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The extended, circular, planar, restricted three-body problem

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0.1 Abstract

This thesis deals with the study of the Planar Circular Restricted Extended Three-Body problem, which is a generalisation of the regular Planar Circular Restricted Three-Body problem in which we replace the classical gravitational field ($F \sim r^{-2}$) with another one such that $F \sim r^{\alpha}$, with arbitrary values of α, with the final objective of performing accurate simulations of the trajectory of the third body.

We begin by precisely stating the problem as well as introducing some concepts that will be needed throughout this thesis for its development. We will then move onto the study of Hill Regions, which are the regions of the plane in which motion of the third body is possible. These regions are related to the value of the Jacobi constant, the only constant variable known for the restricted three-body problem. Following we will deal with the regularisation of the problem, a useful concept that becomes of utmost importance when computing the trajectory of the third body near the collisions. Finally we outline some important points that arose when computing the trajectory and how did we overcome them.

0.2 Acknowledgements

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0.3 Introduction

The three-body problem is a specific instance of the more general n-body problem, which seeks to predict the motion of a set of n bodies interacting with each other. Originally, its study originated from the desire to better understand the motions of celestial bodies, such as planets or stars according to classical Newtonian mechanics. More recently, it has lead into the study of motions of celestial bodies according to the laws of general relativity or the study of the motion of subatomic particles according to the laws of quantum mechanics.

In general, neither the n-body problem nor the three-body problem have analytic solutions, and its evolution is believed to be chaotic in the majority of configurations. Hence, if one seeks study the motion of the individual bodies, one must draw upon numerical methods to get an approximation to the solution.

In this thesis, we will restrict ourselves to the study of the extended, circular, restricted, planar three-body problem, i.e. we will consider a plane in which we have two bodies orbiting in a circular motion around its centre of masses, and we shall study the motion of a third body with negligible mass interacting with the other two. In the extended version, we do not restrict ourselves to the gravitational force, but we consider a generic force field F such that $F \sim r^\alpha$, where $\alpha \in \mathbb{R}$.

Although this may seem as an oversimplification, these assumptions are enough to study real phenomena, such as the motion of a satellite under the gravitational pull of the Earth and the Moon or, in quantum mechanics, the study of the movement of an electron in a hydrogen atom.

In this thesis, we will first of all, introduce the basic definitions needed to comprehend the problem, as well as provide a precise mathematical definition of it. We will also perform a study of the Lagrange points, which are basically stationary solutions of the problem when considering a synodic frame of reference, i.e. a rotational frame of reference in which the two revolving bodies become fixed over the x axis.

Then we will perform an study of the Hill Regions, which are basically the regions of the plane in which the motion of the third body is possible. These regions are delimited by the Zero Velocity Curves, which are, as it name suggests, the curves in which the velocity of the third body becomes null in a synodic frame of reference.

We will move then onto the topic of regularisation. We will try to generalise the method used in the classical field and then, apply it to perform the regularisation for any value of α.

Once this analysis is performed, we will be well equipped to tackle the computation of the orbit of the third body. In the last chapter of this thesis, we will describe briefly how this theoretical analysis is used to solve various problems that arise when trying to perform the computation.

Chapter 1

The extended, circular, planar, restricted three-body problem

We will begin this chapter by providing a precise definition of the extended, planar, restricted, circular three-body problem, which is the object of our study.

After describing the problem, the differential equations of motion in the inertial (sidereal) referential frame, will be derived, and then transformed into the rotating (synodic) referential frame, which provides a great simplification for subsequent calculations.

We shall then introduce the Lagrangian and the Hamiltonian of the system, as well as introduce the concept of the Jacobi integral and the Jacobi constant, which will be a central concept in the understanding of zero velocity curves, which will be studied in depth in chapter 2.

Finally, in the synodic referential frame, we will perform an study of the Lagrange points of the system, i.e. points in which the sum of gravitational forces is 0 (and therefore, a body placed in one of these points shall remain fixed). The Lagrange points colinear with the bodies do not have, in general, analytic solutions, although it is possible to perform a qualitative study and obtain the number of solutions according to the value of a given parameter.

1.1 The extended gravitational field

Let F_α be a vector field such that

$$\begin{aligned} F_\alpha : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} &\to \mathbb{R}^3 \\ (r, \dot{r}, t) &\mapsto F_\alpha(r, \dot{r}, t). \end{aligned}$$

The Newton equation, i.e. the equation that rules the motion of a body under the influence of the field F_α according to classical mechanics is given by

$$\ddot{r} = F_\alpha(r, \dot{r}, t). \tag{1.1}$$

Let us consider two bodies of masses M_1 and M_2. The vector field that describes the gravitational force acting on the first body by effect of the second is $F_\alpha(\bar{R}) = -GM_1M_2\bar{R}^{-2}u_r$,

where $\bar{R}$ is the distance between the two bodies and u_r is the unitary vector from the first body to the second. In this thesis we will consider extend the study of the gravitational force to a family of forces of the form

$$F_\alpha(\bar{R}) = -GM_1M_2\bar{R}^\alpha u_r, \tag{1.2}$$

where $\alpha \in \mathbb{R}$, known as the *extended gravitational forces*, hence generating a family of gravitational potentials of the form

$$V_\alpha(\bar{R}) = \int F_\alpha(\bar{R})\ \mathrm{d}\bar{R} = \frac{1}{\alpha+1}GM_1M_2\bar{R}^{\alpha+1} + K$$

where K is the integration constant. The common consensus in physics is to pick the origin of potentials at the infinity. However, since we are dealing with forces that might not be null at the infinite, we will choose our origin of potentials at radius 1, by means of choosing an appropriate integration constant $K = -\frac{1}{\alpha+1}GM_1M_2$, and thus,

$$V_\alpha(\bar{R}) = \frac{1}{\alpha+1}M_1M_2(\bar{R}^{\alpha+1} - 1) \tag{1.3}$$

1.2 The extended two-body problem

Let us consider two bodies of masses M_1 and M_2 such that $M_1 \geq M_2$. These bodies are named the primary and the secondary body respectively.

