \frac{L^{n+1}}{n!} \int_0^{+\infty} \frac{t^n}{L^{n+1}} e^{-t} dt \overset{(**)}{=} \frac{L^{n+1}}{n!} \frac{\Gamma(n+1)}{L^{n+1}} = 1,\end{aligned}$$

where in $(*)$ we use the change of variables $t = Lr^2$ and $dt = 2Lrdr$; in $(**)$ we observe that $\int_0^\infty t^n e^{-t} dt = n! = \Gamma(n+1)$. On the other hand, for $n \neq m$:

$$\begin{aligned}(e_n, e_m)_{\mathcal{F}_L} &= \frac{L}{\pi} \int_{\mathbb{C}} \frac{\sqrt{L^n}}{\sqrt{n!}} \frac{\sqrt{L^m}}{\sqrt{m!}} z^n \overline{z}^m e^{-L|z|^2} dm(z) \\ &= \frac{L}{\pi} \frac{\sqrt{L^n}}{\sqrt{n!}} \frac{\sqrt{L^m}}{\sqrt{m!}} \int_0^{2\pi} \int_0^{+\infty} r^{n+m+1} e^{i\theta(n-m)} e^{-Lr^2} dr d\theta.\end{aligned}$$

However:

$$\int_0^{2\pi} e^{i\theta(n-m)} d\theta = \left[\frac{e^{i\theta(n-m)}}{i(n-m)} \right]_{\theta=0}^{\theta=2\pi} = 0.$$

Then $(e_n, e_m)_{\mathcal{F}_L} = 0$. For completeness, if for a given function f of $\mathcal{F}_L$ it holds $(f, e_n)_{\mathcal{F}_L} = 0$, then $f \equiv 0$. Following a procedure in [22], we can expand f as a Taylor power series:

$$f(z) = \sum_{m=0}^{+\infty} c_m z^m,$$

and we know that this series converges uniformly over compact subsets of $\mathbb{C}$. We have:

$$\begin{aligned}
(f, e_n)_{\mathcal{F}_L} &= \frac{L\sqrt{L^n}}{\pi\sqrt{n!}} \int_{\mathbb{C}} f(z)\overline{z}^n e^{-L|z|^2} dm(z) \\
&= \frac{L\sqrt{L^n}}{\pi\sqrt{n!}} \lim_{R\to+\infty} \int_0^R \int_0^{2\pi} f(re^{i\theta}) r^{n+1} e^{-in\theta} e^{-Lr^2} d\theta dr \\
&= \frac{L\sqrt{L^n}}{\pi\sqrt{n!}} \lim_{R\to+\infty} \int_0^R \int_0^{2\pi} \sum_{m=0}^{+\infty} c_m r^{n+m+1} e^{i\theta(m-n)} e^{-Lr^2} d\theta dr \\
&= \frac{L\sqrt{L^n}}{\pi\sqrt{n!}} \lim_{R\to+\infty} \sum_{m=0}^{+\infty} \int_0^R \int_0^{2\pi} c_m r^{n+m+1} e^{i\theta(m-n)} e^{-Lr^2} d\theta dr \\
&= 2\frac{L\sqrt{L^n}}{\sqrt{n!}} c_n \lim_{R\to+\infty} \int_0^R r^{2n+1} e^{-Lr^2} dr = 0,
\end{aligned}$$

where the commutativity of the integrals with the sum is due to the uniform convergence of the power series of f over the compact subsets of $\mathbb{C}$. Therefore, if $(f, e_n)_{\mathcal{F}_L} = 0$, the value c_n is the unique one that vanishes. Hence $f \equiv 0$, as we wanted to see. Thus $(e_n)_{n=0}^{+\infty}$ is an orthonormal basis of $\mathcal{F}_L$.

Now by Propositions 1.13 and 1.11 we define the GAF:

$$f(z) = \sum_{n=0}^{+\infty} \zeta_n \frac{\sqrt{L^n}}{\sqrt{n!}} z^n,$$

for all real $L > 0$. Then f has covariance kernel:

$$\mathcal{K}_f(z, w) = \sum_{n=0}^{+\infty} \frac{L^n}{n!} z^n \overline{w}^n = \sum_{n=0}^{+\infty} \frac{(Lz\overline{w})^n}{n!} = e^{Lz\overline{w}}.$$

Once we have calculated this, we are going to see that the functions of the Fock space are invariant under translations, and the points of the function and its translated are almost equal in distribution, noted as $\stackrel{d}{=}$. More precisely:

Proposition 2.1. *Let f be a GAF in $\mathcal{F}_L$ over the complex plane $\mathbb{C}$. The point sets of f are invariant under the transformation*

$$\varphi_a(z) = z - a,$$

where z and a are complex values.

Proof. By hypothesis:

$$f(z) = \sum_{n=0}^{+\infty} \zeta_n \frac{\sqrt{L^n}}{\sqrt{n!}} z^n.$$

We know that the covariance kernel of f is

$$\mathcal{K}_f(z,w) = e^{Lz\overline{w}}.$$

If $f_a(z) = f(\varphi_a(z))$, f_a has covariance kernel:

$$\mathcal{K}_{f_a}(z,w) = \mathcal{K}_f(\varphi_a(z), \varphi_a(w)) = e^{L(z-a)\overline{(w-a)}} = e^{Lz\overline{w}-Lz\overline{a}-La\overline{w}+L|a|^2}.$$

Now, we have that:

$$f(z) \overset{d}{=} f_a(z) e^{Lz\overline{a}-\frac{L}{2}|a|^2}.$$

To show this, if we denote

$$T_a f(z) = f_a(z) e^{Lz\overline{a}-\frac{L}{2}|a|^2},$$

we must prove that

$$\mathcal{K}_f(z,w) = \mathcal{K}_{T_a f}(z,w).$$

Indeed:

$$\begin{aligned}\mathcal{K}_{T_a f}(z,w) &= \mathcal{K}_{f_a}(z,w) e^{Lz\overline{a}-\frac{L}{2}|a|^2} e^{La\overline{w}-\frac{L}{2}|a|^2} = e^{Lz\overline{w}-Lz\overline{a}-La\overline{w}-L|a|^2} e^{Lz\overline{a}+La\overline{w}+L|a|^2} = e^{Lz\overline{w}} \\ &= \mathcal{K}_f(z,w).\end{aligned}$$

Therefore:

$$f(z) \overset{d}{=} T_a f(z).$$

□

Proposition 2.2. *Using the same notation as in the last proof, f and $T_a f$ are isometric, that is:*

$$\| f \|^2_{\mathcal{F}_L} = \| T_a f \|^2_{\mathcal{F}_L}.$$

Proof. Indeed, we have that:

$$\| T_a f \|^2_{\mathcal{F}_L} = \frac{L}{\pi} \int_{\mathbb{C}} |f(z-a)|^2 \left| e^{L\overline{a}z-\frac{L}{2}|a|^2} \right|^2 e^{-L|z|^2} dm(z).$$

The exponential factor can be written as:

$$\left| e^{L\overline{a}z-\frac{L}{2}|a|^2} \right|^2 = \left(e^{L\overline{a}z-\frac{L}{2}|a|^2} \right) \overline{\left(e^{L\overline{a}z-\frac{L}{2}|a|^2} \right)} = \left(e^{L\overline{a}z-\frac{L}{2}|a|^2} \right) \left(e^{La\overline{z}-\frac{L}{2}|a|^2} \right) = e^{L\overline{a}z+La\overline{z}-L|a|^2}.$$

Returning to the integral and applying the change $w = z - a$ and $dm(w) = dm(z)$:

$$\begin{aligned}\| T_a f \|^2_{\mathcal{F}_L} &= \frac{L}{\pi} \int_{\mathbb{C}} |f(z-a)|^2 \left| e^{L\overline{a}z-\frac{L}{2}|a|^2} \right|^2 e^{-L|z|^2} dm(z) \\ &= \frac{L}{\pi} \int_{\mathbb{C}} |f(w)|^2 e^{L\overline{a}(w+a)+La(\overline{w}+\overline{a})-L|a|^2-L|w+a|^2} dm(w) \\ &= \frac{L}{\pi} \int_{\mathbb{C}} |f(w)|^2 e^{L\overline{a}(w+a)+La(\overline{w}+\overline{a})-L|a|^2-L|w|^2-Lw\overline{a}-La\overline{w}-L|a|^2} dm(w) \\ &= \frac{L}{\pi} \int_{\mathbb{C}} |f(w)|^2 e^{-L|w|^2} dm(w) = \| f \|^2_{\mathcal{F}_L}.\end{aligned}$$

□

As a remark, let f be a function of the Fock space. If f is a GAF, $T_a f$ is also a GAF, since it is a composition of f with the translation φ_a and multiplied by the corrector factor $\eta_a(z) = e^{Lz\bar{a}-\frac{L}{2}|a|^2}$. The factor η_a is a deterministic one. It is a term that we impose to get the equality of covariance kernels of f and $T_a f$ and to obtain the isometric property. Moreover, the last results assures us the equality in distribution of f and $T_a f$ for every point z of the region Ω, and that includes the zero sets of both functions. In this case it is simple, because since the exponential function does not vanish at any point, f and f_a have the same zero sets in distribution. In addition, the theorems to find zeros with meromorphic functions such that Rouché's theorem or the Argument Principle can be also applied to $T_a f$. As a conclusion, the last proved results are powerful tools to find zero sets of functions. Indeed, if f is a GAF whose zeros are tough to determine, then, with a translation to the origin, for example, we can erase this difficulty.

2.2 The sphere $\mathbb{S}^2$

For every natural number L, we endow the Riemann's sphere $\mathbb{S}^2$ with the Hilbert space of polynomials:

$$\mathcal{P}_L := \left\{ p \in \mathbb{P}_L[\mathbb{C}] \ : \| \ p \ \|^2_{\mathcal{P}_L} = \frac{L+1}{\pi} \int_{\mathbb{C}} \frac{|p(z)|^2}{(1+|z|^2)^{L+2}} dm(z) < +\infty \right\},$$

where $\mathbb{P}_L[\mathbb{C}]$ is the vectorial space of polynomials of degree at most L with complex coefficients.

Let us denote:

$$e_n = \frac{\sqrt{L(L-1)...(L-n+1)}}{\sqrt{n!}} z^n.$$

Then we have that $(e_n)_{n=0}^{L}$ is an orthonormal basis on $\mathcal{P}_L$. We must prove that $(e_n, e_n)_{\mathcal{P}_L} = 1$ and $(e_n, e_m)_{\mathcal{P}_L} = 0$ for all $n \neq m$.

Before facing this, we should remark some results to clarify future calculations. First of all, it holds the next equality:

$$\frac{L(L-1)...(L-n+1)}{n!} = \frac{\Gamma(L+1)}{\Gamma(n+1)\Gamma(L-n+1)}.$$

Second of all, the beta function is defined, for all $\text{Re}(x), \text{Re}(y) > 0$, as:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 s^{x-1}(1-s)^{y-1} ds.$$

We have the next property:

Proposition 2.3. *For all* $\text{Re}(x), \text{Re}(y) > 0$, *it satisfies:*

$$B(x,y) = \int_0^1 s^{x-1}(1-s)^{y-1} ds = \int_0^{+\infty} \frac{s^{x-1}}{(1+s)^{x+y}} ds.$$

Proof. Using the change of variables $s = \frac{u}{u+1}$ and $ds = \frac{du}{(1+u)^2}$ we get:

$$\int_0^1 s^{x-1}(1-s)^{y-1}ds = \int_0^{+\infty} \frac{u^{x-1}}{(1+u)^{x+y-2}}\frac{du}{(1+u)^2} = \int_0^{+\infty} \frac{u^{x-1}}{(1+u)^{x+y}}du.$$

□

Now we can return to the original problem. On one hand we have:

$$\begin{aligned}(e_n, e_n)_{\mathcal{P}_L} &= \frac{L+1}{\pi}\int_{\mathbb{C}} \frac{L(L-1)...(L-n+1)}{n!}|z|^{2n}\frac{dm(z)}{(1+|z|^2)^{L+2}} \\ &= \frac{L+1}{\pi}\frac{\Gamma(L+1)}{\Gamma(n+1)\Gamma(L-n+1)}\int_0^{2\pi}\int_0^{+\infty}\frac{r^{2n+1}}{(1+r^2)^{L+2}}drd\theta \\ &= \frac{(L+1)\Gamma(L+1)}{\Gamma(n+1)\Gamma(L-n+1)}\int_0^{+\infty}\frac{r^{2n}}{(1+r^2)^{L+2}}2rdr \\ &\overset{(*)}{=} \frac{(L+1)\Gamma(L+1)}{\Gamma(n+1)\Gamma(L-n+1)}\int_0^{+\infty}\frac{t^n}{(1+t)^{L+2}}dt \\ &\overset{(**)}{=} \frac{(L+1)\Gamma(L+1)}{\Gamma(n+1)\Gamma(L-n+1)}\frac{\Gamma(n+1)\Gamma(L-n+1)}{\Gamma(L+2)} \\ &= \frac{(L+1)\Gamma(L+1)}{\Gamma(L+2)} = \frac{(L+1)!}{(L+1)!} = 1,\end{aligned}$$

where in $(*)$ we use the change of variables $t = r^2$ and $dt = 2rdr$. The $(**)$ equality is given by the last proposition.
On the other hand, for all $n \neq m$:

$$\begin{aligned}(e_n, e_m)_{\mathcal{P}_L} &= \frac{L+1}{\pi}\int_{\mathbb{C}}\frac{\sqrt{L(L-1)...(L-n+1)}}{\sqrt{n!}}\frac{\sqrt{L(L-1)...(L-m+1)}}{\sqrt{m!}}\frac{z^n\overline{z}^m}{(1+|z|^2)^{L+2}}dm(z) \\ &= \frac{L+1}{\pi}\frac{\sqrt{L(L-1)...(L-n+1)}}{\sqrt{n!}}\frac{\sqrt{L(L-1)...(L-m+1)}}{\sqrt{m!}}\int_0^{2\pi}\int_0^{+\infty}\frac{r^{n+m+1}e^{i\theta(n-m)}}{(1+r^2)^{L+2}}drd\theta,\end{aligned}$$

but we already know that:

$$\int_0^{2\pi} e^{i\theta(n-m)}d\theta = 0.$$

Hence, $(e_n, e_m)_{\mathcal{P}_L} = 0$. Completeness is direct, because $(z^n)_{n=0}^L$ generates the polynomials of $\mathcal{P}_L$ and forms an orthonormal basis over that space. Thus $(e_n)_{n=0}^L$ is an orthonormal basis of $\mathcal{P}_L$.
To conclude this section, by Propositions 1.13 and 1.11 we define the GAF:

$$f(z) = \sum_{n=0}^{L}\zeta_n\frac{\sqrt{L(L-1)...(L-n+1)}}{\sqrt{n!}}z^n,$$

for all natural L. Then f has covariance kernel:

$$\mathcal{K}_f(z,w) = \sum_{n=0}^{L}\frac{L(L-1)...(L-n+1)}{n!}z^n\overline{w}^n = \sum_{n=0}^{L}\binom{L}{n}z^n\overline{w}^n = (1+z\overline{w})^L.$$

Similarly to the last section, the point sets of functions in the Hilbert space of polynomials of degree at most L on $\mathbb{S}^2$ are invariant under Möbius transformations. The next proposition guarantees this:

Proposition 2.4. *Let f be a GAF in $\mathcal{P}_L$ over the Riemann's sphere $\mathbb{S}^2$. The point sets of f are invariant under the Möbius transformation*

$$\varphi_a(z) = \frac{z-a}{1+\overline{a}z},$$

where z and a are values of $\mathbb{C}$.

Proof. By hypothesis, the GAF f is of the form:

$$f(z) = \sum_{n=0}^{L} \zeta_n \frac{\sqrt{L(L-1)...(L-n+1)}}{\sqrt{n!}} z^n,$$

and has covariance kernel:

$$\mathcal{K}_f(z,w) = (1+z\overline{w})^L.$$

Let f_a be the function:

$$f_a(z) = f(\varphi_a(z)).$$

Then f_a has covariance kernel:

$$\mathcal{K}_{f_a}(z,w) = \mathcal{K}_f(\varphi_a(z),\varphi_a(w)) = \left(1+\frac{z-a}{1+\overline{a}z}\frac{\overline{w}-\overline{a}}{1+a\overline{w}}\right)^L = \left(\frac{(1+|a|^2)(1+z\overline{w})}{(1+\overline{a}z)(1+a\overline{w})}\right)^L.$$

Now, we have that:

$$f(z) \stackrel{d}{=} f_a(z)\left(\frac{1+|a|^2}{(1+\overline{a}z)^2}\right)^{-\frac{L}{2}}.$$

To show this, if we denote

$$T_af(z) = f_a(z)\left(\frac{1+|a|^2}{(1+\overline{a}z)^2}\right)^{-\frac{L}{2}},$$

we must prove that

$$\mathcal{K}_f(z,w) = \mathcal{K}_{T_af}(z,w).$$

Indeed:

$$\begin{aligned}
\mathcal{K}_{T_af}(z,w) &= \mathcal{K}_{f_a}(z,w)\left(\frac{1+|a|^2}{(1+\overline{a}z)^2}\right)^{-\frac{L}{2}}\left(\frac{1+|a|^2}{(1+a\overline{w})^2}\right)^{-\frac{L}{2}} \\
&= \left(\frac{(1+|a|^2)(1+z\overline{w})}{(1+\overline{a}z)(1+a\overline{w})}\right)^L\left(\frac{1+|a|^2}{(1+\overline{a}z)(1+a\overline{w})}\right)^{-L} \\
&= (1+z\overline{w})^L = \mathcal{K}_f(z,w).
\end{aligned}$$

Therefore:

$$f(z) \stackrel{d}{=} T_af(z).$$

□

Proposition 2.5. *Using the same notation as in the last proof, f and $T_a f$ are isometric, that is:*

$$\| f \|^2_{\mathcal{P}_L} = \| T_a f \|^2_{\mathcal{P}_L} .$$

Proof. Indeed, we have that:

$$\| T_a f \|^2_{\mathcal{P}_L} = \frac{L+1}{\pi} \int_{\mathbb{C}} \frac{|T_a f(z)|^2}{(1+|z|^2)^{L+2}} dm(z) = \frac{L+1}{\pi} \int_{\mathbb{C}} \frac{|f_a(z)|^2}{(1+|z|^2)^{L+2}} \left| \frac{1+|a|^2}{(1+\bar{a}z)^2} \right|^{-L} dm(z).$$

By the change of variables

$$w = \frac{z-a}{1+\bar{a}z}$$

this implies:

$$z = \frac{w+a}{1-\bar{a}w}.$$

Also we have that:

$$dm(z) = \left| \frac{\partial}{\partial w} \left(\frac{w+a}{1-\bar{a}w} \right) \right|^2 dm(w) = \frac{(1+|a|^2)^2}{|1-\bar{a}w|^4} dm(w) = \frac{(1+|a|^2)^2}{(1-\bar{a}w)^2(1-a\overline{w})^2} dm(w).$$

Therefore:

$$1+|z|^2 = 1 + \left| \frac{w+a}{1-\bar{a}w} \right|^2 = 1 + \frac{|w|^2 + \bar{a}w + a\overline{w} + |a|^2}{(1-\bar{a}w)(1-a\overline{w})} = \frac{(1+|a|^2)(1+|w|^2)}{(1-\bar{a}w)(1-a\overline{w})}.$$

This implies that:

$$\frac{1}{(1+|z|^2)^{L+2}} = \frac{(1-\bar{a}w)^{L+2}(1-a\overline{w})^{L+2}}{(1+|a|^2)^{L+2}(1+|w|^2)^{L+2}}.$$

Also we have:

$$1+\bar{a}z = 1 + \bar{a}\frac{w+a}{1-\bar{a}w} = \frac{1+|a|^2}{1-\bar{a}w},$$

and then:

$$\frac{1+|a|^2}{(1+\bar{a}z)^2} = \frac{(1-\bar{a}w)^2}{1+|a|^2}.$$

In addition:

$$\left| \frac{(1-\bar{a}w)^2}{1+|a|^2} \right|^{-L} = \frac{|1-\bar{a}w|^{-2L}}{(1+|a|^2)^{-L}} = \frac{(1-\bar{a}w)^{-L}(1-a\overline{w})^{-L}}{(1+|a|^2)^{-L}}.$$

Hence, by the last equalities:

$$\begin{aligned}
\| T_a f \|^2_{\mathcal{P}_L} &= \frac{L+1}{\pi} \int_{\mathbb{C}} \frac{|f_a(z)|^2}{(1+|z|^2)^{L+2}} \left| \frac{1+|a|^2}{(1+\bar{a}z)^2} \right|^{-L} dm(z) \\
&= \frac{L+1}{\pi} \int_{\mathbb{C}} |f(w)|^2 \frac{(1-\bar{a}w)^{L+2}(1-a\overline{w})^{L+2}}{(1+|a|^2)^{L+2}(1+|w|^2)^{L+2}} \left| \frac{(1-\bar{a}w)^2}{1+|a|^2} \right|^{-L} \frac{(1+|a|^2)^2}{(1-\bar{a}w)^2(1-a\overline{w})^2} dm(w) \\
&= \frac{L+1}{\pi} \int_{\mathbb{C}} |f(w)|^2 \frac{(1-\bar{a}w)^{L+2}(1-a\overline{w})^{L+2}(1-\bar{a}w)^{-L}(1-a\overline{w})^{-L}(1+|a|^2)^2}{(1+|a|^2)^{L+2}(1+|w|^2)^{L+2}(1+|a|^2)^{-L}(1-\bar{a}w)^2(1-a\overline{w})^2} dm(w) \\
&= \frac{L+1}{\pi} \int_{\mathbb{C}} \frac{|f(w)|^2}{(1+|w|^2)^{L+2}} dm(w) = \| f \|^2_{\mathcal{P}_L} .
\end{aligned}$$

□

For every function f of $\mathcal{P}_L$, we remark that if f is a GAF, then $T_a f$ is also a GAF because it is a composition with the transformation φ_a and the deterministic corrector factor

$$\eta_a(z) = \left(\frac{1+|a|^2}{(1+\bar{a}z)^2}\right)^{-\frac{L}{2}}.$$

Given the fact that the point sets of f and $T_a f$ are equal in distribution, the zero sets of the last functions also satisfy this. Moreover, the zeros sets of f_a are equal in distribution with f. Indeed, if $\eta_a(z) = 0$, we would have $|a|^2 = -1$, which is a contradiction since the modulus of a complex value is non-negative. As a conclusion, the zero sets of f_a are the same in distribution than f.

2.3 The Hyperbolic Plane $\mathbb{D}$

Giving a real parameter $L > 0$ we define the weighted Bergman space over $\mathbb{D}$ as:

$$\mathcal{B}_L := \left\{ f \in \mathcal{A}(\mathbb{D}) \ : \| f \|^2_{\mathcal{B}_L} = \frac{L}{\pi}\int_{\mathbb{D}} |f(z)|^2 (1-|z|^2)^{L-2} dm(z) < +\infty \right\}.$$

Let us denote:

$$e_n = \frac{\sqrt{L(L+1)...(L+n-1)}}{\sqrt{n!}} z^n.$$

In this case, $(e_n)_{n=0}^{+\infty}$ is an orthonormal basis on $\mathcal{B}_L$. For this we must prove that $(e_n, e_n)_{\mathcal{B}_L} = 1$ and $(e_n, e_m)_{\mathcal{B}_L} = 0$. Before going forward, we remark that:

$$\frac{L(L+1)...(L+n-1)}{n!} = \frac{\Gamma(L+n)}{\Gamma(n+1)\Gamma(L)}.$$

Now, we have that:

$$\begin{aligned}
(e_n, e_n)_{\mathcal{B}_L} &= \frac{L}{\pi}\int_{\mathbb{D}} \frac{L(L+1)...(L+n-1)}{n!} |z|^{2n} (1-|z|^2)^{L-2} dm(z) \\
&= \frac{L}{\pi}\frac{\Gamma(L+n)}{\Gamma(n+1)\Gamma(L)} \int_0^{2\pi}\int_0^1 r^{2n+1}(1-r^2)^{L-2} dr d\theta \\
&= \frac{L\Gamma(L+n)}{\Gamma(n+1)\Gamma(L)} \int_0^1 r^{2n}(1-r^2)^{L-2} 2r dr \\
&\overset{(*)}{=} \frac{L\Gamma(L+n)}{\Gamma(n+1)\Gamma(L)} \int_0^1 t^n (1-t)^{L-2} dt \overset{(**)}{=} \frac{L\Gamma(L+n)}{\Gamma(n+1)\Gamma(L)} \frac{\Gamma(n+1)\Gamma(L-1)}{\Gamma(n+L)} \\
&= L\frac{\Gamma(L-1)}{\Gamma(L)} = \frac{\Gamma(L)}{\Gamma(L)} = 1,
\end{aligned}$$

where in $(*)$ we use the change of variables $t = r^2$ and $dt = 2rdr$. The $(**)$ equality holds by the beta function described in the section before.
Now, for all $n \neq m$:

$$\begin{aligned}
(e_n, e_m)_{\mathcal{B}_L} &= \frac{L}{\pi}\int_{\mathbb{D}} \frac{\sqrt{L(L+1)...(L+n-1)}}{\sqrt{n!}} \frac{\sqrt{L(L+1)...(L+m-1)}}{\sqrt{m!}} z^n \bar{z}^m (1-|z|^2)^{L-2} dm(z) \\
&= \frac{L}{\pi} \frac{\sqrt{L(L+1)...(L+n-1)}}{\sqrt{n!}} \frac{\sqrt{L(L+1)...(L+m-1)}}{\sqrt{m!}} \int_0^{2\pi}\int_0^1 \frac{r^{n+m+1} e^{i\theta(n-m)}}{(1-r^2)^{2-L}} dr d\theta.
\end{aligned}$$

We know that:

$$\int_0^{2\pi} e^{i\theta(n-m)} d\theta = 0.$$

Hence, $(e_n, e_m)_{\mathcal{B}_L} = 0$. For completeness, if for a given function f of $\mathcal{B}_L$ it holds $(f, e_n)_{\mathcal{B}_L} = 0$, then $f \equiv 0$. Following a procedure in [22], we can expand f as a Taylor power series:

$$f(z) = \sum_{m=0}^{+\infty} c_m z^m,$$

and this power series converges uniformly over compact subsets of $\mathbb{D}$. We have:

$$\begin{aligned}
(f, e_n)_{\mathcal{B}_L} &= \frac{L\sqrt{L(L+1)...(L+n-1)}}{\pi\sqrt{n!}} \int_{\mathbb{D}} f(z)\overline{z}^n (1-|z|^2)^{L-2} dm(z) \\
&= \frac{L\sqrt{L(L+1)...(L+n-1)}}{\pi\sqrt{n!}} \lim_{R\to 1} \int_0^R \int_0^{2\pi} f(re^{i\theta}) r^{n+1} e^{-in\theta} (1-r^2)^{L-2} d\theta dr \\
&= \frac{L\sqrt{L(L+1)...(L+n-1)}}{\pi\sqrt{n!}} \lim_{R\to 1} \int_0^R \int_0^{2\pi} \sum_{m=0}^{+\infty} c_m r^{n+m+1} (1-r^2)^{L-2} e^{i\theta(m-n)} d\theta dr \\
&= \frac{L\sqrt{L(L+1)...(L+n-1)}}{\pi\sqrt{n!}} \lim_{R\to 1} \sum_{m=0}^{+\infty} \int_0^R \int_0^{2\pi} c_m r^{n+m+1} (1-r^2)^{L-2} e^{i\theta(m-n)} d\theta dr \\
&= 2\frac{L\sqrt{L(L+1)...(L+n-1)}}{\sqrt{n!}} c_n \lim_{R\to 1} \int_0^R r^{2n+1} (1-r^2)^{L-2} dr = 0,
\end{aligned}$$

where the commutativity of the integrals with the sum is due to the uniform convergence of the power series of f over $\mathbb{D}$. Therefore, if $(f, e_n)_{\mathcal{B}_L} = 0$, the value c_m is the unique one that vanishes. Hence $f \equiv 0$, as we wanted to see. Thus $(e_n)_{n=0}^{+\infty}$ is an orthonormal basis on $\mathcal{B}_L$.

By Propositions 1.13 and 1.11 we define the GAF:

$$f(z) = \sum_{n=0}^{+\infty} \zeta_n \frac{\sqrt{L(L+1)...(L+n-1)}}{\sqrt{n!}} z^n,$$

for all real $L > 0$. Then f has covariance kernel:

$$\mathcal{K}_f(z,w) = \sum_{n=0}^{+\infty} \frac{L(L+1)...(L+n-1)}{n!} z^n \overline{w}^n = \sum_{n=0}^{+\infty} \binom{L+n-1}{n} z^n \overline{w}^n = (1 - z\overline{w})^{-L},$$

where the last equality is given by:

$$\sum_{n=0}^{+\infty} \binom{n+a}{n} x^n = \frac{1}{(1-x)^{a+1}},$$

for real a and $|x| < 1$. The equality remains true since the sum term is the Taylor series of the function of the right side of the equality.

Likewise the other Hilbert spaces, the point sets of functions in $\mathcal{B}_L$ on $\mathbb{D}$ are also invariant under transformations.

Proposition 2.6. *Let f be a GAF in $\mathcal{B}_L$ over the hyperbolic plane $\mathbb{D}$. The point sets of f are invariant under the transformation*

$$\varphi_a(z) = \frac{z-a}{1-\bar{a}z},$$

where z and a are values of $\mathbb{D}$.

Proof. By hypothesis, the GAF f is of the form:

$$f(z) = \sum_{n=0}^{+\infty} \zeta_n \frac{\sqrt{L(L+1)...(L+n-1)}}{\sqrt{n!}} z^n,$$

and has covariance kernel:

$$\mathcal{K}_f(z,w) = (1-z\overline{w})^{-L}.$$

Let f_a be the function:

$$f_a(z) = f(\varphi_a(z)).$$

Then f_a has covariance kernel:

$$\mathcal{K}_{f_a}(z,w) = \mathcal{K}_f(\varphi_a(z),\varphi_a(w)) = \left(1 - \frac{z-a}{1-\bar{a}z}\frac{\overline{w}-\bar{a}}{1-a\overline{w}}\right)^{-L} = \left(\frac{(1-|a|^2)(1-z\overline{w})}{(1-\bar{a}z)(1-a\overline{w})}\right)^{-L}.$$

Now, we have that:

$$f(z) \stackrel{d}{=} f_a(z)\left(\frac{1-|a|^2}{(1-\bar{a}z)^2}\right)^{\frac{L}{2}}.$$

To show this, if we denote

$$T_a f(z) = f_a(z)\left(\frac{1-|a|^2}{(1-\bar{a}z)^2}\right)^{\frac{L}{2}},$$

we must prove that

$$\mathcal{K}_f(z,w) = \mathcal{K}_{T_a f}(z,w).$$

Indeed:

$$\begin{aligned}\mathcal{K}_{T_a f}(z,w) &= \mathcal{K}_{f_a}(z,w)\left(\frac{1-|a|^2}{(1-\bar{a}z)^2}\right)^{\frac{L}{2}}\left(\frac{1-|a|^2}{(1-a\overline{w})^2}\right)^{\frac{L}{2}} \\ &= \left(\frac{(1-|a|^2)(1-z\overline{w})}{(1-\bar{a}z)(1-a\overline{w})}\right)^{-L}\left(\frac{1-|a|^2}{(1-\bar{a}z)(1-a\overline{w})}\right)^{L} = (1-z\overline{w})^{-L} = \mathcal{K}_f(z,w).\end{aligned}$$

Therefore:

$$f(z) \stackrel{d}{=} T_a f(z).$$

□

Proposition 2.7. *Using the same notation as in the last proof, f and $T_a f$ are isometric, that is:*

$$\| f \|^2_{\mathcal{B}_L} = \| T_a f \|^2_{\mathcal{B}_L}.$$

Proof. Indeed, we have that:

$$\begin{aligned}\| T_a f \|^2_{\mathcal{B}_L} &= \frac{L}{\pi}\int_{\mathbb{D}} |T_a f(z)|^2 (1-|z|^2)^{L-2} dm(z)\\ &= \frac{L}{\pi}\int_{\mathbb{D}} |f_a(z)|^2 (1-|z|^2)^{L-2} \left|\frac{1-|a|^2}{(1-\bar{a}z)^2}\right|^L dm(z).\end{aligned}$$

By the change of variables

$$w = \frac{z-a}{1-\bar{a}z}$$

this implies:

$$z = \frac{w+a}{1+\bar{a}w}.$$

Also we have that:

$$dm(z) = \left|\frac{\partial}{\partial w}\left(\frac{w+a}{1+\bar{a}w}\right)\right|^2 dm(w) = \frac{(1-|a|^2)^2}{|1+\bar{a}w|^4} dm(w) = \frac{(1-|a|^2)^2}{(1+\bar{a}w)^2(1+a\overline{w})^2} dm(w).$$

Therefore:

$$1-|z|^2 = 1-\left|\frac{w+a}{1+\bar{a}w}\right|^2 = 1-\frac{|w|^2+\bar{a}w+a\overline{w}+|a|^2}{(1+\bar{a}w)(1+a\overline{w})} = \frac{(1-|a|^2)(1-|w|^2)}{(1+\bar{a}w)(1+a\overline{w})}.$$

This implies that:

$$(1-|z|^2)^{L-2} = \frac{(1-|a|^2)^{L-2}(1-|w|^2)^{L-2}}{(1+\bar{a}w)^{L-2}(1+a\overline{w})^{L-2}}.$$

Also we have:

$$1-\bar{a}z = 1-\bar{a}\frac{w+a}{1+\bar{a}w} = \frac{1-|a|^2}{1+\bar{a}w},$$

and then:

$$\frac{1-|a|^2}{(1-\bar{a}z)^2} = \frac{(1+\bar{a}w)^2}{1-|a|^2}.$$

In addition:

$$\left|\frac{(1+\bar{a}w)^2}{1-|a|^2}\right|^L = \frac{|1+\bar{a}w|^{2L}}{(1-|a|^2)^L} = \frac{(1+\bar{a}w)^L(1+a\overline{w})^L}{(1-|a|^2)^L}.$$

Hence, by the last equalities:

$$\begin{aligned}\| T_a f \|^2_{\mathcal{B}_L} &= \frac{L}{\pi}\int_{\mathbb{D}} |f_a(z)|^2 (1-|z|^2)^{L-2}\left|\frac{1-|a|^2}{(1-\bar{a}z)^2}\right|^L dm(z)\\ &= \frac{L}{\pi}\int_{\mathbb{D}} |f(w)|^2 \frac{(1-|a|^2)^{L-2}(1-|w|^2)^{L-2}}{(1+\bar{a}w)^{L-2}(1+a\overline{w})^{L-2}}\left|\frac{(1+\bar{a}w)^2}{1-|a|^2}\right|^L \frac{(1-|a|^2)^2}{(1+\bar{a}w)^2(1+a\overline{w})^2} dm(w)\\ &= \frac{L}{\pi}\int_{\mathbb{D}} |f(w)|^2 \frac{(1-|a|^2)^{L-2}(1-|w|^2)^{L-2}(1+\bar{a}w)^L(1+a\overline{w})^L(1-|a|^2)^2}{(1+\bar{a}w)^{L-2}(1+a\overline{w})^{L-2}(1-|a|^2)^L(1+\bar{a}w)^2(1+a\overline{w})^2} dm(w)\\ &= \frac{L}{\pi}\int_{\mathbb{D}} |f(w)|^2 (1-|w|^2)^{L-2} dm(w) = \| f \|^2_{\mathcal{B}_L}.\end{aligned}$$

□

For every GAF f of $\mathcal{B}_L$, the function $T_a f$ is also a GAF because it is a composition with the disk automorphism φ_a and multiplied by the deterministic corrector factor:

$$\eta_a(z) = \left(\frac{1-|a|^2}{(1-\bar{a}z)^2} \right)^{\frac{L}{2}}.$$

As we just saw, the point sets of f are equal in distribution than those of $T_a f$, therefore the zero sets are also equal in distribution. Moreover, the zero sets of f are equal in distribution than those of f_a. Indeed, if $\eta_a(z) = 0$, we would have $|a|^2 = 1$, which is a contradiction since a is a value from $\mathbb{D}$. As a conclusion, the zero sets of f_a are the same in distribution than those of f.