Assume that these bodies follow a circular motion around their centre of mass and that a third body, with mass much lower than both M_1 and M_2, moves in the plane defined by the two revolving bodies. Describing the motion of this third body is the *extended restricted circular planar three-body problem*

Let $F_\alpha(\bar{R}) = GM_1M_2\bar{R}^\alpha$ be an attractive force associated with the two bodies, where $\bar{R}$ is the distance between the two. These forces are an extension of the gravitational force, which stands for $\alpha = -2$. Assume that these two bodies revolve in a circular motion around their centre of mass and that a third body moves in the plane defined by the other two bodies such that its motion is affected by them but it itself does not affect the revolving bodies. Describing the motion of this third body is the *extended, restricted, circular, planar three-body problem*.

1.3 The extended, planar and circular restricted three-body problem

Let us restrict the motion of the primary and the secondary bodies such that they describe a circular motion around their centre of masses with constant angular velocity n. In this situation, both bodies remain at a constant distance D and, assuming initial positions $(mD, 0)$ and $((m-1)D, 0)$, where $M = M_1 + M_2$, $m = \frac{M_2}{M}$ and $1 - m = \frac{M_1}{M}$, describe circular orbits with equations

$$(mD\cos(n\bar{t}),\ mD\sin(n\bar{t})) \quad \text{and} \quad ((m-1)D\cos(n\bar{t}),\ (m-1)D\sin(n\bar{t})). \tag{1.4}$$

The *extended planar and circular restricted three-body problem* analyses the movement of a third body under the influence of these two bodies in their same plane and assuming that

its mass is much less than both M_1 and M_2 (by much less we mean that this third body does not perturb the motion of the other two). The differential equations (Newton equation) that describe the motion of this their body are:

$$\begin{aligned}\frac{\mathrm{d}^2\bar{X}}{\mathrm{d}\bar{t}^2} &= -GM(1-m)(\bar{X}-mD\cos(n\bar{t}))\bar{R}_1{}^{\alpha-1} + m(\bar{X}-(m-1)D\cos(n\bar{t}))\bar{R}_2{}^{\alpha-1},\\ \frac{\mathrm{d}^2\bar{Y}}{\mathrm{d}\bar{t}^2} &= -GM(1-m)(\bar{Y}-mD\sin(n\bar{t}))\bar{R}_1{}^{\alpha-1} + m(\bar{Y}-(m-1)D\sin(n\bar{t}))\bar{R}_2{}^{\alpha-1},\end{aligned} \tag{1.5}$$

with

$$\begin{aligned}R_1^2 &= (\bar{X}-mD\cos(n\bar{t}))^2 + (\bar{Y}-mD\sin(n\bar{t}))^2\\ R_2^2 &= (\bar{X}-(m-1)D\cos(n\bar{t}))^2 + (\bar{Y}-(m-1)D\sin(n\bar{t}))^2,\end{aligned}$$

which can also be written in terms of the extended gravitational potentials as

$$\frac{\mathrm{d}^2\bar{X}}{\mathrm{d}\bar{t}^2} = \frac{\partial V_\alpha(\bar{X},\bar{Y},\bar{t})}{\partial X}, \quad \text{and} \quad \frac{\mathrm{d}^2\bar{Y}}{\mathrm{d}\bar{t}^2} = \frac{\partial V_\alpha(\bar{X},\bar{Y},\bar{t})}{\partial \bar{Y}}$$

1.3.1 The synodical coordinate system

Lets consider now a system of coordinates (X, Y) in which the primary and the secondary bodies are fixed. The coordinate transformation from the Cartesian coordinate system is simply a rotation:

$$X = \bar{X}\cos(n\bar{t}) - \bar{Y}\sin(n\bar{t}) \tag{1.6}$$

$$Y = \bar{X}\sin(n\bar{t}) + \bar{Y}\cos(n\bar{t}), \tag{1.7}$$

which may be written in complex form as

$$Z = \bar{Z}e^{in\bar{t}}, \tag{1.8}$$

such that $\bar{Z} = \bar{X} + i\bar{Y}$, $Z = X + iY$. In complex form, the distances R_1 and R_2 are given by

$$R_1 = |Z - Z_1|, \quad \text{and} \quad R_2 = |Z - Z_2|, \tag{1.9}$$

where $Z_1 = mDe^{in\bar{t}}$ and $Z_2 = -(m-1)De^{in\bar{t}}$. To obtain the equations of motion in this system of coordinates, we must find the expression of $\frac{\mathrm{d}^2 Z}{\mathrm{d}\bar{t}^2}$:

$$\begin{aligned}\frac{\mathrm{d}^2 Z}{\mathrm{d}\bar{t}^2} &= \frac{\mathrm{d}}{\mathrm{d}\bar{t}}\frac{\mathrm{d}(\bar{Z}e^{in\bar{t}})}{\mathrm{d}\bar{t}} = \frac{\mathrm{d}}{\mathrm{d}\bar{t}}\left(\frac{\mathrm{d}\bar{Z}}{\mathrm{d}\bar{t}}e^{in\bar{t}} + in\bar{Z}e^{in\bar{t}}\right)\\ &= \frac{\mathrm{d}^2\bar{Z}}{\mathrm{d}\bar{t}^2}e^{in\bar{t}} + in\frac{\mathrm{d}\bar{Z}}{\mathrm{d}\bar{t}}e^{in\bar{t}} + in\frac{\mathrm{d}^2\bar{Z}}{\mathrm{d}\bar{t}^2}e^{in\bar{t}} - n^2\bar{Z}e^{in\bar{t}}\\ &= \left(\frac{\mathrm{d}^2\bar{Z}}{\mathrm{d}\bar{t}^2} + 2in\frac{\mathrm{d}\bar{Z}}{\mathrm{d}\bar{t}} - n^2\bar{Z}\right)e^{in\bar{t}}.\end{aligned}$$

Therefore, the complex form of the equations of motion in the synodical system is

$$\begin{aligned}&\frac{\mathrm{d}^2\bar{Z}}{\mathrm{d}\bar{t}^2} + 2in\frac{\mathrm{d}\bar{Z}}{\mathrm{d}\bar{t}} - n^2\bar{Z}\\ &\quad = -GM\left((1-m)(\bar{Z}-mD)|\bar{Z}-mD|^{\alpha-1} + m(\bar{Z}+(m-1)D)|\bar{Z}+(m-1)D|^{\alpha-1}\right),\end{aligned}$$

with real and imaginary parts

$$\begin{aligned}\frac{\mathrm{d}^2\bar{X}}{\mathrm{d}\bar{t}^2} - 2n\frac{\mathrm{d}\bar{Y}}{\mathrm{d}\bar{t}} - n^2\bar{X} &= -GM\left((1-m)(\bar{X}-mD)R_1^{\alpha-1} + m(\bar{X}+(m-1)D)R_2^{\alpha-1}\right)\\ \frac{\mathrm{d}^2\bar{Y}}{\mathrm{d}\bar{t}^2} + 2n\frac{\mathrm{d}\bar{X}}{\mathrm{d}\bar{t}} - n^2\bar{Y} &= -GM\left((1-m)\bar{Y}R_1^{\alpha-1} + m\bar{Y}R_2^{\alpha-1}\right).\end{aligned} \tag{1.10}$$

Upon finding these expressions its clear that the left hand side becomes noticeably more complicated, with the presence of first order derivatives, and one may question whether this transformation serves any interest or rather it only confuses the matter. As we will see in section 1.3.3, equations 1.10 have a useful integral.