2.4 The Paley - Wiener space

For a real parameter $L > 0$, the Paley - Wiener space is given by:

$$PW_L := \left\{ f \in \mathcal{A}(\mathbb{C}) \ : \ |f(z)| \leq Ce^{L|\mathrm{Im}(z)|}, \ \| f \|_{PW_L}^2 = \int_{\mathbb{R}} |f(x)|^2 dx < +\infty \right\}.$$

Let us denote:

$$e_n = \frac{\sin \pi(n-Lz)}{\pi(n-Lz)}.$$

We have that $(e_n)_{n\in\mathbb{Z}}$ is an orthonormal basis on PW_L. For this we will use Fourier transformations, defined as

$$\hat{f}(z) = \int_{-\infty}^{+\infty} f(x)e^{-2\pi i x z} dx,$$

where f is an integrable function in $\mathbb{R}$; and we will use also the Parseval's identity to extend the definition to $L^2(\mathbb{R})$. First of all we will see that the function $f(x) = \frac{1}{L}\chi_{[-\frac{L}{2},\frac{L}{2}]}(x)e^{\frac{2\pi i n x}{L}}$ is the Fourier transformation of e_n for all integer n, real x and complex z. Indeed:

$$\begin{aligned} \hat{f}\,(z) &= \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} e^{\frac{2\pi i n x}{L}} e^{-2\pi i x z} dx = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} e^{\frac{2\pi i n x}{L}} e^{-\frac{2\pi i L x z}{L}} dx = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} e^{\frac{2\pi i x(n-Lz)}{L}} dx \\ &= \frac{L}{2\pi i L(n-Lz)} \left[e^{\frac{2\pi i x(n-Lz)}{L}} \right]_{x=-\frac{L}{2}}^{x=\frac{L}{2}} = \frac{\sin \pi(n-Lz)}{\pi(n-Lz)}. \end{aligned}$$

Returning to the problem, we must see that $(e_n, e_n)_{PW_L} = 1$ and $(e_n, e_m)_{PW_L} = 0$ for all integer numbers $n \neq m$. Indeed:

$$(e_n, e_n)_{PW_L} = \int_{-\infty}^{+\infty} \left| \frac{\sin \pi(n-Lz)}{\pi(n-Lz)} \right|^2 dx \overset{(*)}{=} \frac{1}{L}\int_{-\infty}^{+\infty} \chi_{[-\frac{L}{2},\frac{L}{2}]}(x)|e^{\frac{2\pi i x n}{L}}|^2 dx = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} dx = 1,$$

where in $(*)$ we use the Parseval's identity.
Now, for all integer $n \neq m$:

$$\begin{aligned} (e_n, e_m)_{PW_L} &= \int_{-\infty}^{+\infty} \frac{\sin \pi(n-Lz)}{\pi(n-Lz)} \overline{\frac{\sin \pi(n-Lz)}{\pi(n-Lz)}} dx \overset{(**)}{=} \frac{1}{L^2}\int_{-\infty}^{+\infty} \chi_{[-\frac{L}{2},\frac{L}{2}]}(x)e^{\frac{2\pi i(n-m)x}{L}} dx \\ &= \frac{1}{L^2}\int_{-\frac{L}{2}}^{\frac{L}{2}} e^{\frac{2\pi i(n-m)x}{L}} dx = \frac{e^{\pi i(n-m)} - e^{-\pi i(n-m)}}{2\pi i L(n-m)} = \frac{\sin \pi(n-m)}{\pi L(n-m)} = 0, \end{aligned}$$

where in $(**)$ we use the Parseval's identity again. For completeness, if f is a function of PW_L and $(f, e_n)_{PW_L} = 0$, then $f \equiv 0$. Indeed, the space of Paley - Wiener can be written as:

$$CS_L := \left\{ f \in L^2(\mathbb{R}) \ : \ \text{supp}\hat{f} \subset \left[-\frac{L}{2}, \frac{L}{2} \right] \right\},$$

and this is true by the next theorem:

Theorem 2.8. *The complex Fourier transformation establishes an isometry between PW_L and CS_L.*

The proof of this result is long, but in pages 566 to 570 of reference [4] it is well - explained and the isometry is a consequence of Theorem 12.11, page 570.
Therefore, by Parseval's identity:

$$(f, e_n)_{L^2(\mathbb{R})} = (\hat{f}, \hat{e_n})_{L^2(\mathbb{R})} = \left(\hat{f}, \frac{1}{L} \chi_{\left[-\frac{L}{2}, \frac{L}{2}\right]} e^{-\frac{2\pi i n x}{L}} \right)_{L^2(\mathbb{R})}.$$

However, we know that the collection $(e^{\frac{2\pi i n x}{L}})_{n \in \mathbb{Z}}$ is an orthonormal basis on $L^2(\left[-\frac{L}{2}, \frac{L}{2}\right])$. Hence, $(\hat{f}, \frac{1}{L}\chi_{\left[-\frac{L}{2}, \frac{L}{2}\right]} e^{-\frac{2\pi i n x}{L}})_{L^2(\mathbb{R})} = 0$, and that implies that $\hat{f} \equiv 0$. So, the function f also vanishes. As a conclusion, $(e_n)_{n \in \mathbb{Z}}$ is an orthonormal basis on PW_L.
By Propositions 1.13 and 1.11 we define the GAF:

$$f(z) = \sum_{n=-\infty}^{+\infty} \zeta_n \frac{\sin \pi(n - Lz)}{\pi(n - Lz)},$$

where $(\zeta_n)_n$ are complex i.i.d. Gaussian random variables with zero mean and unit variance and $L > 0$ is a real parameter. Denoting $g \equiv \hat{f}$, the covariance kernel of f can be calculated as:

$$(g, \mathcal{K}_f)_{PW_L} = g(z) = \left(f, \frac{1}{L} e^{\frac{2\pi i z x}{L}} \right)_{L^2(\left[-\frac{L}{2}, \frac{L}{2}\right])}.$$

Therefore:

$$\mathcal{K}_f(z, w) = \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} e^{\frac{2\pi i (z - \overline{w}) x}{L}} dx = \frac{1}{2\pi i (z - \overline{w})} \left(e^{\pi i (z - \overline{w})} - e^{-\pi i (z - \overline{w})} \right) = \frac{\sin \pi (z - \overline{w})}{\pi (z - \overline{w})}.$$

As we did before, we will prove the invariance under transformations of the point sets of a GAF in PW_L. But there is a difference from the rest of the cases: the transformation is under the real line. Then, we have the next proposition:

Proposition 2.9. *Let f be a GAF in PW_L. The point sets of f are invariant under the transformation*

$$\varphi_a(z) = z - a,$$

where z is a complex value and a is a real value. Also we have that

$$\| f \|_{PW_L}^2 = \| f_a \|_{PW_L}^2 .$$

Proof. By hypothesis, the GAF f is of the form:

$$f(z) = \sum_{n=-\infty}^{+\infty} \zeta_n \frac{\sin \pi(n - Lz)}{\pi(n - Lz)},$$

and has covariance kernel:

$$\mathcal{K}_f(z, w) = \frac{\sin \pi(z - \overline{w})}{\pi(z - \overline{w})}.$$

If we denote $f_a(z) = f(\varphi_a(z))$, then f_a has covariance kernel:

$$\mathcal{K}_{f_a}(z, w) = \mathcal{K}_f(\varphi_a(z), \varphi_a(w)) = \frac{\sin \pi(z - a - \overline{w} + \overline{a})}{\pi(z - a - \overline{w} + \overline{a})} \overset{(*)}{=} \frac{\sin \pi(z - \overline{w})}{\pi(z - \overline{w})},$$

where in $(*)$ we use that $a = \overline{a}$ since a is real. Therefore it holds that:

$$\mathcal{K}_f(z, w) = \mathcal{K}_{f_a}(z, w),$$

and the point sets of f are equal in distribution than f_a. Also it is obvious that:

$$\| f \|_{PW_L}^2 = \| f_a \|_{PW_L}^2 .$$

□

As a direct consequence of this proposition, the zero sets of f are equal in distribution than f_a thanks to φ_a.

Chapter 3

Distribution and intensity of zeros of a GAF

Once we saw the main properties of Gaussian analytic functions and the invariance under Möbius transformation of the GAFs considered before, we would like to study the zero sets of those GAFs. We will see that the average of the distribution of the zero points is directly determined by the covariance kernel. And what is more impressive, if two GAFs have the same first intensity, then these two functions are equal in distribution.

3.1 The Edelman - Kostlan formula

Let Ω be a region of $\mathbb{C}$ and let $\mathcal{H}$ be a Hilbert space of analytic functions on Ω. If f is a GAF of $\mathcal{H}$, we define $\mathcal{Z}_f$ as the zero set of f. In other words:

$$\mathcal{Z}_f = f^{-1}(0).$$

Intuitively, we can understand the counting measure ν_f as a way to know how many zeros f has. In this context, ν_f is called the empirical measure. Thus, if A is a set of Ω:

$$\nu_f(A) = \#\left(\mathcal{Z}_f \cap A\right).$$

Definition 3.1. *Let $\mathcal{X}$ be a point process in Ω. If for all φ of $C_c^\infty(\Omega)$ satisfies:*

$$\mathbb{E}\left(\int_\Omega \varphi d\nu_f\right) = \int_\Omega \varphi d\rho_1,$$

then the deterministic measure ρ_1 is called the first intensity*.*

Our main goal in this section is to determine a formula to express the first intensity of a GAF. We are going to compute this using Green's second identity. Although, we should introduce a definition from [4] (page 315) before going deeper. There, $\mathbb{R}^n$ is considered, but since $\mathbb{C}$ is isomorphic to $\mathbb{R}^2$, we can state:

Definition 3.2. *Let Ω be a region of $\mathbb{C}$ and let μ be a measure such that is finite over compact subsets of Ω. We say that a function u of $L^1_{\text{loc}}(\Omega)$ is a solution of*

$$\Delta u = \mu$$

on Ω in a distributional sense if for all function ψ of $C_c^\infty(\Omega)$ it holds

$$\int_\Omega u(z)\Delta\psi(z)dm(z) = \int_\Omega \psi(z)d\mu(z).$$

The Laplacian Δ should be understood in the distributional sense.

We are going to show a proposition for analytic functions, not only for GAFs.

Proposition 3.3. *Using the same notation as before, it satisfies:*

$$\nu_f = \frac{1}{2\pi}\Delta \log |f|,$$

where f is a function of $\mathcal{A}(\Omega)$ and Δ is taken in the distributional sense.

Proof. (See Subsection 2.4.1. in [2], page 24). By the last definition, we must see that:

$$\int_\Omega \frac{1}{2\pi} \log |f(z)|\Delta\psi(z)dm(z) = \int_\Omega \psi(z)d\nu_f(z),$$

where ψ is a function of $\mathcal{C}_c^\infty(\Omega)$. Since ψ is compactly supported, the zeros of f that are laying in the compact support of ψ are finite. Thus, in a neighbourhood of the support of ψ:

$$f(z) = g(z)\prod_{k=1}^{n}(z - z_k)^{m_k},$$

where z_k are the zeros of f with multiplicity m_k and g is an analytic function, since f is, with no zeros in the support of ψ. Therefore it holds:

$$\log |f(z)| = \log |g(z)| + \sum_{k=1}^{n} m_k \log |z - z_k|,$$

and also we have:

$$\Delta \log |f(z)| = \Delta \log |g(z)| + \sum_{k=1}^{n} m_k \Delta \log |z - z_k|.$$

Since g is an element of $\mathcal{A}(\Omega)$ and $g(z) \neq 0$ for all z of Ω in a neighbourhood in the compact support of ψ, we have that $\log |g(z)|$ is a harmonic function, that is, $\Delta \log |g(z)| = 0$. Indeed, let $u \equiv u(x,y)$ and $v \equiv v(x,y)$ be real functions of $\mathcal{C}^2(\Omega)$. Then we can write $g(z) = g(x,y) = u(x,y) + iv(x,y)$. We want to see that:

$$\Delta \log |g(x,y)| = \frac{\partial^2}{\partial x^2} \log |g(x,y)| + \frac{\partial^2}{\partial y^2} \log |g(x,y)| = 0.$$

We have:

$$\frac{\partial^2}{\partial x^2}\log|g(x,y)| = \frac{(u_x^2+uu_{xx}+v_x^2+vv_{xx})(u^2+v^2)-(uu_x+vv_x)(2uu_x+2vv_x)}{(u^2+v^2)^2},$$

and:

$$\frac{\partial^2}{\partial y^2}\log|g(x,y)| = \frac{(u_y^2+uu_{yy}+v_y^2+vv_{yy})(u^2+v^2)-(uu_y+vv_y)(2uu_y+2vv_y)}{(u^2+v^2)^2}.$$

By the analyticity of g, we can use the Cauchy - Riemann equations and from here it holds that $u_{xx}+u_{yy}=0$ and $v_{xx}+v_{yy}=0$. Making suitable changes, we conclude that $\Delta\log|g(z)|=0$.
Now, let us declare $G(z)=\frac{1}{2\pi}\log|z-z_k|$, which is integrable in a neighbourhood of the compact support of ψ. Indeed, if we consider a disk D centered in z_k and radii ρ, we have:

$$\begin{aligned}\int_D G(z)dm(z) &= \frac{1}{2\pi}\int_D \log|z-z_k|dm(z) = \int_0^\rho r\log r dr = \left[\frac{r^2}{4}(2\log(r)-1)\right]_{r=0}^{r=\rho}\\ &= \frac{\rho^2}{4}(2\log(\rho)-1) < +\infty.\end{aligned}$$

Now we are going to see that:

$$\int_\Omega \frac{1}{2\pi}\log|z|\Delta\psi(z)dm(z) = \psi(0).$$

Since we are operating with a logarithm, we should extract the disk $\mathbb{D}(0,\varepsilon)$, with $\varepsilon>0$, from Ω. Let us denote $\Omega_\varepsilon=\Omega\setminus\mathbb{D}(0,\varepsilon)$. Using Green's second identity:

$$\begin{aligned}\int_{\Omega_\varepsilon}\frac{1}{2\pi}\log|z|\Delta\psi(z)dm(z) &= \int_{\Omega_\varepsilon}\frac{1}{2\pi}\Delta\log|z|\psi(z)dm(z)\\ &\quad -\int_{\partial\mathbb{D}(0,\varepsilon)^-}\left(\frac{1}{2\pi}\log\varepsilon\frac{\partial\psi}{\partial\mathbf{n}}(z)-\frac{1}{2\pi}\frac{\partial\log|z|}{\partial\mathbf{n}}_{\big|_{|z|=\varepsilon}}\psi(z)\right)ds,\end{aligned}$$

where the integral over $\partial\Omega_\varepsilon$ is 0 because ψ is compactly supported. The first integral of the right side of the equality is zero because $\Delta\log|z|=0$. For the second integral we have:

$$\int_{\partial\mathbb{D}(0,\varepsilon)^-}\frac{1}{2\pi}\log\varepsilon\frac{\partial\psi}{\partial\mathbf{n}}(z)ds = \frac{1}{2\pi}\log\varepsilon\int_{\partial\mathbb{D}(0,\varepsilon)^-}\frac{\partial\psi}{\partial\mathbf{n}}(z)ds \le \varepsilon\log\varepsilon \parallel \nabla\psi\parallel_\infty \xrightarrow{\varepsilon\to 0} 0,$$

and:

$$\begin{aligned}\int_{\partial\mathbb{D}(0,\varepsilon)^-}\frac{1}{2\pi}\frac{\partial\log|z|}{\partial\mathbf{n}}_{\big|_{|z|=\varepsilon}}\psi(z)ds &= \frac{1}{2\pi}\int_0^{2\pi}\psi(\varepsilon e^{i\theta})d\theta = \frac{1}{2\pi}\int_0^{2\pi}\Big(\psi(\varepsilon e^{i\theta})-\psi(0)+\psi(0)\Big)d\theta\\ &= \psi(0)+\frac{1}{2\pi}\int_0^{2\pi}\Big(\psi(\varepsilon e^{i\theta})-\psi(0)\Big)d\theta.\end{aligned}$$

Notice that $\lim_{\varepsilon\to 0}\psi(\varepsilon e^{i\theta})=\psi(0)$ and that $\psi(\varepsilon e^{i\theta})$ is bounded because ψ is from $\mathcal{C}_c^\infty(\Omega)$. Then, by the Dominated Convergence Theorem, we have:

$$\lim_{\varepsilon\to 0}\left(\psi(0)+\frac{1}{2\pi}\int_0^{2\pi}\Big(\psi(\varepsilon e^{i\theta})-\psi(0)\Big)d\theta\right)=\psi(0).$$

Since $\Omega_\varepsilon \xrightarrow{\varepsilon\to 0} \Omega$, it holds:

$$\int_\Omega \frac{1}{2\pi} \log|z| \Delta\psi(z) dm(z) = \psi(0).$$

Evaluating the last integral in $\psi(z+z_k)$, we get:

$$\int_\Omega \frac{1}{2\pi} \log|z-z_k| \Delta\psi(z) dm(z) = \psi(z_k).$$

Now, we know that $f(z) = g(z)\prod_{k=1}^n (z-z_k)^{m_k}$. Since g has no zeros in a neighbourhood of the compact support of ψ, g is a harmonic function in there. Taking logarithms and the Laplacian operator and using again Green's second identity, $\int_\Omega \log|g(z)|\Delta\psi(z)dm(z)$ vanishes. Then, we get

$$\int_\Omega \frac{1}{2\pi} \log|f(z)| \Delta\psi(z) dm(z) = \int_\Omega \psi(z) d\nu_f(z),$$

and then:

$$\nu_f = \frac{1}{2\pi}\Delta \log|f|,$$

as we wanted to see. □

The next theorem is one of the keystones of this chapter. It will explain that the expectation of the distributed zeros values of a GAF is directly related to the covariance kernel of the GAF that we are considering. The formula is called the *Edelman - Kostlan formula.*

Theorem 3.4. (The Edelman - Kostlan formula) *Let $\mathcal{H}$ be a Hilbert space of analytic functions on a region Ω of $\mathbb{C}$. Let f be a Gaussian analytic function with mean zero and covariance kernel $\mathcal{K}_f(z,w)$, for all z and w of Ω. The first intensity of the zeros of f is given by:*

$$\rho_1(z) = \frac{1}{4\pi}\Delta \log \mathcal{K}_f(z,z),$$

where the Laplacian Δ must be interpreted in the distributional sense.

Proof. (See Subsection 2.4.1. in [2], pages 24 and 25). By Proposition 3.3 and using the same notation than its proof we have:

$$\int_\Omega \psi(z) d\nu_f(z) = \int_\Omega \frac{1}{2\pi} \log|f(z)| \Delta\psi(z) dm(z).$$

Taking expectations at both sides we have:

$$\mathbb{E}\left[\int_\Omega \psi(z) d\nu_f(z)\right] = \mathbb{E}\left[\int_\Omega \frac{1}{2\pi} \log|f(z)| \Delta\psi(z) dm(z)\right]. \tag{3.1}$$

To finish the procedure we must be able to use Fubini's theorem at the right side of the equation. Therefore, we must see that:

$$\mathbb{E}\left[\int_\Omega \left|\frac{1}{2\pi} \log|f(z)| \Delta\psi(z) dm(z)\right|\right] < +\infty.$$

By the linearity of the expectation we obtain:

$$\mathbb{E}\left[\int_{\Omega}\left|\frac{1}{2\pi}\log|f(z)|\Delta\psi(z)dm(z)\right|\right] = \frac{1}{2\pi}\int_{\Omega}|\Delta\psi(z)|\mathbb{E}\left[|\log|f(z)||\right]dm(z).$$

For a fixed z of Ω, $f(z)$ is a complex Gaussian random variable with zero mean and variance $\mathcal{K}_f(z,z)$. Hence:

$$\begin{aligned}\mathbb{E}\left[|\log|f(z)||\right] &= \mathbb{E}\left[\left|\log\left|\frac{f(z)}{\sqrt{\mathcal{K}_f(z,z)}}\sqrt{\mathcal{K}_f(z,z)}\right|\right|\right]\\ &= \mathbb{E}\left[\left|\log\left|\frac{f(z)}{\sqrt{\mathcal{K}_f(z,z)}}\right|\right|\right] + \log\left|\sqrt{\mathcal{K}_f(z,z)}\right|.\end{aligned}$$

Denoting $\zeta = \frac{f(z)}{\sqrt{\mathcal{K}_f(z,z)}}$, the random variable ζ is a standard complex Gaussian random variable for all z of Ω. Therefore:

$$\begin{aligned}\mathbb{E}\left[|\log|f(z)||\right] &= \mathbb{E}\left[|\log|\zeta||\right] + \log\left|\sqrt{\mathcal{K}_f(z,z)}\right|\\ &= \int_{\mathbb{C}}|\log|z||\frac{e^{-|z|^2}}{\pi}dm(z) + \frac{1}{2}\log|\mathcal{K}_f(z,z)|\\ &\overset{(*)}{=} \int_0^{+\infty} 2r|\log(r)|e^{-r^2}dr + \frac{1}{2}\log|\mathcal{K}_f(z,z)|\\ &\overset{(**)}{=} \int_0^{+\infty}|\log(\rho)|e^{-\rho}d\rho + \frac{1}{2}\log|\mathcal{K}_f(z,z)| = C + \frac{1}{2}\log|\mathcal{K}_f(z,z)|,\end{aligned}$$

where C is a constant value, in $(*)$ we use a polar coordinate change and in $(**)$ we apply the change of variable $\rho = r^2$ and $d\rho = 2rdr$. The factor $\log|\mathcal{K}_f(z,z)|$ is locally integrable for all z of Ω, but there is a problem when $\mathcal{K}_f(z_0,z_0) = 0$ for z_0 of Ω. In such a case, $\mathcal{K}_f$ can be expressed as:

$$\mathcal{K}_f(z,z) = |z-z_0|^{2m}H(z,z),$$

where H is a function that is different of zero at z_0 and m a natural number. Indeed, we know that the covariance kernel of a GAF f is:

$$\mathcal{K}_f(z,z) = \sum_{n=1}^{+\infty}|f_n(z)|^2,$$

and each f_n can be expressed as:

$$f_n(z) = (z-z_0)^{m_n}h_n(z),$$

where m_n is the multiplicity of the value z_0 and h_n is a function that does not vanish at z_0. Hence:

$$\mathcal{K}_f(z,z) = \sum_{n=1}^{+\infty}|z-z_0|^{2m_n}|h_n(z)|^2.$$

Now, if we denote m by the minimum value that takes the sequence $(m_n)_{n\in\mathbb{N}}$, we have:

$$\mathcal{K}_f(z,z) = |z-z_0|^{2m}\sum_{n=1}^{+\infty}|z-z_0|^{2m_n-2m}|h_n(z)|^2 = |z-z_0|^{2m}H(z,z),$$

where H is a function that is not zero at z_0 because h_n is not either and there is a value m_n that is equal to m. Therefore it holds that:

$$\mathcal{K}_f(z,z) = |z-z_0|^{2m}H(z,z).$$

With this, $\log\mathcal{K}_f(z,z)$ is integrable in a neighbourhood of z_0. Thus:

$$\mathbb{E}\left[\int_\Omega\left|\frac{1}{2\pi}\log|f(z)|\Delta\psi(z)dm(z)\right|\right] < +\infty,$$

and we can apply Fubini's theorem at (3.1) to obtain:

$$\mathbb{E}\left[\int_\Omega\psi(z)d\nu_f(z)\right] = \int_\Omega\frac{1}{2\pi}\mathbb{E}\left[\log|f(z)|\right]\Delta\psi(z)dm(z) = \int_\Omega\frac{1}{2\pi}\Delta\mathbb{E}\left[\log|f(z)|\right]\psi(z)dm(z).$$

Now, since $\zeta = \frac{f(z)}{\sqrt{\mathcal{K}_f(z,z)}}$ is a standard complex Gaussian random variable:

$$\begin{aligned}\mathbb{E}\left[\log|f(z)|\right] &= \mathbb{E}\left[\log\left|\frac{f(z)}{\sqrt{\mathcal{K}_f(z,z)}}\sqrt{\mathcal{K}_f(z,z)}\right|\right] = \mathbb{E}\left[\log\left|\frac{f(z)}{\sqrt{\mathcal{K}_f(z,z)}}\right|\right] + \log\sqrt{\mathcal{K}_f(z,z)}\\ &= \mathbb{E}\left[\log|\zeta|\right] + \frac{1}{2}\log\mathcal{K}_f(z,z) = \tilde{C} + \frac{1}{2}\log\mathcal{K}_f(z,z).\end{aligned}$$

where $\tilde{C}$ is a constant value. Therefore:

$$\mathbb{E}\left[\int_\Omega\psi(z)d\nu_f(z)\right] = \int_\Omega\frac{1}{4\pi}\Delta\log\mathcal{K}_f(z,z)\psi(z)dm(z),$$

and by the definition of the first intensity we have, with respect to the Lebesgue measure:

$$\rho_1(z) = \frac{1}{4\pi}\Delta\log\mathcal{K}_f(z,z).$$

□

The coefficients of the linear combination of holomorphic functions of a GAF in the Paley - Wiener can be real or complex Gaussian random variables. The Edelman - Kostlan formula is valid in the case of using complex random variables. The real case is slightly different. The paper [8], by Naomi D. Feldheim, will give us all the tools to prove the other version of the Edelman - Kostlan formula. Before introducing the theorem, we must define what is a *symmetric* GAF.

Definition 3.5. *Let f be a random analytic function on a symmetric domain Ω, that is $\Omega = \overline{\Omega}$. We say that f is a* symmetric GAF *if $f = \sum_{n=1}^{+\infty}\zeta_n f_n$, where ζ_n are independent random variables with $N_{\mathbb{R}}(0,1)$ distribution and $(f_n)_n$ is a sequence of analytic functions such that $\sum_{n=1}^{+\infty}|f_n(z)|^2$ converges uniformly on compact sets of the domain Ω and $(f_n)_n$ is symmetric with respect to the real axis, that is $\overline{f_n(z)} = f_n(\overline{z})$ for all z of Ω.*

Theorem 3.6. *Let f be a symmetric Gaussian analytic function with mean zero and covariance kernel $\mathcal{K}_f(z,w)$, for all z and w of Ω. The first intensity of the zeros of f is given by:*

$$\rho_1(z) = \frac{1}{4\pi}\Delta \log\left(\mathcal{K}_f(z,z) + \sqrt{\mathcal{K}_f(z,z)^2 - |\mathcal{K}_f(z,\overline{z})|^2}\right),$$

where the Laplacian Δ must be interpreted in the distributional sense.

As a remark, the covariance kernel in the real field is equal to the complex one if we consider $(\zeta_n)_n \sim N_{\mathbb{R}}(0,1)$ or $(\zeta_n)_n \sim N_{\mathbb{C}}(0,1)$.

Proof. (see Section 4 in [8]) We must see that:

$$\begin{aligned}\int_\Omega \psi(z)\rho_1(z) = \mathbb{E}\left[\int_\Omega \psi(z) d\nu_f(z)\right] &= \int_\Omega \frac{1}{2\pi}\mathbb{E}\left[\Delta \log|f(z)|\psi(z)\right] dm(z)\\ &= \int_\Omega \frac{1}{2\pi}\Delta\mathbb{E}\left[\log|f(z)|\right]\psi(z) dm(z),\end{aligned}$$

where we applied Fubini's theorem. The integrability of the terms of the integrals is due to the same reason than in Theorem 3.4. We have to compute $\mathbb{E}\left[\log|f(z)|\right]$. First of all, if f is a GAF as described and $(f_n)_n$ is a sequence of analytic functions in Ω , we can write:

$$f(z) = \sum_{n=1}^{+\infty}\zeta_n f_n(z) = \sum_{n=1}^{+\infty}\zeta_n u_n(z) + i\sum_{n=1}^{+\infty}\zeta_n v_n(z) = u(z) + iv(z),$$

where $(\zeta_n)_n$ are random variables that follow a real standard Gaussian distribution and u_n and v_n are real functions that denote the real and imaginary part of f_n respectively. Therefore the random vector (u,v) follows a real centered Gaussian distribution with covariance matrix:

$$\begin{aligned}\Lambda &= \begin{pmatrix}\mathbb{E}\left[\left(\sum_{n=1}^{+\infty}\zeta_n u_n\right)\left(\sum_{n=1}^{+\infty}\zeta_n u_n\right)\right] & \mathbb{E}\left[\left(\sum_{n=1}^{+\infty}\zeta_n u_n\right)\left(\sum_{n=1}^{+\infty}\zeta_n v_n\right)\right]\\ \mathbb{E}\left[\left(\sum_{n=1}^{+\infty}\zeta_n u_n\right)\left(\sum_{n=1}^{+\infty}\zeta_n v_n\right)\right] & \mathbb{E}\left[\left(\sum_{n=1}^{+\infty}\zeta_n v_n\right)\left(\sum_{n=1}^{+\infty}\zeta_n v_n\right)\right]\end{pmatrix}\\ &\overset{(*)}{=} \begin{pmatrix}\sum_{n=1}^{+\infty} u_n^2 & \sum_{n=1}^{+\infty} u_n v_n\\ \sum_{n=1}^{+\infty} u_n v_n & \sum_{n=1}^{+\infty} v_n^2\end{pmatrix},\end{aligned}$$

where in $(*)$ we applied the linearity of the mean and that $(\zeta_n)_n \sim N_{\mathbb{R}}(0,1)$.
Now, we claim that the eigenvalues of Λ are

$$\lambda_1 = \frac{\mathcal{K}_f(z,z) \mp |\mathcal{K}_f(z,\overline{z})|}{2}, \quad \lambda_2 = \frac{\mathcal{K}_f(z,z) \pm |\mathcal{K}_f(z,\overline{z})|}{2}.$$

Indeed, for every $f_n = u_n + iv_n$ we have the equations, which are simple to check:

$$u_n^2 = \frac{1}{2}\left(|f_n|^2 + \mathrm{Re}(f_n^2)\right), \;\; v_n^2 = \frac{1}{2}\left(|f_n|^2 - \mathrm{Re}(f_n^2)\right), \;\; u_n v_n = \frac{1}{2}\mathrm{Im}(f_n^2).$$

Therefore the covariance matrix can be expressed as:

$$\Lambda = \begin{pmatrix}\frac{1}{2}\left(\sum_{n=1}^{+\infty}|f_n|^2 + \mathrm{Re}\left(\sum_{n=1}^{+\infty} f_n^2\right)\right) & \frac{1}{2}\mathrm{Im}\left(\sum_{n=1}^{+\infty} f_n^2\right)\\ \frac{1}{2}\mathrm{Im}\left(\sum_{n=1}^{+\infty} f_n^2\right) & \frac{1}{2}\left(\sum_{n=1}^{+\infty}|f_n|^2 - \mathrm{Re}\left(\sum_{n=1}^{+\infty} f_n^2\right)\right)\end{pmatrix}.$$

Thus:

$$\det\Lambda = \frac{1}{4}\left(\left(\sum_{n=1}^{+\infty}|f_n|^2\right)^2 - \left(\mathrm{Re}\left(\sum_{n=1}^{+\infty}f_n^2\right)\right)^2 - \left(\mathrm{Im}\left(\sum_{n=1}^{+\infty}f_n^2\right)\right)^2\right) \tag{3.2}$$

$$= \frac{1}{4}\left(\mathcal{K}_f(z,z)^2 - |\mathcal{K}_f(z,\overline{z})|^2\right), \tag{3.3}$$

$$\mathrm{tr}\Lambda = \sum_{n=1}^{+\infty}|f_n|^2 = \mathcal{K}_f(z,z). \tag{3.4}$$

And knowing the fact that $\det\Lambda = \lambda_1\lambda_2$ and $\mathrm{tr}\Lambda = \lambda_1+\lambda_2$ we have that $\det\Lambda = (\mathrm{tr}\Lambda - \lambda_2)\,\lambda_2$. Then:

$$\begin{aligned}\frac{1}{4}\left(\mathcal{K}_f(z,z)^2 - |\mathcal{K}_f(z,\overline{z})|^2\right) &= \mathcal{K}_f(z,z)\lambda_2 - \lambda_2^2\\ &\Longrightarrow \lambda_2^2 - \mathcal{K}_f(z,z)\lambda_2 + \frac{1}{4}\left(\mathcal{K}_f(z,z) - |\mathcal{K}_f(z,\overline{z})|^2\right) = 0\\ &\Longrightarrow \lambda_2 = \frac{\mathcal{K}_f(z,z) \pm \sqrt{\mathcal{K}_f(z,z) - \mathcal{K}_f(z,z) + |\mathcal{K}_f(z,\overline{z})|^2}}{2}\\ &\Longrightarrow \lambda_2 = \frac{\mathcal{K}_f(z,z) \pm |\mathcal{K}_f(z,\overline{z})|}{2},\end{aligned}$$

and this implies that:

$$\lambda_1 = \frac{\mathcal{K}_f(z,z) \mp |\mathcal{K}_f(z,\overline{z})|}{2}.$$

Now:

$$\mathbb{E}\left[\log|f(z)|\right] = \frac{1}{2\pi\sqrt{\det\Lambda}}\int_{\mathbb{R}^2}\log\left(\sqrt{x^2+y^2}\right)e^{-\frac{1}{2}(x\ y)\Lambda^{-1}(x\ y)^t}dxdy. \tag{3.5}$$

To compute the factor $(x\ y)\Lambda^{-1}(x\ y)^t$ we will use the change of variable:

$$(\tilde{x}\ \tilde{y})^t = M(x\ y)^t \Longrightarrow M^{-1}(\tilde{x}\ \tilde{y})^t = (x\ y)^t,$$

where M is a two - dimensional orthogonal matrix, that is, M satisfies $M^t = M^{-1}$. Also we have that $\tilde{x}^2+\tilde{y}^2 = x^2+y^2$, $d\tilde{x}d\tilde{y} = dxdy$ and the determinant of the Jacobian matrix is 1. Knowing the fact that the covariance matrix Λ can be written as $\Lambda = M^tDM$, where D is a diagonal matrix whose elements are the eigenvalues of Λ, then:

$$\begin{aligned}(x\ y)\Lambda^{-1}(x\ y)^t &= (\tilde{x}\ \tilde{y})M^{-t}\Lambda^{-1}M^{-1}(\tilde{x}\ \tilde{y})^t = (\tilde{x}\ \tilde{y})M\Lambda^{-1}M^t(\tilde{x}\ \tilde{y})^t\\ &= \frac{1}{\lambda_1\lambda_2}(\tilde{x}\ \tilde{y})\begin{pmatrix}\lambda_2 & 0\\ 0 & \lambda_1\end{pmatrix}(\tilde{x}\ \tilde{y})^t = \frac{\lambda_2\tilde{x}^2+\lambda_1\tilde{y}^2}{\lambda_1\lambda_2} = \lambda_1^{-1}\tilde{x}^2 + \lambda_2^{-1}\tilde{y}^2.\end{aligned}$$