1.3.2 Transformation into nondimensional units

Let us apply adequate Galilean transformations of space and time:

$$\bar{X} = DX, \quad \bar{Y} = DY, \quad n\bar{t} = t.$$

Applying these transformations we get

$$\begin{aligned}\frac{\mathrm{d}^2\bar{X}}{\mathrm{d}\bar{t}^2} &= \frac{\mathrm{d}^2(DX)}{\mathrm{d}(\frac{t}{n})^2} = n^2D\frac{\mathrm{d}^2X}{\mathrm{d}t^2},\\ \frac{\mathrm{d}^2\bar{Y}}{\mathrm{d}\bar{t}^2} &= \frac{\mathrm{d}^2(DY)}{\mathrm{d}(\frac{t}{n})^2} = n^2D\frac{\mathrm{d}^2Y}{\mathrm{d}t^2},\\ \bar{R}_1^2 &= (DX - mD\cos(t))^2 + (DY - mD\sin(t))^2\\ &= D^2\left((X - m\cos(t))^2 + (Y - m\sin(t))^2\right) = D^2R_1^2,\\ \bar{R}_2^2 &= (DX - (m-1)D\cos(t))^2 + (DY - (m-1)D\sin(t))^2\\ &= D^2\left((X - (m-1)\cos(t))^2 + (Y - (m-1)\sin(t))^2\right) = D^2R_1^2,\end{aligned} \tag{1.11}$$

and therefore,

$$\begin{aligned}n^2D\frac{\mathrm{d}^2X}{\mathrm{d}t^2} &= -GMD^\alpha((1-m)(X - m\cos(t))R_1^{\alpha-1} + m(X-(m-1)\cos(t))R_2^{\alpha-1}),\\ n^2D\frac{\mathrm{d}^2Y}{\mathrm{d}t^2} &= -GMD^\alpha((1-m)(Y - m\sin(t))R_1^{\alpha-1} + m(Y-(m-1)\sin(t))R_2^{\alpha-1}).\end{aligned} \tag{1.12}$$

Taking into account the extended Kepler's third law, $n^2D = GMD^\alpha$, we may simplify equations to

$$\begin{aligned}\frac{\mathrm{d}^2X}{\mathrm{d}t^2} &= -(1-m)(X - m\cos(t)R_1^{\alpha-1} + m(X-(m-1)\cos(t))R_2^{\alpha-1},\\ \frac{\mathrm{d}^2Y}{\mathrm{d}t^2} &= -(1-m)(Y - m\sin(t))R_1^{\alpha-1} + m(Y-(m-1)\sin(t))R_2^{\alpha-1}.\end{aligned} \tag{1.13}$$

Similarly,the equations of motion in the nondimensional synodical system, corresponding to equations 1.10, become

$$\begin{aligned}\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 2\frac{\mathrm{d}y}{\mathrm{d}t} - x &= (1-m)(x-m)r_1^{\alpha-1} - m(x-m+1)r_2^{\alpha-1},\\ \frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 2\frac{\mathrm{d}x}{\mathrm{d}t} - y &= (1-m)yr_1^{\alpha-1} - myr_2^{\alpha-1},\end{aligned} \tag{1.14}$$

with $r_1^2 = (x-m)^2 + y^2$ and $r_2^2 = (x-m+1)^2 + y^2$

These scaling to nondimensional units produce the following simplifications in the equations of movement of the third body:

1. The scaled masses have an unitary sum: $1-m$, m.
2. The gravitational constant does not appear in the equations.
3. The distance between the two masses is 1 and the time to revolve once around their common centre of masses is 2π.

1.3.3 An invariant relation: The Jacobi integral

In this section, we will derive an invariant relation, which will prove useful in the upcoming sections to study the motion of the system, as well as a powerful way to check our numerical computations, since it must be constant regardless of the position and the velocity of the third body. We shall first introduce the concept of an integral of a dynamical system:

Let us consider a dynamical system of n degrees of freedom with coordinates $q_1, q_2, \ldots, q_n$, and write the equations of motion as

$$\frac{\mathrm{d}^2 q_i}{\mathrm{d}t^2} = Q_i(q_1, q_2, \ldots, q_n, \dot{q_1}, \dot{q_2}, \ldots, \dot{q_n}, t), \tag{1.15}$$

where $\dot{q}_i$ denotes the derivative of q_i with respect to t.

This system of n differential equations of second-order may also be written as a $2n$ system of first-order differential equations, by means of introducing variables $x_i = q_i$ and $x_{n+i} = \dot{q}_i$. In this case, the system of differential equations 1.15 becomes

$$\begin{aligned} \frac{\mathrm{d}x_i}{\mathrm{d}t} &= x_{n+1}, \\ \frac{\mathrm{d}x_{n+i}}{\mathrm{d}t} &= Q_i(x_1, x_2, \ldots, x_{2n}, t), \end{aligned} \tag{1.16}$$

or simply

$$\dot{x}_k = P_k(x_1, \ldots, x_m, t), \tag{1.17}$$

where $m = 2n$ and $k = 1, \ldots, m$, and such that P_k and their partial derivatives are defined and continuous in a certain domain.