From (3.5) we obtain:

$$
\begin{aligned}
\mathbb{E}\left[\log|f(z)|\right] &= \frac{1}{2\pi\sqrt{\det\Lambda}}\int_{\mathbb{R}^2}\log\left(\sqrt{\tilde{x}^2+\tilde{y}^2}\right)e^{-\frac{1}{2}\lambda_1^{-1}\tilde{x}^2+\lambda_2^{-1}\tilde{y}^2}d\tilde{x}d\tilde{y}\\
&\overset{(*)}{=} \frac{1}{2\pi}\int_{\mathbb{R}^2}\log\left(\sqrt{\lambda_1u^2+\lambda_2v^2}\right)e^{-\frac{1}{2}(u^2+v^2)}dudv\\
&\overset{(**)}{=} \frac{1}{2\pi}\int_0^{+\infty}\int_0^{2\pi}\log\left(\sqrt{\lambda_1r^2\cos^2\theta+\lambda_2r^2\sin^2\theta}\right)re^{-\frac{r^2}{2}}d\theta dr\\
&= \int_0^{+\infty}r\log(r)e^{-\frac{r^2}{2}}dr+\frac{1}{2\pi}\underbrace{\int_0^{+\infty}re^{-\frac{r^2}{2}}dr}_{1}\int_0^{2\pi}\log\left(\sqrt{\lambda_1\cos^2\theta+\lambda_2\sin^2\theta}\right)d\theta\\
&= \frac{1}{2\pi}\int_0^{2\pi}\log\left(\sqrt{\lambda_1\cos^2\theta+\lambda_2\sin^2\theta}\right)d\theta+C_1,
\end{aligned}
$$

where in $(*)$ we use the change of variable $\tilde{x}=u\sqrt{\lambda_1}$, $\tilde{y}=v\sqrt{\lambda_2}$ with Jacobian $\sqrt{\lambda_1\lambda_2}=\det\Lambda$. In $(**)$ we apply polar coordinates $u=r\cos\theta$ and $v=r\sin\theta$. The factor C_1 is a constant value that will vanish with the Laplacian operator. We have that:

$$
\sqrt{\lambda_1\cos^2\theta+\lambda_2\sin^2\theta}=\left|\sqrt{\lambda_1}\cos\theta+i\sqrt{\lambda_2}\sin\theta\right|,
$$

and:

$$
\begin{aligned}
\sqrt{\lambda_1}\cos\theta+i\sqrt{\lambda_2}\sin\theta &= \sqrt{\lambda_1}\frac{e^{i\theta}+e^{-i\theta}}{2}+i\sqrt{\lambda_2}\frac{e^{i\theta}-e^{-i\theta}}{2i}\\
&= \frac{1}{2}\left[e^{i\theta}(\sqrt{\lambda_1}+\sqrt{\lambda_2})+e^{-i\theta}(\sqrt{\lambda_1}-\sqrt{\lambda_2})\right]\\
&= \frac{\sqrt{\lambda_1}+\sqrt{\lambda_2}}{2}e^{-i\theta}\left(e^{2i\theta}+\frac{\sqrt{\lambda}-\sqrt{\lambda_2}}{\sqrt{\lambda_1}+\sqrt{\lambda_2}}\right).
\end{aligned}
$$

Therefore:

$$
\left|\sqrt{\lambda_1}\cos\theta+i\sqrt{\lambda_2}\sin\theta\right|=\left|\frac{\sqrt{\lambda_1}+\sqrt{\lambda_2}}{2}\right|\left|e^{2i\theta}+\frac{\sqrt{\lambda}-\sqrt{\lambda_2}}{\sqrt{\lambda_1}+\sqrt{\lambda_2}}\right|.
$$

This leads to:

$$
\begin{aligned}
\mathbb{E}\left[\log|f(z)|\right] &= \frac{1}{2\pi}\int_0^{2\pi}\log\left(\sqrt{\lambda_1\cos^2\theta+\lambda_2\sin^2\theta}\right)d\theta+C_1\\
&= \frac{1}{2\pi}\int_0^{2\pi}\log\left(\frac{\sqrt{\lambda_1}+\sqrt{\lambda_2}}{2}\left|e^{2i\theta}+\frac{\sqrt{\lambda}-\sqrt{\lambda_2}}{\sqrt{\lambda_1}+\sqrt{\lambda_2}}\right|\right)d\theta+C_1\\
&= \log\left(\sqrt{\lambda_1}+\sqrt{\lambda_2}\right)+\frac{1}{2\pi}\int_0^{2\pi}\log\left|e^{2i\theta}+\frac{\sqrt{\lambda}-\sqrt{\lambda_2}}{\sqrt{\lambda_1}+\sqrt{\lambda_2}}\right|d\theta+C_2,
\end{aligned}
$$

where C_2 is another constant that will also vanish with the Laplacian operator. To compute the last integral we use Jensen's formula. Let us denote $h(z)=z^2+k$, where $k=\frac{\sqrt{\lambda}-\sqrt{\lambda_2}}{\sqrt{\lambda_1}+\sqrt{\lambda_2}}<1$. Hence, since h has two simple roots in $\mathbb{D}$, let them be z_1 and z_2, we have:

$$
\frac{1}{2\pi}\int_0^{2\pi}\log|h(e^{i\theta})|d\theta=\log|h(0)|-\log|z_1|-\log|z_2|=\log|k|-2\log\left(\sqrt{|k|}\right)=0.
$$

Then, by (3.3) and (3.4), it is straightforward that:

$$\begin{aligned}\mathbb{E}\left[\log|f(z)|\right] &= \log\left(\sqrt{\lambda_1}+\sqrt{\lambda_2}\right) = \frac{1}{2}\log\left(\lambda_1+\lambda_2+\sqrt{4\lambda_1\lambda_2}\right) \\ &= \frac{1}{2}\log\left(\mathcal{K}_f(z,z)+\sqrt{\mathcal{K}_f(z,z)^2-|\mathcal{K}_f(z,\overline{z})|^2}\right).\end{aligned}$$

Therefore:

$$\mathbb{E}\left[\int_\Omega \psi(z)d\nu_f(z)\right] = \int_\Omega \frac{1}{4\pi}\Delta\log\left(\mathcal{K}_f(z,z)+\sqrt{\mathcal{K}_f(z,z)^2-|\mathcal{K}_f(z,\overline{z})|^2}\right)\psi(z)dm(z),$$

and by the definition of the first intensity we have, with respect to the Lebesgue measure:

$$\rho_1(z) = \frac{1}{4\pi}\Delta\log\left(\mathcal{K}_f(z,z)+\sqrt{\mathcal{K}_f(z,z)^2-|\mathcal{K}_f(z,\overline{z})|^2}\right).$$

□

Now we can compute the first intensity for all the Hilbert spaces of analytic functions described in the last chapter.

3.1.1 The Fock space in $\mathbb{C}$

The covariance kernel of a GAF f in the Fock space is

$$\mathcal{K}_f(z,w) = e^{Lz\overline{w}},$$

for all real $L>0$. Thus:

$$\Delta\log\left(\mathcal{K}_f(z,z)\right) = 4\frac{\partial}{\partial z}\frac{\partial}{\partial\overline{z}}(Lz\overline{z}) = 4L,$$

and the first intensity is

$$\rho_1(z) = \frac{L}{\pi}.$$

As a remark, in Chapter 2 we saw that the zero set of a GAF in the Fock space was invariant under translations. Thus, it is not a surprise that the first intensity does not depend of the point we are computing this value. In addition, the first intensity is also invariant under translations in $\mathbb{C}$ and it is proportional to the Lebesgue measure.

3.1.2 The space of polynomials in $\mathbb{S}^2$

The covariance kernel of a GAF f in the space of polynomial of the degree at most L, for a natural L, is

$$\mathcal{K}_f(z,w) = (1+z\overline{w})^L.$$

Thus:

$$\Delta\log\left(\mathcal{K}_f(z,z)\right) = 4L\frac{\partial}{\partial z}\frac{\partial}{\partial\overline{z}}\left[\log(1+|z|^2)\right] = 4L\frac{\partial}{\partial z}\left(\frac{z}{1+|z|^2}\right) = \frac{4L}{(1+|z|^2)^2},$$

and the first intensity is

$$\rho_1(z) = \frac{L}{\pi}\frac{1}{(1+|z|^2)^2}.$$

In Chapter 2 we proved that the zero set of a GAF in the space of polynomials with finite degree was invariant under the Möbius transformation

$$\varphi_a(z) = \frac{z-a}{1+\overline{a}z},$$

for z and a values of $\mathbb{C}$. In this case, if we multiply the first intensity by the Lebesgue measure over the sphere and we apply the inverse mapping of the stereographic projection, we get that the first intensity is invariant under rotations. Indeed, following the notation of the proof of Proposition 2.5, we have that:

$$\frac{dm(z)}{(1+|z|^2)} = \frac{(1+|a|^2)^2(1-\overline{a}w)^2(1-a\overline{w})^2}{(1-\overline{a}w)^2(1-a\overline{w})^2(1+|a|^2)^2(1+|w|^2)^2}dm(w) = \frac{dm(w)}{(1+|w|^2)^2}.$$

3.1.3 The weighted Bergman space in $\mathbb{D}$

The covariance kernel of a GAF f in the weighted Bergman space is

$$\mathcal{K}_f(z,w) = (1-zw)^{-L},$$

for all real $L > 0$. Thus:

$$\Delta \log\left(\mathcal{K}_f(z,z)\right) = -4L\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}}\left[\log(1-|z|^2)\right] = 4L\frac{\partial}{\partial z}\left(\frac{z}{1-|z|^2}\right) = \frac{4L}{(1-|z|^2)^2},$$

and the first intensity is

$$\rho_1(z) = \frac{L}{\pi}\frac{1}{(1-|z|^2)^2}.$$

In this case, the first intensity in the weighted Bergman space is proportional to the hyperbolic measure

$$\frac{dm(z)}{(1-|z|^2)^2}.$$

It is straightforward that ρ_1 is invariant under automorphisms in $\mathbb{D}$. The proof of this is analogous than the one in the space of polynomials in $\mathbb{S}^2$.

3.1.4 The Paley - Wiener space with $N_{\mathbb{C}}(0,1)$ random variables

The covariance kernel of a GAF f in the Paley - Wiener space is

$$\mathcal{K}_f(z,w) = \frac{\sin\pi(z-\overline{w})}{\pi(z-\overline{w})}.$$

First of all we should rewrite the covariance kernel $\mathcal{K}_f(z,z)$ into another expression. If we denote z as $x+iy$ for all real values $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$, we have:

$$\frac{\sin\pi(z-\overline{z})}{\pi(z-\overline{z})} = \frac{\sin(2\pi iy)}{2\pi iy} = \frac{e^{2\pi y}-e^{-2\pi y}}{4\pi y} = \frac{\sinh(2\pi y)}{2\pi y}.$$

For the following we are going to consider only that $y > 0$, because the last function is an even one. Taking logarithms we have:

$$\log \mathcal{K}_f(y,y) = \log \sinh(2\pi y) - \log(2\pi y),$$

and applying the Laplacian $\Delta_{(x,y)} \equiv \partial^2_{xx} + \partial^2_{yy}$ at both sides:

$$\begin{aligned}
\Delta \log \mathcal{K}_f(y,y) &= \Delta \log \sinh(2\pi y) - \Delta \log(2\pi y) = \frac{\partial^2}{\partial y^2} \log \sinh(2\pi y) - \frac{\partial^2}{\partial y^2} \log(2\pi y) \\
&= \frac{\partial}{\partial y} 2\pi \frac{\cosh(2\pi y)}{\sinh(2\pi y)} - \frac{\partial}{\partial y}\frac{1}{y} = \frac{\partial}{\partial y} 2\pi \coth(2\pi y) - \frac{\partial}{\partial y}\frac{1}{y} \\
&\overset{(*)}{=} -4\pi^2 \text{csch}^2(2\pi y) + \frac{1}{y^2},
\end{aligned}$$

where in $(*)$ we use that $d/dx(\coth x) = -\text{csch}^2 x$. Therefore, the first intensity is:

$$\rho_1(y) = \frac{1}{4\pi y^2} - \pi \text{csch}^2(2\pi y),$$

and we can see that is invariant under translations on the real line because it only depends of $\text{Im}(z)$. At first glance, it seems we have a potential problem at $y = \text{Im}(z) = 0$. However we can specify the value of the first intensity at that point. The series of $\text{csch}^2(2\pi y)$ at a neighbourhood of 0 is:[1]

$$\text{csch}^2(2\pi y) = \frac{1}{4\pi^2 y^2} - \frac{1}{3} + \frac{4\pi^2 y^2}{15} + \mathcal{O}(y^4).$$

Then:

$$\lim_{y\to 0} \rho_1(y) = \lim_{y \to 0} \left(\frac{1}{4\pi y^2} - \frac{1}{4\pi y^2} + \frac{\pi}{3} - \frac{4\pi^3 y^2}{15} + \mathcal{O}(y^4) \right) = \frac{\pi}{3}.$$

3.1.5 The Paley - Wiener space with $N_{\mathbb{R}}(0,1)$ random variables

The covariance kernel of a GAF f in the Paley - Wiener space is

$$\mathcal{K}_f(z,w) = \frac{\sin \pi(z - \overline{w})}{\pi(z-\overline{w})}.$$

By Theorem 3.6, the first intensity is:

$$\rho_1(z) = \frac{1}{4\pi} \Delta \log \left(\mathcal{K}_f(z,z) + \sqrt{\mathcal{K}_f(z,z)^2 - |\mathcal{K}_f(z,\overline{z})|^2} \right).$$

We are going to calculate the first intensity in terms of the imaginary part of z. Denoting $\text{Im}(z) = y$ for short, $\kappa(y) = \mathcal{K}_f(z,z)$, $\mathcal{K}_f(z,\overline{z}) = 1$ (this value is due to the definition of the cardinal sine) and following a procedure of [8], we have:

$$\rho_1(y) = \frac{1}{4\pi} \Delta \log \left(\kappa(y) + \sqrt{\kappa(y)^2 - 1} \right) = \frac{1}{4\pi} \frac{\partial}{\partial y} \frac{\kappa'(y)}{\sqrt{\kappa(y)^2 - 1}}.$$

[1] Expression extracted from [29].

Indeed:

$$\frac{\partial^2}{\partial y^2}\log\left(\kappa(y)+\sqrt{\kappa(y)^2-1}\right)=\frac{\partial}{\partial y}\frac{\kappa'(y)+\frac{2\kappa(y)\kappa'(y)}{2\sqrt{\kappa(y)^2-1}}}{\kappa(y)+\sqrt{\kappa(y)^2-1}}$$
$$=\frac{\partial}{\partial y}\frac{\kappa'(y)\sqrt{\kappa(y)^2-1}+\kappa(y)\kappa'(y)}{(\kappa(y)+\sqrt{\kappa(y)^2-1})(\sqrt{\kappa(y)^2-1})}$$
$$=\frac{\partial}{\partial y}\frac{\kappa'(y)}{\sqrt{\kappa(y)^2-1}}.$$

In our case, if

$$\kappa(y)=\frac{\sin(2\pi i y)}{2\pi i y},$$

we have, using the identities $\sinh y=-i\sin iy$ and $\cosh y=\cos iy$, that

$$\kappa'(y)=\frac{-4\pi^2 y\cos(2\pi i y)-2\pi i\sin(2\pi i y)}{-4\pi^2y^2}=\frac{2\pi y\cosh(2\pi y)-\sinh(2\pi y)}{2\pi y^2},$$

and

$$\rho_1(y)=\frac{1}{4\pi}\frac{\partial}{\partial y}\frac{2\pi y\cosh(2\pi y)-\sinh(2\pi y)}{y\sqrt{\sinh^2(2\pi y)-4\pi^2y^2}}.$$

Thus:[2]

$$\rho_1(y)=\frac{(3+48\pi^2y^2+64\pi^4y^4)\sinh(2\pi y)-\sinh(6\pi y)-64\pi^3y^3\cosh(2\pi y)}{16y^3(4\pi^3y^2-\pi\sinh^2(2\pi y))\sqrt{\frac{\sinh^2(2\pi y)}{y^2}-4\pi^2}},$$

and it is invariant under translations on the real line because it only depends of $\mathrm{Im}(z)$. To see what is the value of the first intensity at neighbourhood of 0, we can write κ as a Taylor series near of 0. Since κ is an even function, we can consider only $\mathrm{Im}(z)=y>0$. We have that:

$$\kappa(y)=1+\frac{2}{3}\pi^2y^2+\mathcal{O}(y^4),\quad \kappa^2(y)=1+\frac{4}{3}\pi^2y^2+\mathcal{O}(y^4).$$

Then:

$$\rho_1(y)=\frac{1}{4\pi}\Delta\log\left(1+\frac{2}{3}\pi^2y^2+\mathcal{O}(y^4)+\sqrt{1+\frac{4}{3}\pi^2y^2-1+\mathcal{O}(y^4)}\right)$$
$$=\frac{1}{4\pi}\Delta\log\left(1+\frac{2}{3}\pi^2y^2+\mathcal{O}(y^4)+\frac{2}{\sqrt{3}}\pi y\sqrt{1+\mathcal{O}(y^2)}\right)$$
$$=\frac{1}{4\pi}\Delta\log\left(1+\frac{2}{3}\pi^2y^2+\mathcal{O}(y^4)+\frac{2}{\sqrt{3}}\pi y\left[1+\frac{1}{2}\mathcal{O}(y^2)\right]\right)$$
$$=\frac{1}{4\pi}\Delta\log\left(1+\frac{2}{\sqrt{3}}\pi y+\frac{2}{3}\pi^2y^2+\mathcal{O}(y^3)\right)=\frac{1}{4\pi}\Delta\left(\frac{2}{\sqrt{3}}\pi y+\mathcal{O}(y^3)\right),$$

[2]Expression extracted from [30].

where in the last equality we applied the Taylor series of $\log(1+y)$ for y in a neighbourhood of 0, and we omitted long computations. At the end, the first intensity in a neighborhood of the origin takes low values. Considering any value of y at a neighbourhood of 0, it is clear that the expression $\frac{2}{\sqrt{3}}\pi|y| + \mathcal{O}(y^3)$ is not differentiable. However, using the concept of derivation in the distributional sense we will conclude that $d^2/dx^2(|x|) = 2\delta_0$. The delta of Dirac δ with a test function τ is

$$(\delta, \tau) = \tau(0),$$

and the derivative in a distributional sense for a locally integrable function f is:

$$(f', \tau) = \int_{\mathbb{R}} f'(x)\tau(x)dx = -\int_{\mathbb{R}} f(x)\tau'(x)dx = -(f, \tau').$$

The derivative of the absolute value can be described as:

$$H(x) = \operatorname{sgn}(x) = \frac{d|x|}{dx} = \begin{cases} 1, & \text{if } x > 0, \\ -1, & \text{if } x < 0. \end{cases}$$

The function H is the so-called *Heaviside function*. Now we have:

$$(H', \tau) = -(H, \tau') = -\int_{\mathbb{R}} H(x)\tau'(x)dx = -2\int_0^{+\infty} \tau'(x)dx = 2\tau(0) = 2(\delta, \tau).$$

Then the derivative of the absolute value behaves like $2\delta_0$, and the first intensity we wanted to compute at $y = 0$ is $\frac{1}{\sqrt{3}}\delta_0$.

3.2 Calabi's rigidity

In this section we will enunciate and prove the Calabi's rigidity. It guarantees that two GAFs with the same first intensity are equal in distribution. Therefore, this theorem of uniqueness concludes that the zero set of a GAF is determined by its first intensity.
Before going to the main result, we have to see the next lemma:

Lemma 3.7. *Let $F(z, w)$ be a function in a region Ω that is holomorphic in z and anti-holomorphic in w, for z and w of Ω. If $F(z, z) = 0$ for all z of Ω, then $F(z, w) = 0$ for z and w of Ω.*

Proof. (See proof of Lemma 2.5.1 in [2], page 30) It is only necessary to see that F is zero in a neighbourhood of $(0, 0)$ in Ω. Since F is a real analytic function, we can express F as:

$$F(z, w) = \sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty} c_{n,m} z^n \overline{w}^m, \tag{3.6}$$

for complex values $c_{n,m}$. Therefore it also holds that:

$$F(z, z) = \sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty} c_{n,m} z^n \overline{z}^m.$$

We state that:

$$\frac{\partial^{n+m}}{\partial z^n \partial \overline{z}^m} z^j \overline{z}^k \mid_{z=0} = \delta_{(n,m)(j,k)} n!m!,$$

where

$$\delta_{(n,m)(j,k)} = \begin{cases} 1, & \text{if } j = n \text{ and } k = m, \\ 0, & \text{otherwise.} \end{cases}$$

Rewriting (3.6) as:

$$F(z,z) = \sum_{j=1}^{+\infty} \sum_{k=1}^{+\infty} c_{j,k} z^j \overline{z}^k,$$

and using the $F(z,z) = 0$ hypothesis, we arrive to:

$$0 = \frac{\partial^{n+m}}{\partial z^n \partial \overline{z}^m} F(z,z) \mid_{z=0} = c_{n,m} n!m!$$

Thus, $c_{n,m} = 0$ and $F(z,w) = 0$ for all z and w of Ω. □

Now we are ready for Calabi's rigidity theorem.

Theorem 3.8. (Calabi's rigidity) *Let Ω be a simply connected domain of $\mathbb{C}$. Let f and g be GAFs in Ω. If the first intensity of f and g are equal, there exists a deterministic analytic function ψ on Ω such that it does not vanish at any point and it holds that $f \overset{d}{=} \psi g$. In addition, $\mathcal{Z}_f \overset{d}{=} \mathcal{Z}_g$. As a conclusion, the first intensity of a GAF determines its zero set.*

Proof. (See proof of Theorem 2.5.2. in [2], pages 30 and 31) For all z of Ω, we have that z is almost surely a zero of a GAF f if and only if it is a zero of $\mathcal{K}_f(z,z)$, and the multiplicity of z is the same for f and the covariance kernel. Indeed; let $(f_n)_n$ be a sequence of analytic functions on Ω. Since:

$$\mathcal{K}_f(z,z) = \sum_{n=1}^{+\infty} f_n(z)\overline{f_n(z)} = \sum_{n=1}^{+\infty} |f_n(z)|^2 \text{ and } f(z) = \sum_{n=1}^{\infty} \zeta_n f_n(z),$$

for $\zeta_n \sim N_{\mathbb{C}}(0,1)$, we have, almost surely, that:

$$\sum_{n=1}^{+\infty} |f_n(z)|^2 = 0 \Leftrightarrow f_n(z) = 0 \Leftrightarrow f(z) = \sum_{n=1}^{\infty} \zeta_n f_n(z) = 0 \Leftrightarrow f(z) = 0.$$

Since we assumed that f and g have the same first intensity, the discrete set of deterministic zeros, let us call it D, is the same for both GAFs, whatever are the multiplicities of these zeros. However, we are interested in the random zeros of f and g. Then, we subtract the set D of Ω and we conclude that f and g do not vanish anywhere in $\Omega \setminus D$, or which is the same, $\mathcal{K}_f(z,z)$ and $\mathcal{K}_g(z,z)$ are not zero for all z of $\Omega \setminus D$.
Since $\mathcal{K}_f(z,z)$ and $\mathcal{K}_g(z,z)$ are not zero for all z of $\Omega \setminus D$, by hypothesis we have that:

$$\frac{1}{4\pi}\Delta \log \mathcal{K}_f(z,z) = \frac{1}{4\pi}\Delta \log \mathcal{K}_g(z,z) \Longrightarrow \Delta \left[\log(\mathcal{K}_f(z,z)) - \log(\mathcal{K}_g(z,z))\right] = 0$$

$$\Longrightarrow \Delta \log \left(\frac{\mathcal{K}_f(z,z)}{\mathcal{K}_g(z,z)}\right) = 0,$$

and this implies that $\log\left(\frac{\mathcal{K}_f(z,z)}{\mathcal{K}_g(z,z)}\right)$ is a harmonic function in $\Omega \setminus D$. Let us denote the last expression as u, which is obvious that u is a harmonic function in $\Omega \setminus D$. Then:

$$\log\left(\frac{\mathcal{K}_f(z,z)}{\mathcal{K}_g(z,z)}\right) = u(z) \Longrightarrow \frac{\mathcal{K}_f(z,z)}{\mathcal{K}_g(z,z)} = e^{u(z)} \Longrightarrow \mathcal{K}_f(z,z) = e^{u(z)}\mathcal{K}_g(z,z).$$

Since Ω is simply connected there exists an analytic function h in $\Omega \setminus D$ such that $2\mathrm{Re}(h) = u$. Thus, by

$$e^{u(z)} = e^{2\mathrm{Re}(h(z))} = |e^{h(z)}|^2 = e^{h(z)}\overline{e^{h(z)}} = \psi(z)\overline{\psi(z)},$$

we have that

$$\mathcal{K}_f(z,z) = \psi(z)\overline{\psi(z)}\mathcal{K}_g(z,z), \tag{3.7}$$

Then, the functions $\mathcal{K}_f(z,w)$ and $\psi(z)\overline{\psi(w)}\mathcal{K}_g(z,w)$ are equal on the diagonal by (3.7). Since both last functions satisfy the hypothesis of Lemma 3.7, it holds true that:

$$\mathcal{K}_f(z,w) = \psi(z)\overline{\psi(w)}\mathcal{K}_g(z,w),$$

and this implies that $f \stackrel{d}{=} \psi g$. Since ψ is the exponential function, it never vanishes, and we get that $\mathcal{Z}_f \stackrel{d}{=} \mathcal{Z}_g$. □

Chapter 4

GAF computation

In this last chapter of the project we will obtain the first experimental results using C++ coding and the plotting program `gnuplot`. The codes are specified in *Annex*.

4.1 GAF in the finite space of polynomials endowed with the norm $\|\cdot\|^2_{\mathcal{F}_L}$

In this section we will consider the finite space of polynomials endowed with the norm of the Fock space, $\|\cdot\|^2_{\mathcal{F}_L}$, so let m be a natural number such that the orthonormal basis is $(e_n)_{n=0}^m$, where:

$$e_n = \frac{\sqrt{L^n}}{\sqrt{n!}} z^n.$$

Before calculating the first intensity of a GAF f, we should introduce the *incomplete gamma function*, that is, for a given positive real a and non-negative x:

$$\Gamma(a,x) = \int_x^{+\infty} t^{a-1} e^{-t} dt.$$

If n is a natural number, then:

$$\Gamma(n,x) = (n-1)! e^{-x} \sum_{k=0}^{n-1} \frac{x^k}{k!}.$$

Now, let us calculate the first intensity step by step. We have:

$$\mathcal{K}_f(z,z) = \sum_{n=0}^{m} \frac{L^n}{n!} |z|^{2n} = \sum_{n=0}^{m} \frac{(L|z|^2)^n}{n!} = \frac{\Gamma(m+1, L|z|^2)}{\Gamma(m+1)} e^{L|z|^2}.$$

Taking logarithms at both sides:

$$\log \mathcal{K}_f(z,z) = \log\left(\frac{\Gamma(m+1, L|z|^2)}{\Gamma(m+1)} e^{L|z|^2}\right) = \log\left(\frac{\Gamma(m+1, L|z|^2)}{\Gamma(m+1)}\right) + L|z|^2,$$

and taking the Laplacian operator at each side of the equality:

$$\Delta \log \mathcal{K}_f(z,z) = \Delta \log \left(\frac{\Gamma(m+1, L|z|^2)}{\Gamma(m+1)} \right) + 4L.$$

Thus, the first intensity is:

$$\rho_1(z) = \frac{1}{4\pi} \Delta \log \left(\frac{\Gamma(m+1, L|z|^2)}{\Gamma(m+1)} \right) + \frac{L}{\pi}.$$

Since m is a natural number, $\Gamma(m+1)$ will vanish with the Laplacian operator. Hence:

$$\rho_1(z) = \frac{1}{4\pi} \Delta \log \Gamma(m+1, L|z|^2) + \frac{L}{\pi}. \tag{4.1}$$

We have that:

$$\begin{aligned} \frac{\partial}{\partial \bar{z}} \Gamma(m+1, L|z|^2) &= -Lm! z e^{-L|z|^2} \sum_{n=0}^{m} \frac{L^n z^n \bar{z}^n}{n!} + m! e^{-L|z|^2} \sum_{n=1}^{m} \frac{L^n z^n \bar{z}^{n-1}}{(n-1)!} \\ &= -Lzm! e^{-L|z|^2} \sum_{n=0}^{m} \frac{L^n |z|^{2n}}{n!} + Lzm! e^{-L|z|^2} \sum_{n=0}^{m-1} \frac{L^n |z|^{2n}}{n!} \\ &= Lzm\Gamma(m, L|z|^2) - Lz\Gamma(m+1, L|z|^2). \end{aligned}$$

Then:

$$\begin{aligned} \Delta \log \Gamma(m+1, L|z|^2) &= 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} \log \Gamma(m+1, L|z|^2) && (4.2) \\ &= 4 \frac{\partial}{\partial z} \left(\frac{Lzm\Gamma(m, L|z|^2) - Lz\Gamma(m+1, L|z|^2)}{\Gamma(m+1, L|z|^2)} \right) && (4.3) \\ &= 4Lm \frac{\partial}{\partial z} \left(z \frac{\Gamma(m, L|z|^2)}{\Gamma(m+1, L|z|^2)} \right) - 4L. && (4.4) \end{aligned}$$

From now on, we are going to use the following notation to shorten further expressions:

$$\Gamma_k(z) \equiv \Gamma(k, L|z|^2),$$

for whatever natural number k is. Now:

$$\frac{\partial}{\partial z} \left(z \frac{\Gamma_m(z)}{\Gamma_{m+1}(z)} \right) = \frac{\Gamma_m(z)}{\Gamma_{m+1}(z)} + z \frac{\partial}{\partial z} \frac{\Gamma_m(z)}{\Gamma_{m+1}(z)}.$$

The derivative in the right side is:

$$\begin{aligned} \frac{\partial}{\partial z} \frac{\Gamma_m(z)}{\Gamma_{m+1}(z)} &= \frac{\partial_z \Gamma_m(z) \Gamma_{m+1}(z) - \Gamma_m(z) \partial_z \Gamma_{m+1}(z)}{\Gamma_{m+1}^2(z)} \\ &= \frac{(-Lz\Gamma_m(z) + L\bar{z}(m-1)\Gamma_{m-1}(z))\Gamma_{m+1}(z) - \Gamma_m(z)(L\bar{z}m\Gamma_m(z) - L\bar{z}\Gamma_{m+1}(z))}{\Gamma_{m+1}^2(z)} \\ &= \frac{L\bar{z}(m-1)\Gamma_{m-1}(z)\Gamma_{m+1}(z) - L\bar{z}m\Gamma_m^2(z)}{\Gamma_{m+1}^2(z)}. \end{aligned}$$

Thus:

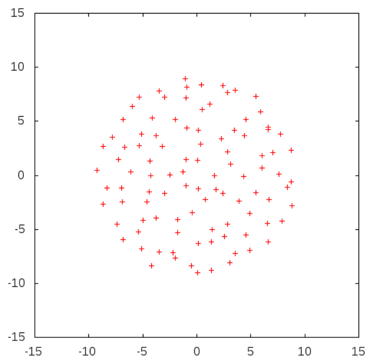
$$\frac{\partial}{\partial z}\left(z\frac{\Gamma_m(k)}{\Gamma_{m+1}(z)}\right) = \frac{\Gamma_m(z)}{\Gamma_{m+1}(z)} + \frac{L|z|^2(m-1)\Gamma_{m-1}(z)\Gamma_{m+1}(z) - L|z|^2 m\Gamma_m^2(z)}{\Gamma_{m+1}^2(z)} \tag{4.5}$$

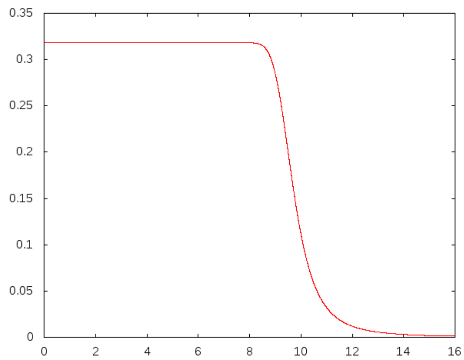
$$= \frac{\Gamma_m(z)\Gamma_{m+1}(z) + L|z|^2(m-1)\Gamma_{m-1}(z)\Gamma_{m+1}(z) - L|z|^2 m\Gamma_m^2(z)}{\Gamma_{m+1}^2(z)}. \tag{4.6}$$

Finally, using (4.6) to (4.4) and plugging the result in (4.1) we obtain:

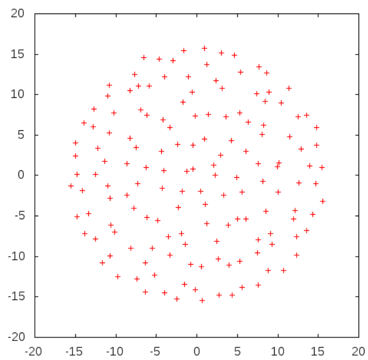
$$\rho_1(z) = \frac{Lm}{\pi}\frac{\Gamma_m(z)\Gamma_{m+1}(z) + L|z|^2(m-1)\Gamma_{m-1}(z)\Gamma_{m+1}(z) - L|z|^2 m\Gamma_m^2(z)}{\Gamma_{m+1}^2(z)}.$$

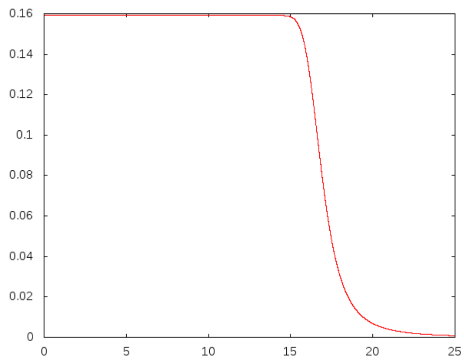
Using the program in 5.2 for the zero set (left) and 5.3 for the first intensity (right) and setting the values `n = 100` and `L = 1`, we obtain:



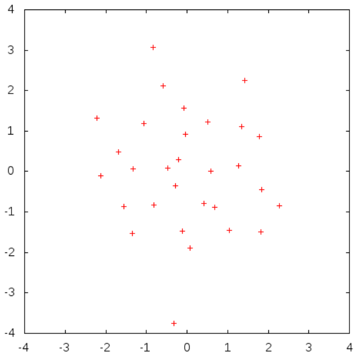


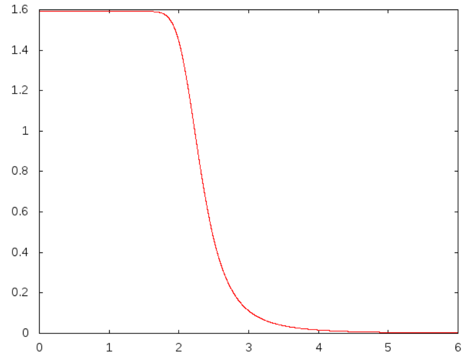
For `n = 150` and `L = 0.5`:





For `n = 30` and `L = 5`:

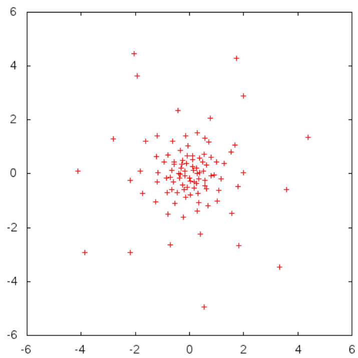


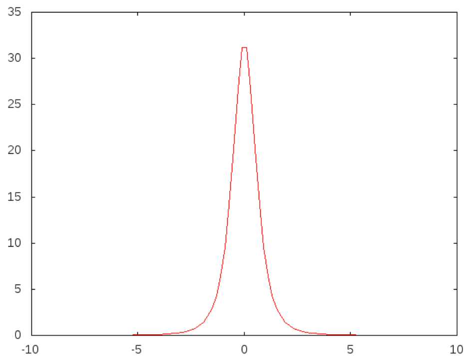


As we can see in every example, the first intensity is constant at the region where we can find zeros almost surely. At the boundary of the circle that generates the zero set, the curve decays to the origin line, indicating that there are no zeros beyond the circle almost surely.