Definition 1.1. *Consider a system of m first-order differential equations such that*

$$\dot{x}_k = P_k(x_1, \ldots, x_n, t). \tag{1.18}$$

Let $G(x_1, \ldots, x_m, t)$ be a differentiable function such that its partial derivatives with respect to $x_1, x_2, \ldots, x_m, t$ are continuous, and such that

$$\frac{\mathrm{d}G}{\mathrm{d}t} = 0 \tag{1.19}$$

for every set of solutions $x_1(t), x_2(t), \ldots, x_m(t)$. Then, we call $G(x_1, x_2, \ldots, x_m, t)$ an integral *of the system of equations 1.18.*

We shall find now an integral for the restricted problem of three bodies. We will derive it in the nondimensional synodical system, since this is the system we will mainly use throughout this thesis. To begin with, we will rewrite equations 1.14 by means of a potential function $\bar{\Omega}$:

$$\begin{aligned} \frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 2\frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{\partial\bar{\Omega}}{\partial x} \\ \frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 2\frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{\partial\bar{\Omega}}{\partial y} \end{aligned} \qquad \text{or} \qquad \begin{aligned} \ddot{x} - 2\dot{y} &= \bar{\Omega}_x \\ \ddot{y} - 2\dot{x} &= \bar{\Omega}_y. \end{aligned} \tag{1.20}$$

Multiplying now the first equation by $\dot{x}$ and the second one by $\dot{y}$ and summing them we obtain

$$\ddot{x} + \ddot{y} = \bar{\Omega}_x\dot{x} + \bar{\Omega}_y\dot{y}, \tag{1.21}$$

and integrating this new expression yields

$$\dot{x}^2 + \dot{y}^2 = 2\bar{\Omega} + C, \tag{1.22}$$

where C is the integration constant. Observe that

$$J(x, y, \dot{x}, \dot{y}) = \dot{x}^2 + \dot{y}^2 - 2\bar{\Omega}(x, y) \tag{1.23}$$

is an integral, known as the *Jacobi integral*, and C is known as the *Jacobi constant*.

Let us find the expression of the function $\bar{\Omega}$ explicitly, since it will be useful in upcoming sections. From the system of equations 1.20 we have

$$\begin{aligned} \frac{\partial\bar{\Omega}}{\partial x} &= x - (1-m)(x-m)r_1^{\alpha-1} - m(x-m+1)r_2^{\alpha-1} \\ \frac{\partial\bar{\Omega}}{\partial y} &= y - (1-m)yr_1^{\alpha-1} - myr_2^{\alpha-1}. \end{aligned} \tag{1.24}$$

Therefore,

$$\begin{aligned} \bar{\Omega} &= \int \frac{\partial\bar{\Omega}}{\partial x}\mathrm{d}x + \phi(y) \\ &= \int (x - (1-m)(x-m)r_1^{\alpha-1} - m(x-m+1)r_2^{\alpha-1})\mathrm{d}x + \phi(y) \\ &= \frac{x^2}{2} - (1-m)\frac{1}{2}\int 2(x-m)((x-m)^2+y^2)^{\frac{\alpha-1}{2}}\mathrm{d}x \\ &\quad - m\frac{1}{2}\int 2(x-m+1)((x-m+1)^2+y^2)^{\frac{\alpha-1}{2}}\mathrm{d}x + \phi(y) \\ &= \frac{x^2}{2} - \frac{1-m}{\alpha+1}((x-m)^2+y^2)^{\frac{\alpha+1}{2}} - \frac{m}{\alpha+1}((x-m+1)^2+y^2)^{\frac{\alpha+1}{2}} + \phi(y) + C + D \\ &= \frac{x^2}{2} - (1-m)\frac{1}{\alpha+1}(r_1^{\alpha+1}-1) - m\frac{1}{\alpha+1}(r_2^{\alpha+1}-1) + \phi(y) + K_1, \end{aligned} \tag{1.25}$$

and also

$$\begin{aligned}
\bar{\Omega} &= \int \frac{\partial \bar{\Omega}}{\partial y} \mathrm{d}y + \psi(x) \\
&= \int (y - (1-m) y r_1^{\alpha-1} - m y r_2^{\alpha-1}) \mathrm{d}y + \psi(x) \\
&= \frac{y^2}{2} - (1-m)\frac{1}{2}\int 2y((x-m)^2 + y^2)^{\frac{\alpha+1}{2}} \mathrm{d}y \\
&\quad - m\frac{1}{2}\int 2y((x-m+1)^2 + y^2)^{\frac{\alpha+1}{2}} \mathrm{d}y + \psi(x) \\
&= \frac{y^2}{2} - \frac{1-m}{\alpha+1}((x-m)^2 + y^2)^{\frac{\alpha+1}{2}} - \frac{m}{\alpha+1}((x-m+1)^2 + y^2)^{\frac{\alpha+1}{2}} + \psi(x) + C + D \\
&= \frac{y^2}{2} - (1-m)\frac{1}{\alpha+1}(r_1^{\alpha+1} - 1) - m\frac{1}{\alpha+1}(r_2^{\alpha+1} - 1) + \psi(y) + K_2.
\end{aligned} \tag{1.26}$$

Note that in the last step of both computations, we have chosen appropriate integration constants C and D such that the origin of potentials is at radius 1, as mentioned in section 1.1. Taking $K_1 = K_2 = 0$, $\phi(y) = \frac{y^2}{2}$ and $\psi(x) = \frac{x^2}{2}$, we arrive at

$$\bar{\Omega} = \frac{1}{2}(x^2 + y^2) - (1-m)\frac{1}{\alpha+1}(r_1^{\alpha+1} - 1) - m\frac{1}{\alpha+1}(r_2^{\alpha+1} - 1) \tag{1.27}$$

Observe that the equations of motion are solely determined by the partial derivatives of this function $\bar{\Omega}$, and hence, an addition of a constant value will not modify them. We may therefore, consider the function

$$\Omega = \bar{\Omega} + \frac{1}{2}m(1-m), \tag{1.28}$$

which also defines unequivocally the equations of motion, but offers a more symmetric form, for it may be written solely in terms of r_1 and r_2:

$$\Omega = \frac{1}{2}((1-m)r_1^2 + m r_2^2) - (1-m)\frac{1}{\alpha+1}(r_1^{\alpha+1} - 1) - m\frac{1}{\alpha+1}(r_2^{\alpha+1} - 1), \tag{1.29}$$

or

$$\Omega = (1-m)\left(\frac{r_1^2}{2} - \frac{1}{\alpha+1}(r_1^{\alpha+1} - 1)\right) + m\left(\frac{r_2^2}{2} - \frac{1}{\alpha+1}(r_2^{\alpha+1} - 1)\right). \tag{1.30}$$

Conservation of energy

Upon studying the Jacobi integral the question arises whether there are more of this invariant relations in the restricted problem of three bodies. As it turns out, the Jacobi integral is the only known invariant in the restricted problem of three bodies, and for example, the total energy is not constant, as we shall now prove.