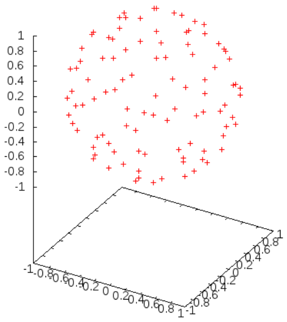
4.2 GAF in the finite space of polynomials endowed with the norm $\| \cdot \|^2_{\mathcal{P}_L}$

Since the space of polynomials $\mathcal{P}_L$, for L a natural number, is already finite, the first intensity for a GAF in the finite space of polynomials endowed with the norm $\| \cdot \|^2_{\mathcal{P}_L}$ is the one we calculated in the last chapter. Now, using the program 5.4 for the zero set and `gnuplot` for the first intensity and setting `n=100`, we get:





The first intensity, which is the picture of the right, matches with the zero set distribution (left picture), because all the zeros of the GAF are gathered nearby the origin and the function takes its maximum values at a neighbourhood of zero.
In the left picture it seems that there is no pattern in the distribution of the zeros, that there is only a cluster of points. However, using the inverse of the stereographic projection map, we obtain a sphere whose points are distributed uniformly over the surface of $\mathbb{S}^2$.



Now it is visually comprehensible that the distribution of zeros of a GAF in this space is invariant under rotations.

4.3 GAF in the finite space of polynomials endowed with the norm $\|\cdot\|^2_{\mathcal{B}_L}$

For any z in $\mathbb{D}$, we define the *hypergeometric function*:

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{+\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, $$

where $(a)_n$ is called the *Pochhammer symbol* and it is defined as:

$$ (a)_n := \begin{cases} a(a+1)...(a+n-1), & n > 0, \\ 1, & n = 0. \end{cases} $$

We want to compute the covariance kernel of a GAF in the finite space of polynomials endowed with the norm $\|\cdot\|^2_{\mathcal{B}_L}$. Let m be a natural number. We have:

$$ (1-|z|^2)^{-L} = \sum_{n=0}^{+\infty} \binom{L+n-1}{n} |z|^{2n} = \sum_{n=0}^{m} \binom{L+n-1}{n} |z|^{2n} + \sum_{n=m+1}^{+\infty} \binom{L+n-1}{n} |z|^{2n}, $$

therefore:

$$ \sum_{n=0}^{m} \binom{L+n-1}{n} |z|^{2n} = (1-|z|^2)^{-L} - \sum_{n=m+1}^{+\infty} \binom{L+n-1}{n} |z|^{2n}. $$

We must rewrite the second sum to obtain the hypergeometric function. Making the change of variables $k = n - m - 1$ and knowing that $(1)_k = k! = \Gamma(k+1)$:

$$\begin{aligned}\sum_{n=m+1}^{+\infty} \binom{L+n-1}{n} |z|^{2n} &= \sum_{k=0}^{+\infty} \binom{L+k+m}{k+m+1} |z|^{2k+2m+2} = |z|^{2m+2} \sum_{k=0}^{+\infty} \binom{L+k+m}{k+m+1} |z|^{2k} \\ &= |z|^{2m+2} \sum_{k=0}^{+\infty} \binom{L+m}{m+1} \frac{(L+m+1)_k}{(m+2)_k} |z|^{2k} \\ &= |z|^{2m+2} \binom{L+m}{m+1} \sum_{k=0}^{+\infty} \frac{(1)_k (L+m+1)_k}{(m+2)_k} \frac{|z|^{2k}}{k!} \\ &= |z|^{2m+2} \binom{L+m}{m+1} {}_2F_1(1, L+m+1; m+2; |z|^2).\end{aligned}$$

Thus, the covariance kernel of a GAF f in this space is:

$$\mathcal{K}_f(z,z) = \sum_{n=0}^{m} \binom{L+n-1}{n} |z|^{2n} = (1-|z|^2)^{-L} - |z|^{2m+2} \binom{L+m}{m+1} {}_2F_1(1, L+m+1; m+2; |z|^2).$$

Let us calculate the first intensity. First of all:

$$\log \mathcal{K}_f(z,z) = \log \left((1-|z|^2)^{-L} - |z|^{2m+2} \binom{L+m}{m+1} {}_2F_1(1, L+m+1; m+2; |z|^2) \right).$$

The following equations are long to compute, so we are going to use another notation to simplify terms, and also we are going to use polar coordinates. We have $|z|^2 = r^2$, for $0 < r < 1$. Let us denote:

$${}_2F_1(r) \equiv {}_2F_1(1, L+m+1; m+2; r^2), \quad \partial_r \equiv \frac{\partial}{\partial r}, \quad \partial^2_{rr} \equiv \frac{\partial^2}{\partial r^2}.$$

Now,

$$\partial_r \log \mathcal{K}_f(r) = \frac{2Lr(1-r^2)^{-L-1} - \binom{L+m}{m+1} \left[(2m+2) r^{2m+1} {}_2F_1(r) + r^{2m+2} \partial_{r\,2}F_1(r) \right]}{\mathcal{K}_f(r)}.$$

In the disk of convergence, we have:

$$\begin{aligned}\partial_{r\,2}F_1(r) &= \sum_{k=1}^{+\infty} \frac{(L+m+1)_k}{(m+2)_k} 2k r^{2k-1} = \frac{2}{r} \sum_{k=1}^{+\infty} \frac{(L+m+1)_k}{(m+2)_k} k r^{2k} \\ &\overset{(*)}{=} \frac{2}{r} \sum_{l=0}^{+\infty} \frac{(L+m+1)_{l+1}}{(m+2)_{l+1}} (l+1) r^{2l+2} = 2r \sum_{l=0}^{+\infty} \frac{(L+m+1)_{l+1}}{(m+2)_{l+1}} (l+1) r^{2l} \\ &= 2r \frac{L+m+1}{m+2} \sum_{l=0}^{+\infty} \frac{(L+m+2)_l}{(m+3)_l} \frac{(2)_l}{l!} r^{2l} = 2r \frac{L+m+1}{m+2} {}_2F_1(2, L+m+2; m+3; r^2),\end{aligned}$$

where in $(*)$ we use the change $l = k-1$ and remarking that $(2)_l = (l+1)! = \Gamma(l+2)$. Following an analogous reasoning we have that:

$$\partial^2_{rr\,2}F_1(r) = 2\frac{L+m+1}{m+2}\,{}_2F_1(2, L+m+2; m+3; r^2) + 8r^2\frac{(L+m+1)(L+m+2)}{(m+2)(m+3)}\,{}_2F_1(3, L+m+3; m+4; r^2).$$

And also we get:

$$\partial_r\mathcal{K}_f(r) = 2Lr(1-r^2)^{-L-1} - \binom{L+m}{m+1}\left[(2m+2)r^{2m+1}{}_2F_1(r) + r^{2m+2}\partial_{r\,2}F_1(r)\right].$$

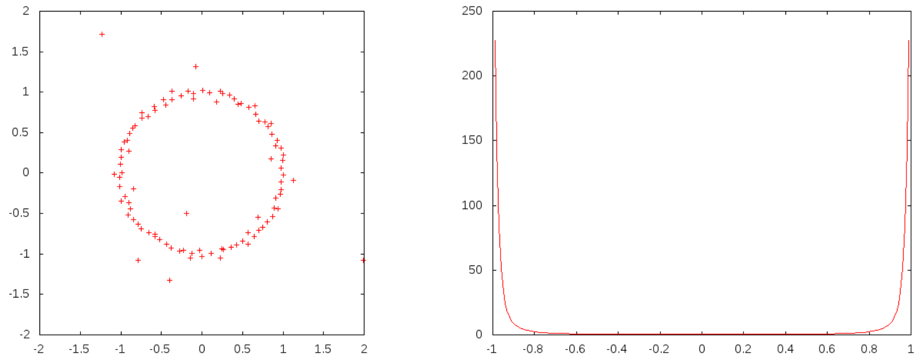
Then, since the function that we want to compute does not depend of the angle, the Laplacian is $\Delta_{(r,\theta)} \equiv \partial^2_{rr} + r^{-1}\partial_r$, and:

$$\begin{aligned}\Delta\log\mathcal{K}_f(r) = &\frac{2L\left[(1-r^2)^{-L-1} + 2(L+1)r^2(1-r^2)^{-L-2}\right]}{\mathcal{K}_f(r)}\\
&- \frac{\binom{L+m}{m+1}\left[(2m+2)(2m+1)r^{2m}{}_2F_1(r) + (2m+2)r^{2m+1}\partial_{r\,2}F_1(r)\right]}{\mathcal{K}_f(r)}\\
&- \frac{\binom{L+m}{m+1}\left[(2m+2)r^{2m+1}\partial_{r\,2}F_1(r) + r^{2m+2}\partial^2_{rr\,2}F_1(r)\right]}{\mathcal{K}_f(r)} - \frac{\left(\partial_r\mathcal{K}_f(r)\right)^2}{\left(\mathcal{K}_f(z,z)\right)^2}\\
&+ \frac{2L(1-r^2)^{-L-1} - \binom{L+m}{m+1}\left[(2m+2)r^{2m}{}_2F_1(r) + r^{2m+1}\partial_{r\,2}F_1(r)\right]}{\mathcal{K}_f(r)}.\end{aligned}$$

and the first intensity is:

$$\begin{aligned}\rho_1(r) = &\frac{2L\left[(1-r^2)^{-L-1} + 2(L+1)r^2(1-r^2)^{-L-2}\right]}{4\pi\mathcal{K}_f(r)}\\
&- \frac{\binom{L+m}{m+1}\left[(2m+2)(2m+1)r^{2m}{}_2F_1(r) + (2m+2)r^{2m+1}\partial_{r\,2}F_1(r)\right]}{4\pi\mathcal{K}_f(r)}\\
&- \frac{\binom{L+m}{m+1}\left[(2m+2)r^{2m+1}\partial_{r\,2}F_1(r) + r^{2m+2}\partial^2_{rr\,2}F_1(r)\right]}{4\pi\mathcal{K}_f(r)} - \frac{\left(\partial_r\mathcal{K}_f(r)\right)^2}{4\pi\left(\mathcal{K}_f(z,z)\right)^2}\\
&+ \frac{2L(1-r^2)^{-L-1} - \binom{L+m}{m+1}\left[(2m+2)r^{2m}{}_2F_1(r) + r^{2m+1}\partial_{r\,2}F_1(r)\right]}{4\pi\mathcal{K}_f(r)}.\end{aligned}$$

Using the programs 5.5 and 5.6 for `n = 100` and `L = 1`, we obtain:



As we can observe, most of the zero points of a GAF in the finite space of polynomials endowed with the norm $\| \cdot \|^2_{\mathcal{B}_L}$ are distributed at the boundary of $\mathbb{D}$. The first intensity, represented at the picture of the right, takes elevated values when the curve is near of -1 and 1, which is obvious with what we explained before.

4.4 GAF in the finite Paley - Wiener space

For the left picture in Figure 4.1 we use the program 5.7 for `n = 15` and `L = 1`. The right one was made in `gnuplot`. For Figure 4.2, the left picture was made with the program 5.9 and the other with the program 5.8. In the left picture of Figure 4.1, the red dots are

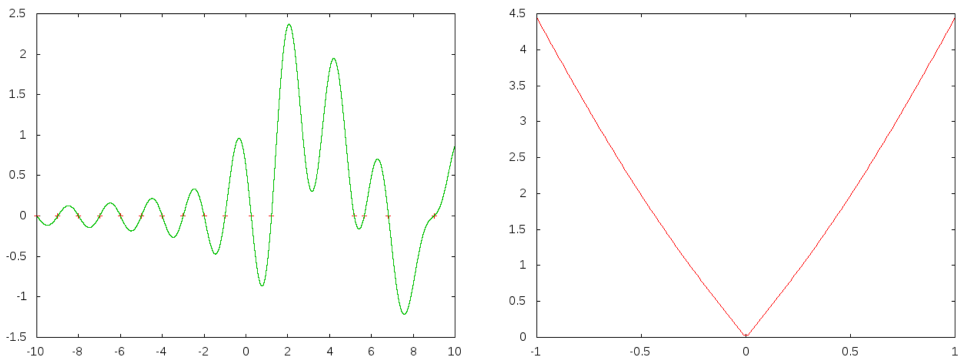


Figure 4.1: [Left picture] Plot of a GAF with $N_{\mathbb{R}}(0,1)$ random variables.
[Right picture] Plot of $\log\left(\kappa(y) + \sqrt{\kappa(y)^2 - 1}\right)$, where $\kappa(y) = \sin(2\pi i y)/(2\pi i y)$.

the real zeros of a GAF with $N_{\mathbb{R}}(0,1)$ random variables, and such a function is represented in green. It is highly remarkable that considering $N_{\mathbb{R}}(0,1)$ or $N_{\mathbb{C}}(0,1)$ random variables could change the first intensity, as we also saw in the last chapter. In Figure 4.2 there is the first intensity for both cases. We see that in the complex one there are no real zeros, but in the real one there are, as we can spot in the picture of the left in Figure 4.1. But, what generates this phenomenon? In Figure 4.2, right picture, we see that the curve intersects with the imaginary axis at the origin point, and this generates the real zeros since the

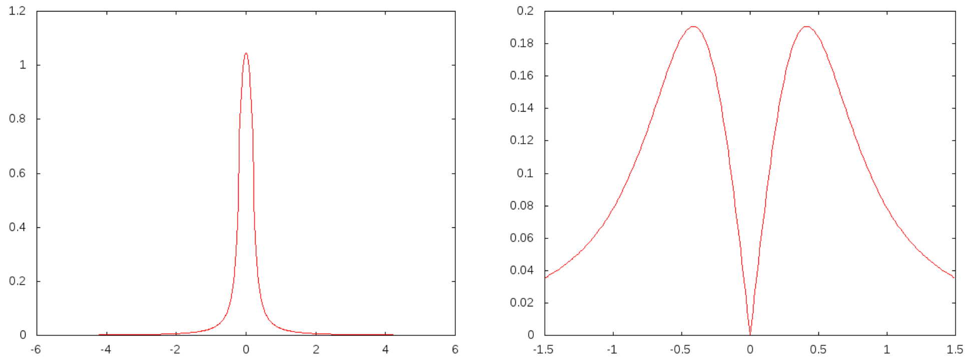


Figure 4.2: [Left picture] First intensity for $N_{\mathbb{C}}(0,1)$ random variables.
[Right picture] First intensity for $N_{\mathbb{R}}(0,1)$ random variables.

curve $\log\left(\kappa(y)+\sqrt{\kappa(y)^2-1}\right)$, where $\kappa(y) = \sin(2\pi i y)/(2\pi i y)$, behaves as an absolute value (see right picture in Figure 4.1), and this implies that the first intensity at $y = 0$ is proportional to a delta of Dirac at 0. From here it emerges two observations. The first one is the form of the curve of the first intensity for $N_{\mathbb{R}}(0,1)$ random variables. Comparing both plots of Figure 4.2, in the real case there are two horns near the origin point. Since the intersection generates the real zeros of this GAF, the complex ones move away from a neighbourhood of the origin. Heuristically, the complex zeros are making room for the real ones. The second remark is the expected number of zeros in a real interval. If we observe again Figure 4.1, the left picture, it seems that for almost every interval of length two, there are two real zeros. With this, one could think that the expected number of zeros in an interval follows this rule: if a and b are real values such that $a < b$, then $\rho_1([a,b]) = C|b-a|$, where C is a positive real constant and $|b-a|$ is the length of the interval $[a,b]$.

Chapter 5

Annex

In this annex we will discuss and expose the C++ programs used to illustrate the chapter *GAF computation*.

5.1 Preliminary explanations and concepts

Let f be a GAF in the finite space of polynomials endowed with one of the following norms: $\|\cdot\|^2_{\mathcal{F}_L}$, $\|\cdot\|^2_{\mathcal{P}_L}$, or $\|\cdot\|^2_{\mathcal{B}_L}$. We know that f is expressed as:

$$f(z) = \zeta_n e_n z^n + \zeta_{n-1} e_{n-1} z^{n-1} + ... + \zeta_1 e_1 z + \zeta_0 e_0,$$

where $(\zeta_n)_n \sim N_{\mathbb{C}}(0,1)$, z is a value that depends of the space before mentioned and $(e_n)_n$ is the orthonormal basis of the last spaces. What we want to compute is the zero set of f. For this we must introduce the next:

Definition 5.1. *Let p be a monic polynomial of degree n:*

$$p(x) = x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0,$$

where $(a_n)_n$ can be complex or real values. The companion matrix *of p is an $n \times n$ square matrix*

$$M = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \ddots & \cdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix}$$

such that the eigenvalues of M are the zeros points of p.

With this definition, one of the strategies to determine the zero set of a GAF could be generate the coefficients of every factor $(z_k)_k$, for $0 \leq k \leq n$, obtain the companion matrix M and calculate the eigenvalues of this matrix. For this, the C++ library `Armadillo`[1]

[1] Web page: `http://arma.sourceforge.net/`

is a highly useful linear algebra tool that is based on LAPACK (focused on linear algebra operations). For the finite Paley - Wiener space is rough to explicitly compute the zero set due to the sine function, independently of using complex or real standard Gaussian random variables. Then, we will calculate the position of the zeros of a GAF only with real Gaussian random variables by the bisection method.
In the program to obtain the first intensity in the finite space of polynomials endowed by the $\|\cdot\|^2_{\mathcal{B}_L}$ norm I used the numerical library gsl [2] (GNU Scientific Library) to use the hypergeometric function.
In the program to obtain the first intensity in the finite space of polynomials endowed by the $\|\cdot\|^2_{\mathcal{F}_L}$ norm I used the library boost [3] to use the incomplete gamma function.
Comments of some non-standard functions that I used in the programs:

- `eig_gen(eigval,M)`
  Let `eigval` be a vector and let `M` be a general square matrix. This function determines the eigenvalues of `M` and it stores them in `eigval`. This function is from the library `Armadillo`.

- `boost::math::tgamma (double a, double x)`
  Returns the incomplete gamma function described at the beginning of Section 4.1. This function is from the library `boost`.

- `gsl_sf_hyperg_2F1(double a, double b, double c, double x)`
  Returns the hypergeometric function described at the beginning of Section 4.3. This function is from the library `gsl`.

5.2 Zeros of a GAF in the finite space of polynomial endowed with the norm $\|\cdot\|^2_{\mathcal{F}_L}$

```
// Name: Alexis Arraz Almirall
// Project: ZEROS OF RANDOM ANALYTIC FUNCTIONS



#include <iostream>
#include <armadillo>
#include <random>
#include <complex>
#include <cmath>
#include <chrono>

using namespace std;
using namespace arma;
```

[2] Web page: https://www.gnu.org/software/gsl/
[3] Web page: https://www.boost.org/

```
int main()
{
    int i, n;
    double L, prod;
    default_random_engine gen;
    gen.seed(std::chrono::system_clock::now().time_since_epoch().count());
    normal_distribution <double> re_rv(0,1./sqrt(2.)), im_rv(0,1./sqrt(2.));
    ofstream d;
    d.open("fock_zeros.dad");

    cout << "Degree: ";
    cin >> n;

    cout << "Parameter: ";
    cin >> L;

    cx_vec z(n);

    z.zeros();

    complex<double> val(re_rv(gen), im_rv(gen));

    z(0) = val;

    prod = 1;

    for(i=1; i<n; i++){

        complex<double> val(re_rv(gen), im_rv(gen));

        z(i) = val * sqrt(prod);

        prod *= L/i;
    }

    for(i=0; i<(n-1); i++){
        z(i) = z(i)/z(n-1);
    }

    cx_mat M(n,n);
```

```
    M.zeros();
    M.diag(-1).ones();
    for(i=0; i<(n-1); i++){
        M.row(i).col(n-1)= -z(i);
    }

    cx_vec eigval;

    eig_gen(eigval,M);

    for(i=0; i<n; i++){
        d << real(eigval(i)) << "\t" << imag(eigval(i)) << endl;
    }
    d.close();

    return 0;
}
```

5.3 First intensity of a GAF in the finite space of polynomials endowed with the norm $\|\cdot\|^2_{\mathcal{F}_L}$

```
// Name: Alexis Arraz Almirall
// Project: ZEROS OF RANDOM ANALYTIC FUNCTIONS


#include <iostream>
#include <armadillo>
#include <random>
#include <boost/math/special_functions/gamma.hpp>
#include <cmath>

#define _USE_MATH_DEFINES

using namespace std;
using namespace arma;

int main()
{
    int n;
    double L, gn, g, gN;

    ofstream d;
```

```
    d.open("fock_first_int.dad");

    cout << "Degree: ";
    cin >> n;

    cout << "Parameter: ";
    cin >> L;

    double x = 0, r;

    do{
        gn = boost::math::tgamma(n-1,L*x*x);
        g = boost::math::tgamma(n,L*x*x);
        gN = boost::math::tgamma(n+1,L*x*x);

        r = ((L*n)/M_PI)*((g*gN + L*x*x*(n-1)*gn*gN-L*x*x*n*g*g)/(gN*gN));

        d << x << "\t" << r << endl;

        x += 0.001;

    }while(x<=6);

    d.close();

    return 0;
}
```

As a remark, the condition at the `do - while` varies at every set zero we want to compute. In this case, the number 6 corresponds to `n = 30` and `L = 5`. As you can see, the library `boost` has been used here to apply the incomplete gamma function.

5.4 Zeros of a GAF in the finite space of polynomials endowed with the norm $\|\cdot\|^2_{\mathcal{P}_L}$

```
// Name: Alexis Arraz Almirall
// Project: ZEROS OF RANDOM ANALYTIC FUNCTIONS


#include <iostream>
#include <armadillo>
#include <random>
```

```
#include <complex>
#include <cmath>
#include <chrono>

using namespace std;
using namespace arma;

int main()
{
    int i;
    double n;
    default_random_engine gen;
    gen.seed(std::chrono::system_clock::now().time_since_epoch().count());
    normal_distribution <double> re_rv(0, 1./sqrt(2.)), im_rv(0, 1./sqrt(2.));
    ofstream d, g;
    d.open("poly_zeros.dad");
    g.open("poly_zeros_3D.dad");

    cout << "Degree: ";
    cin >> n;

    cx_vec z(n);

    z.zeros();

    double p = 1;

    for(i=1; i<=n; i++){

        complex<double> val(re_rv(gen), im_rv(gen));

        p *= (n-i+1)/i;
        z(i-1) = val*sqrt(p);
    }

    for(i=0; i<(n-1); i++){
        z(i) = z(i)/z(n-1);
    }

    cx_mat M(n,n);

    M.zeros();
```

```
    M.diag(-1).ones();
    for(i=0; i<(n-1); i++){
        M.row(i).col(n-1)= -z(i);
    }

    cx_vec eigval;

    eig_gen(eigval,M);

    double comp;

    for(i=0; i<n; i++){
        d << real(eigval(i)) << "\t" << imag(eigval(i)) << endl;
        comp = real(eigval(i))*real(eigval(i)) + imag(eigval(i))*imag(eigval(i));
        g << 2*real(eigval(i))/(1+comp) << "\t" << 2*imag(eigval(i))/(1+comp)
        << "\t" << (comp-1)/(1+comp) << endl;
    }
    d.close();
    g.close();

    return 0;
}
```

As a remark, I divided above the line `g « 2*real(eigval(i))/(1+comp) « "\t" « 2*imag(eigval(i))/(1+comp) « "\t" « (comp-1)/(1+comp) « endl;` into two due to its length.

5.5 Zeros of a GAF in the finite space of polynomials endowed with the norm $\|\cdot\|^2_{\mathcal{B}_L}$

```
// Name: Alexis Arraz Almirall
// Project: ZEROS OF RANDOM ANALYTIC FUNCTIONS


#include <iostream>
#include <armadillo>
#include <random>
#include <complex>
#include <cmath>
#include <chrono>
```

```
using namespace std;
using namespace arma;

int main()
{
    int i, n;
    double L;
    default_random_engine gen;
    gen.seed(std::chrono::system_clock::now().time_since_epoch().count());
    normal_distribution <double> re_rv(0, 1./sqrt(2.)), im_rv(0, 1./sqrt(2.));
    ofstream d;
    d.open("bergman_zeros.dad");

    cout << "Degree: ";
    cin >> n;

    cout << "Parameter: ";
    cin >> L;

    cx_vec z(n);

    z.zeros();

    double p = 1;

    for(i=1; i<=n; i++){

        complex<double> val(re_rv(gen), im_rv(gen));

        p *= (L+i-1)/i;
        z(i-1) = val*sqrt(p);

    }

    for(i=0; i<(n-1); i++){
        z(i) = z(i)/z(n-1);
    }

    cx_mat M(n,n);

    M.zeros();
    M.diag(-1).ones();
```

```
    for(i=0; i<(n-1); i++){
        M.row(i).col(n-1)= -z(i);
    }

    cx_vec eigval;

    eig_gen(eigval,M);

    for(i=0; i<n; i++){
        d << real(eigval(i)) << "\t" << imag(eigval(i)) << endl;
    }
    d.close();

    return 0;
}
```

5.6 First intensity of a GAF in the finite space of polynomial endowed with the norm $\|\cdot\|^2_{\mathcal{B}_L}$

```
// Name: Alexis Arraz Almirall
// Project: ZEROS OF RANDOM ANALYTIC FUNCTIONS


#include <iostream>
#include <armadillo>
#include <random>
#include <cmath>
#include <gsl/gsl_sf_hyperg.h>

#define _USE_MATH_DEFINES

using namespace std;
using namespace arma;

double comb (double, int);
double dhyge (double, double, int);
double ddhyge (double, double, int);
double k (double, double, int);
double dk (double, double, int);
double n1 (double, double, int);
double n2 (double, double, int);
```

```
double n3 (double, double, int);
double n4 (double, double, int);
double frac1 (double, double, int);
double frac2 (double, double, int);


int main()
{
    int n;
    double L;

    ofstream d;
    d.open("bergman_first_intensity.dad");

    cout << "Degree: ";
    cin >> n;

    cout << "Parameter: ";
    cin >> L;

    double x = -0.99;

    do{

        d << x << "\t" << frac1(x,L,n) + frac2(x,L,n) << endl;

        x += 0.01;

    }while(x<1);

    d.close();

    return 0;
}

double comb (double l, int m){

    return tgamma(l+m+1)/(tgamma(m+2)*tgamma(l));
}

double dhyge (double z, double l, int m){
```

```
    return 2*z*((l+m+1)/(m+2))*gsl_sf_hyperg_2F1(2,l+m+2,m+3,z*z);
}

double ddhyge (double z, double l, int m){

    return 2*((l+m+1)/(m+2))*gsl_sf_hyperg_2F1(2,l+m+2,m+3,z*z)
    + 8*z*z*(((l+m+1)*(l+m+2))/((m+2)*(m+3)))*gsl_sf_hyperg_2F1(3,l+m+3,m+4,z*z);
}

double k (double z, double l, int m){

    return pow(1-z*z,-l) - pow(z,2*m+2) * comb(l,m)
    * gsl_sf_hyperg_2F1(1,l+m+1,m+2,z*z);
}

double dk (double z, double l, int m){

    return 2*l*z*pow(1-z*z,-l-1)-comb(l,m)*(2*m+2)*pow(z,2*m+1)
    *gsl_sf_hyperg_2F1(1,l+m+1,m+2,z*z)-comb(l,m)*pow(z,2*m+2)*dhyge(z,l,m);
}

double n1 (double z, double l){

    return 2*l*pow(1-z*z,-l-1)+4*z*z*l*(l+1)*pow(1-z*z,-l-2);
}

double n2 (double z, double l, int m){

    return (2*m+2)*(2*m+1)*pow(z,2*m)*gsl_sf_hyperg_2F1(1,l+m+1,m+2,z*z)
    + (2*m+2)*pow(z,2*m+1)*dhyge(z,l,m);
}

double n3 (double z, double l, int m){

    return (2*m+2)*pow(z,2*m+1)*dhyge(z,l,m)+pow(z,2*m+2)*ddhyge(z,l,m);
}

double n4 (double z, double l, int m){

    return dk(z,l,m)*dk(z,l,m);
}
```

```
double frac1 (double z, double l, int m){

    return ((n1(z,l)-comb(l,m)*n2(z,l,m)-comb(l,m)*n3(z,l,m))*k(z,l,m)
    - n4(z,l,m))/(4*M_PI*k(z,l,m)*k(z,l,m));
}

double frac2 (double z, double l, int m){

    return dk(z,l,m)/(4*M_PI*z*k(z,l,m));
}
```

Remark: I separated into two the `return` line in the functions `ddhyge`, `k`, `dk`, `n2` and `frac1` here due to their length. In this program I used the `gsl` library to apply the hypergeometric function. If one implements the definition of this function, the error in the operations is remarkable due to the multiple number of sums and products. Using the library `gsl` for this function, the error decreases significantly because it uses a recursive algorithm, instead of power series.

5.7 Zeros of a GAF in the finite Paley - Wiener space with $N_{\mathbb{R}}(0,1)$ random variables

```
// Name: Alexis Arraz Almirall
// Project: ZEROS OF RANDOM ANALYTIC FUNCTIONS


#include <iostream>
#include <random>
#include <armadillo>
#include <cmath>
#include <chrono>

#define _USE_MATH_DEFINES

using namespace std;
using namespace arma;

double bisection (double, double, int, double, vec);
double f(double, int, double, vec);

int main()
{
    int i, n;
```

```
double L, h = 0.0001;
default_random_engine gen;
gen.seed(std::chrono::system_clock::now().time_since_epoch().count());
normal_distribution <double> rv(0,1);
ofstream d, g;
d.open("pw_real_zeros.dad");
g.open("pw_real_graphic.dad");

cout << "Degree: ";
cin >> n;

cout << "Parameter: ";
cin >> L;

vec rvec(n);

for(i=0; i<n; i++){
    rvec(i) = rv(gen);
}

double z0 = -10, z1 = z0 + h, p;

do{

    g << z0 << "\t" << f(z0,n,L,rvec) << endl;

    if(f(z0,n,L,rvec)*f(z1,n,L,rvec) > 0) {
        z0 = z1;
        z1 += h;
    }else{
        p = bisection(z0,z1,n,L,rvec);
        d << p << "\t" << f(p,n,L,rvec) << endl;
        z0 = z1;
        z1 += h;
    }

}while(z1 <= 10);

d.close();
g.close();

return 0;
```

```
}

double f(double z, int n, double L, vec v){

    int i;
    double sum = 0.0;

    for(i=0; i<n; i++){

        if(fabs(M_PI*(i-L*z)) > 1.e-10){
           sum += (v(i)*(sin(M_PI*(i-L*z)))/(M_PI*(i-L*z)));
        }else{
           sum += v(i);
        }
    }

    return sum;
}

double bisection (double a, double b, int n, double L, vec v){

    int bn = 0;
    double c;

    do{

        c = (a+b)*0.5;

        if(fabs(f(c,n,L,v)) < 1.e-10) {
            return c;
        }
        else{
            if(f(a,n,L,v)*f(c,n,L,v) < 0) b = c;
            if(f(c,n,L,v)*f(b,n,L,v) < 0) a = c;
        }

        bn += 1;

    }while(fabs(a-b) > 1.e-10 || bn <= 200);


}
```

5.8 First intensity of a GAF in the finite Paley - Wiener space with $N_{\mathbb{R}}(0,1)$ random variables

```
// Name: Alexis Arraz Almirall
// Project: ZEROS OF RANDOM ANALYTIC FUNCTIONS


#include <iostream>
#include <armadillo>
#include <cmath>
#include <math.h>

#define _USE_MATH_DEFINES

using namespace std;
using namespace arma;

double num (double);
double den (double);

int main()
{

    ofstream d;
    d.open("pw_real_first_int.dad");

    double x = -1.5;

    do{

        d << x << "\t" << num(x)/den(x) << endl;

        x += 0.01;

    }while(x<=1.5);

    d.close();

    return 0;
}
```

```
double num (double x){

    return -64*M_PI*M_PI*M_PI*x*x*x*cosh(2*M_PI*x)
    +(64*M_PI*M_PI*M_PI*M_PI*x*x*x*x+48*M_PI*M_PI*x*x+3)*sinh(2*M_PI*x)
    -sinh(6*M_PI*x);
}


double den (double x){

    return 16*x*x*x*(4*M_PI*M_PI*M_PI*x*x-M_PI*sinh(2*M_PI*x)
    *sinh(2*M_PI*x)) * sqrt(((sinh(2*M_PI*x)*sinh(2*M_PI*x))/(x*x))
    -4*M_PI*M_PI);
}
```

Remark: The `return` in both functions has been separated here in multiple lines due to their length.

5.9 First intensity of a GAF in the finite Paley - Wiener space with $N_{\mathbb{C}}(0,1)$ random variables

```
// Name: Alexis Arraz Almirall
// Project: ZEROS OF RANDOM ANALYTIC FUNCTIONS


#include <iostream>
#include <armadillo>
#include <cmath>
#include <math.h>

#define _USE_MATH_DEFINES

using namespace std;
using namespace arma;

int main()
{

    ofstream d;
    d.open("pw_cx_first_int.dad");
```

```
    double x = -6;

    do{
        if(fabs(x)>0.2)
           d << x << "\t" << (1./(4*M_PI*M_PI*x*x))-M_PI*
           (1./(sinh(2*M_PI*M_PI*x)*sinh(2*M_PI*M_PI*x))) << endl;
        else
           d << x << "\t" << (M_PI/3) - ((4*M_PI*M_PI*M_PI*x*x)/15) << endl;
        x += 0.01;

    }while(x<=6);

    d.close();

    return 0;
}
```

As a remark, I separated into two lines here the exit of the `if` due to its length.

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`log((sin(2*pi*i*y))%2F(2*pi*i*y)+%2B+sqrt(((sin(2*pi*i*y))%2F(2*pi*i*y))%5E2-1))`

[The last line has been separated into two due to its length]

[All the electronic sources of the project have been visited for the last time on June, 26th 2018.]