Let m_3 be the mass of the third body and let h_3 be the total energy of the third body per unit of mass, such that

$$h_3 = \frac{1}{2}(\ddot{x} + \ddot{y}) - \Omega \tag{1.31}$$

If we denote by H_{12} the total energy of the primary and secondary bodies combined, it results that

$$H_{12} = \frac{1}{2} - m(1-m), \tag{1.32}$$

where the first term corresponds to the kinetic energy and the second to the potential energy between the two bodies. Observe that H_{12} is a constant.

The total energy of the system H is the sum of the energy of each of its particles, and therefore

$$H = m_3 h_3 + H_{12}, \tag{1.33}$$

which is clearly not a constant.

One may find such a result a violation of the energy conservation principle. However, this is not the case, inasmuch as one of the hypothesis of the restricted problem is that the mass of the third body is negligible, thus not affecting the primary and the secondary body, whereas in reality this cannot be the case. Therefore, we created a situation which, strictly speaking, is only valid when $m_3 = 0$, and in that case, equation 1.33 does certainly become a constant.

1.4 Hamiltonian formulation

A dynamics system with n degrees of freedom and n related independent coordinates $(q_1, q_2, \ldots, q_n)$ can be described at any time provided that one derives every q_i as a function of time. This functions of time can be found applying $2n$ initial conditions, of the form $q_i = q_{i_0}$, $\dot{q}_i = \dot{q}_{i_0}$ at $t = t_0$, and thus,

$$q_i = q_i(t, q_{i_0}, \dot{q}_{i_0}). \tag{1.34}$$

The configuration space, i.e. the space formed by this n q_i coordinates, has dimension n. Note that a point in the configuration space is not uniquely related with a particular motion, for every point $(q_1, q_2, \ldots, q_n)$ is related to infinitely many orbits, depending of the values of the n initial velocities considered.

The Lagrangian dynamics are related to the configuration space, where the equations of motion appear as n second-order differential equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \qquad \text{for } i = 1, \ldots, n, \tag{1.35}$$

where $L = T - V$, the kinetic energy minus the potential energy.

Introducing n more variables, $p_1, p_2, \ldots, p_n$, the generalised momenta, defined as

$$p_i = \frac{\partial L(q_j, \dot{q}_j, t)}{\partial \dot{q}_i}. \tag{1.36}$$

The Hamiltonian of a dynamical system is defined by

$$H(q, p, t) = \sum_{i=0}^{n} \dot{q}_i p_i - L(q, \dot{q}, t). \tag{1.37}$$

The n second-order differential equation system 1.35 is transformed in the Hamiltonian system into a $2n$ first-order differential equation system:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \tag{1.38}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}. \tag{1.39}$$

We shall now compute this Hamiltonian for the synodical coordinate system, although to do so it will be necessary to find it in the sidereal coordinate system. In this system we will consider $X = q_1$ and $Y = q_2$. The Lagrangian function is therefore

$$L = \frac{1}{2}(\dot{q}_1^2 \dot{q}_2^2) + V_\alpha(q_1, q_2, t), \tag{1.40}$$

where in this case,

$$V_\alpha(X, Y, t) = (1-m)\frac{1}{\alpha+1}(R_1^{\alpha+1}(t) - 1) + m\frac{1}{\alpha+1}(R_2^{\alpha+1}(t) - 1). \tag{1.41}$$

The momenta $p_i = \frac{\partial L}{\partial \dot{q}_i}$ become

$$p_i = q_i, \tag{1.42}$$

and thus the Hamiltonian is given by

$$H(q, p, t) = \frac{1}{2}(p_1^2 + p_2^2) - V_\alpha(q_1, q_2, t). \tag{1.43}$$

1.4.1 Canonical transformations in the phase space

Our goal now is to transform the Hamiltonian found for the sidereal coordinate system in equation 1.43 into the synodic coordinate system. To do so, we must first introduce the canonical transformations of the phase space.

Let $q_1, \ldots, q_n, p_1, \ldots, p_n$ and $Q_1, \ldots, Q_n, P_1, \ldots, P_n$ be two sets of $2n$ canonical variables such that

$$\begin{aligned} Q_i &= Q_i(q_1, \ldots, q_n, p_1, \ldots, p_n, t), \\ P_i &= P_i(q_1, \ldots, q_n, p_1, \ldots, p_n, t). \end{aligned} \tag{1.44}$$

Assume that these transformations are such that $Q_1, \ldots, Q_n, P_1, \ldots, P_n$ are canonical. Then, they give place to a new Hamiltonian $\bar{H}$ such that

$$\dot{Q}_i = \frac{\partial \bar{H}}{\partial P_i}, \qquad \dot{P}_i = -\frac{\partial \bar{H}}{\partial Q_i}. \tag{1.45}$$

One may derive some useful transformations from Hamilton's principle, which states that

$$\delta \int_{t_1}^{t_2} L \mathrm{d}t = 0. \tag{1.46}$$

Since $H = \sum_{i=1}^{n} p_i \dot{q}_i - L$, we have

$$\delta \int_{t_1}^{t_2} \left(\sum_{i=1}^{n} p_i \dot{q}_i - H \right) \mathrm{d}t = 0, \tag{1.47}$$

and also, after applying the transformations,

$$\delta \int_{t_1}^{t_2} \left((\sum_{i=1}^{n} P_i \dot{Q}_i - \bar{H} \right) \mathrm{d}t = 0 \tag{1.48}$$

The difference between the two variations is therefore,

$$\delta \int_{t_1}^{t_2} \left(\sum_{i=1}^{n} (p_i \mathrm{d}q_i - P_i \mathrm{d}Q_i) - H + \bar{H} \right) \mathrm{d}t = 0. \tag{1.49}$$

Let now

$$\sum_{i=1}^{n} (p_i \mathrm{d}q_i - P_i \mathrm{d}Q_i) - H + \bar{H} = \frac{\mathrm{d}W}{\mathrm{d}t}, \tag{1.50}$$

or also,

$$\sum_{i=1}^{n} (p_i \mathrm{d}q_i - P_i \mathrm{d}Q_i) = \mathrm{d}W + (H - \bar{H})\mathrm{d}t, \tag{1.51}$$

where W is named the *generating function*. In principle, $W(q_i, Q_i, p_i, P_i, t)$ depends on $4n + 1$ variables. However, since there are $2n$ relationships given by 1.44 between the $4n + 1$ variables, W depends actually on $2n + 1$ variables. We may therefore consider W functions of the form

$$\begin{aligned} W_1 &= W_1(q, Q, t), \\ W_2 &= W_2(q, P, t), \\ W_3 &= W_3(p, Q, t), \\ W_4 &= W_4(p, P, t). \end{aligned} \tag{1.52}$$

For W_3, for example, we have

$$\mathrm{d}W_3 = \frac{\partial W_3}{\partial p_i} \mathrm{d}p_i + \frac{\partial W_3}{\partial Q_i} \mathrm{d}Q_i + \frac{\partial W_3}{\partial t} \mathrm{d}t. \tag{1.53}$$

Substituting 1.53 into 1.51 gives

$$\sum_{i=1}^{n} (p_i \mathrm{d}q_i - P_i \mathrm{d}Q_i) = \frac{\partial W_3}{\partial p_i} \mathrm{d}p_i + \frac{\partial W_3}{\partial Q_i} \mathrm{d}Q_i + (\frac{\partial W_3}{\partial t} + H - \bar{H})\mathrm{d}t, \tag{1.54}$$

and therefore,

$$q_i = -\frac{\partial W_3}{\partial p_i}, \qquad P_i = -\frac{\partial W_3}{\partial Q_i}, \tag{1.55}$$

and

$$\bar{H} = H + \frac{\partial W_3}{\partial t}. \tag{1.56}$$

These $2n + 1$ equations give the new $2n$ canonical variables and the new Hamiltonian.

1.4.2 Transformation into the synodical coordinate system

Let us now consider a canonical transformation to transform the Hamiltonian from the sidereal coordinate system into the synodical coordinate system. We will choose the transformation given by

$$W_3(p_1, p_2, Q_1, Q_2, t) = -a_{ij}(t)p_iQ_j, \tag{1.57}$$

where

$$(a_{ij}) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}, \tag{1.58}$$

which represents a rotation with angular velocity $\omega = 1$. As seen in the previous section, we obtain that

$$q_i = -\frac{\partial W_3}{\partial p_i}, \qquad P_i = -\frac{\partial W_3}{\partial Q_i}, \tag{1.59}$$

and

$$\bar{H} = H + \frac{\partial W_3}{\partial t}. \tag{1.60}$$

Therefore, we obtain that

$$q_i = a_{ij}Q_j, \qquad P_i = a_{ji}p_j, \tag{1.61}$$

and

$$\bar{H}(Q_1, Q_2, P_1, P_2) = \frac{1}{2}(P_1^2 + P_2^2) + Q_2P_1 - Q_1P_2 - \Omega(Q_1, Q_2). \tag{1.62}$$

or

$$\bar{H}(x, y, p_x, p_y) = \frac{1}{2}(p_x^2 + p_y^2) + yp_x - xp_y - \Omega(x, y). \tag{1.63}$$

Note that V_α is transformed into Ω is obvious, since it is the same process that we applied in section 1.3.1.

1.5 Colinear equilibrium points

The *colinear equilibrium points* are the equilibrium points on the x axis, and as such, they fulfil the *colinear equilibrium equation*:

$$x - (1-m)(x-m)r_1^{\alpha-1} - m(x-m+1)r_2^{\alpha-1} = 0.$$

Since $r_1^2 = (x-m)^2+y^2$ and $r_2^2 = (x-m+1)^2+y^2$ (and in this case, $y = 0$), the aforementioned equation is equivalent to:

$$x - (1-m)(x-m)|x-m|^{\alpha-1} - m(x-m+1)|x-m+1|^{\alpha-1} = 0. \tag{1.64}$$

Note that for $\alpha > 0$, $L_0 = (m-1, 0)$ and $L_{-1} = (m, 0)$ are two solutions for the equation 1.64, located at the positions of the primary and the secondary body, and so they are called the *collision points*.

According to the signs of $x-m$ and $x-m+1$, we can find three other colinear Lagrangian equilibrium points: L_1, placed between the two bodies, when $x-m>0$ and $x-m+1<0$, L_2, placed beyond the secondary body, when $x-m<0$ and $x-m+1<0$, and L_3, beyond the primary body, when $x-m>0$ and $x-m+1>0$.

If we define $a = x-m$, the colinear equation becomes

$$F_{\alpha,m}(a) = a + m - (1-m)a|a|^{\alpha-1} - m(a+1)|a+1|^{\alpha-1} = 0,$$

which can be written in each of the intervals as

$$F_{\alpha,m}(a) = \begin{cases} a+m+(1-m)(-a)^\alpha + m(a+1)^\alpha = 0, & \text{if } a < -1, \\ a+m+(1-m)(-a)^\alpha - m(a+1)^\alpha = 0, & \text{if } -1 < a < 0, \\ a+m-(1-m)a^\alpha - m(a+1)^\alpha = 0, & \text{if } 0 < a. \end{cases} \tag{1.65}$$

Solutions of the colinear equilibrium equations

Note that in general, it is not possible to find an analytic solution to the colinear equilibrium equation. It is possible, however, to study the number of its solutions in terms of α, in which similar qualitative behaviour is observed.

We shall study which of the Lagrangian points mentioned in the previous section exist for a given value of α.

Theorem 1.2. *Let us consider three intervals corresponding to the intervals of the $F_{\alpha,m}$ function and values a_1, a_2 and a_3 such that $\alpha_2 \in (-\infty, -1)$, $\alpha_1 \in (-1, 0)$ and $\alpha_3 \in (0, \infty)$.*

- *L_0 and L_{-1}: For $\alpha \geq 0$, $a = 0$ and $a = -1$ are solutions for the colinear equilibrium equation. For $\alpha < 0$, $a = 0$ and $a = -1$ are singularities of $F_{\alpha,m}$.*

  Proof. If $\alpha < 0$, $\lim_{a\to -1} F_{\alpha,m}(a) = \lim_{a\to -1} m(-a-1)^\alpha = \infty$, and $\lim_{a\to 0} F_{\alpha,m}(a) = \lim_{a\to 0} \pm a^\alpha - m = \pm\infty$.

  If $\alpha \geq 0$, $F_{\alpha,m}(-1) = 0$ and $F_{\alpha,m}(0) = 0$, hence fulfilling the colinear equation and $L_{-1} = (m-1, 0)$, $L_0 = (m, 0)$ □

- *L_1: For $\alpha \leq \frac{1}{1-m}$ and $\alpha \geq \frac{1}{m}$, there exists a solution $a_1 \in (-1, 0)$ for the colinear equilibrium equation.*

  Proof. If $\alpha < 0$, then in the interval $(-\infty, -1)$,

  $$F'_{\alpha,m}(a_2) = 1 - \alpha(1-m)(-a)^{\alpha-1} + m\alpha(a+1)^{\alpha-1} > 0, \tag{1.66}$$

  and thus, $F_{\alpha,m}$ is strictly increasing in the interval $(-\infty, -1)$. Since $F_{\alpha,m}$ is a continuous function in this interval, $\lim_{a\to -\infty} F_{\alpha,m}(a) = -\infty$ and $F_{\alpha,m}(-1) = \infty$ as seen previously, it follows immediately, applying Bolzano's theorem that there exists exactly one solution $a_2 \in (-\infty, -1)$.

If $0 < \alpha < 1$,

$$\lim_{a \to -\infty} F'_{\alpha,m}(a) = 1 \tag{1.67}$$

$$\lim_{a \to -1} F'_{\alpha,m}(a) = -\infty, \tag{1.68}$$

and, in the interval $(-\infty, -1)$,

$$F''_{\alpha,m}(a) = \alpha(\alpha-1)(1-m)(-a)^{\alpha-2} + m\alpha(\alpha-1)(a+1)^{\alpha-2} < 0, \tag{1.69}$$

which implies the existence of exactly one maximum in the interval $(-\infty, -1)$. Moreover, since $\lim_{a\to-\infty} F_{\alpha,m}(a) = -\infty$ and $F_{\alpha,m}(-1) = 0$, it is clear that $F_{\alpha,m}(a)$ has exactly one zero in the interval $(-\infty, -1)$.

If $1 < \alpha < \frac{1}{1-m}$, then observe that,

$$\lim_{a \to -\infty} F'_{\alpha,m}(a) = -1 \tag{1.70}$$

$$\lim_{a \to -1} F'_{\alpha,m}(a) = 1 - (1-m)\alpha > 0. \tag{1.71}$$

and, in the interval $(-\infty, -1)$,

$$F''_{\alpha,m}(a) > 0. \tag{1.72}$$

This implies that there exists exactly one minimum in the interval aforementioned interval. Furthermore, since $\lim_{a\to-\infty} F_{\alpha,m}(a) = \infty$ and $F_{\alpha,m}(-1) = 0$, it is clear that there exists exactly one solution in the interval $(-\infty, -1)$.

For $\alpha > \frac{1}{m}$, we may observe that

$$\lim_{a \to -\infty} F'_{\alpha,m}(a) = -1 \tag{1.73}$$

$$\lim_{a \to -1} F'_{\alpha,m}(a) = 1 - (1-m) < 0, \tag{1.74}$$

and, in the interval $(-\infty, -1)$,

$$F''_{\alpha,m}(a) > 0. \tag{1.75}$$

Furthermore, $\lim_{a\to-\infty} F_{\alpha,m}(a) = \infty$ and $F_{\alpha,m}(-1) = 0$, it is clear to see that there exists exactly one solution in the interval $(-\infty, -1)$ as well. □

- *L_2: For $\alpha \leq \frac{1}{1-m}$, there exists a solution $a_2 \in (-1, 0)$ for the colinear equilibrium equation.*

Proof. If $\alpha < 0$, it is clear that $F_{\alpha,m}(a)$ is strictly increasing in the interval $(-1, 0)$. Furthermore,

$$\lim_{a \to -1} F_{\alpha,m}(a) = -\infty \tag{1.76}$$

$$\lim_{a \to 0} F_{\alpha,m}(a) = \infty, \tag{1.77}$$

and, in the interval $(-1, 0)$,

$$F'_{\alpha,m}(a) = 1 - \alpha(1-m)(-a)^{\alpha-1} - m\alpha(a+1)^{\alpha-1} > 0. \tag{1.78}$$

These results, together with the fact that $F_{\alpha,m}$ is a continuous function in the interval $(-1,0)$, imply, by Bolzano's theorem, that there exists exactly a single solution for the function $F_{\alpha,m}$ in the interval $(-1,0)$.

If $0<\alpha<1$, we have that

$$F''_{\alpha,m}(a)=\alpha(\alpha-1)(1-m)(-a)^{\alpha-2}-m\alpha(\alpha-1)(a+1)^{\alpha-2}, \tag{1.79}$$

and, therefore,

$$\lim_{a\to-1}F''_{\alpha,m}(a)=\infty, \tag{1.80}$$

$$\lim_{a\to0}F''_{\alpha,m}(a)=-\infty, \tag{1.81}$$

Then, the third derivative of the function in the interval $(-1,0)$ is given by the following expression:

$$F'''_{\alpha,m}=-\alpha(\alpha-1)(\alpha-2)(1-m)(-a)^{\alpha-3}-m\alpha(\alpha-1)(\alpha-2)(a+1)^{\alpha-3}<0. \tag{1.82}$$

These conditions indicate the existence exactly one inflection point in the interval $(-1,0)$. Furthermore, since

$$\lim_{a\to-1}F'_{\alpha,m}(a)=-\infty, \tag{1.83}$$

$$\lim_{a\to0}F'_{\alpha,m}(a)=-\infty \tag{1.84}$$

it follows the existence of one and only one minimum and one and only one maximum in the interval $(-1,0)$, with negative and positive values respectively. Therefore the existence of a single solution of the colinear equation for $(-1,0)$ follows immediately. If $1<\alpha<\frac{1}{1-m}$,

$$\lim_{a\to-1}F''_{\alpha,m}(a)=-\infty \tag{1.85}$$

$$\lim_{a\to0}F''_{\alpha,m}(a)=\infty \tag{1.86}$$

and

$$F'''_{\alpha,m}(a)>0. \tag{1.87}$$

These facts indicate the existence of one and only one inflexion point in the interval $(-1,0)$. Furthermore,

$$\lim_{a\to-1}F''_{\alpha,m}(a)=1-\alpha(1-m)>0, \tag{1.88}$$

$$\lim_{a\to0}F''_{\alpha,m}(a)=1-m>0, \tag{1.89}$$

indicating the existence of one and only one maximum and one and only one minimum in the interval $(-1,0)$ with positive and negative values respectively. This implies that $F_{\alpha,m}$ has exactly one zero in the interval $(-1,0)$. □

- *L_3: For $\alpha \leq \frac{1}{m}$, there exists one and only one solution $a_3 \in (0, \infty)$ for the colinear equilibrium equation.*

  Proof. If $\alpha < 0$,

$$\lim_{a \to 0} F_{\alpha,m}(a) = -\infty, \tag{1.90}$$

$$\lim_{a \to \infty} F_{\alpha,m}(a) = \infty \tag{1.91}$$

and, in the interval $(0, \infty)$,

$$F'_{\alpha,m}(a) = 1 - \alpha(1-m)a^{\alpha-1} - \alpha(a+1)^{\alpha-1} > 0. \tag{1.92}$$

Therefore, it follows immediately that the colinear equation $F_{\alpha,m}$ has one and only one zero in the interval $(0, \infty)$.

If $0 < \alpha < 1$,

$$\lim_{a \to 0} F'_{\alpha,m}(a) = -\infty, \tag{1.93}$$

$$\lim_{a \to \infty} F'_{\alpha,m}(a) = 1, \tag{1.94}$$

and, in the interval $(0, \infty)$

$$F''_{\alpha,m}(a) = -\alpha(\alpha-1)a^{\alpha-2} - \alpha(\alpha-1)(a+1)^{\alpha-1} > 0. \tag{1.95}$$

These facts together indicate the existence of one and only one maximum in the interval $(0, \infty)$. Together with the fact that $F_{\alpha,m}(0) = 0$ and $\lim_{a\to\infty} F_{\alpha,m}(a) = \infty$, it follows that there must exist one and only one solution of the colinear equation in the interval $(0, \infty)$.

If $1 < \alpha < \frac{1}{1-m}$, we have that

$$\lim_{a \to 0} F'_{\alpha,m}(a) = 1 - \alpha m > 0, \tag{1.96}$$

$$\lim_{a \to \infty} F'_{\alpha,m}(a) = -1, \tag{1.97}$$

and, in the interval $(0, \infty)$,

$$F''_{\alpha,m}(a) < 0. \tag{1.98}$$

It follows immediately that there exists one and only one maximum in the interval $(0, \infty)$. Furthermore, $F_{\alpha,m}(0) = 0$ and $\lim_{a\to\infty} F_{\alpha,m}(a) = -\infty$ and whence, $F_{\alpha,m}$ has exactly one zero in the interval $(0, \infty)$.

If $\frac{1}{1-m} < \alpha < \frac{1}{m}$,

$$\lim_{a \to 0} F'_{\alpha,m}(a) = 1 - \alpha m > 0, \tag{1.99}$$

$$\lim_{a \to \infty} F'_{\alpha,m}(a) = -1, \tag{1.100}$$

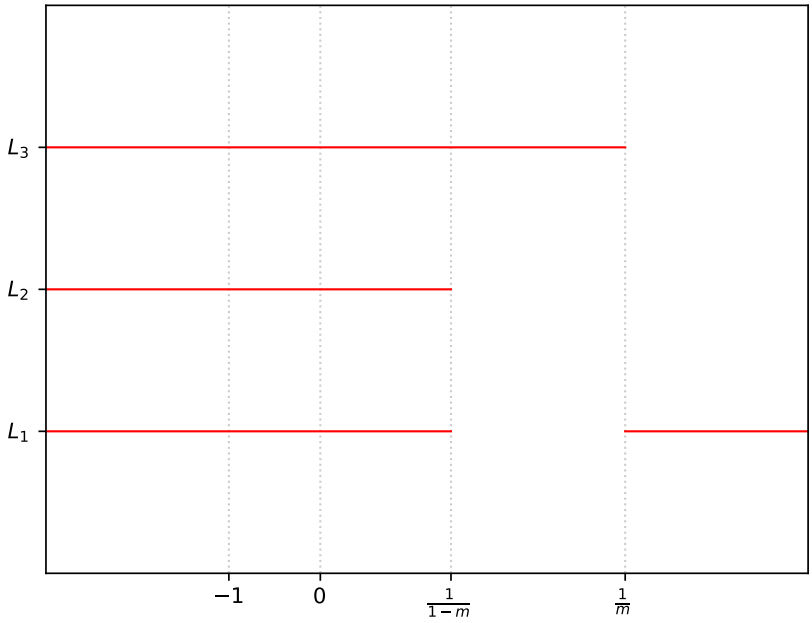

Figure 1.1: Summary of the intervals in which the Lagrange points L_1, L_2 and L_3 exists.

and, in the interval $(0, \infty)$,

$$F''_{\alpha,m} > 0. \tag{1.101}$$

Therefore, it follows that there exists one and only one maximum of $F_{\alpha,m}$ in the interval $(0, \infty)$. Moreover, since $F_{\alpha,m}(0) = 0$ and $\lim_{a\to\infty} F_{\alpha,m}(a) = -\infty$, it is clear that there must exist exactly one zero of $F_{\alpha,m}$ in the interval $(0, \infty)$. □

The preceding result is summarised in figure 1.1.

Particular cases

We shall now study the limit cases $\alpha = 0$ and $\alpha = -1$, as well as an expansion for the cases in which α is close to 1.

- **Case** $\alpha = -1$
  In this case, the three separate cases of the colinear equilibrium equations reduce to a single one

  $$F_{-1,m}(a) = a + m - (1-m)\frac{1}{a} - m\frac{1}{a+1} = 0,$$

  which is actually a cubic equation:

  $$a^3 + (m+1)a^2 - (1-m)a - (1-m) = 0.$$

In this case, the solutions, and hence the Lagrange equilibrium points could be explicitly computed, although the resulting expressions are not simple at all and therefore we will not explicit them.

- **Case** $\alpha = 0$
  In this case, the colinear equilibrium equation becomes:

$$F_{0,m}(a) = \begin{cases} a+1+m & \text{if } a < -1, \\ a+1-m & \text{if } -1 < a < 0, \\ a-1+m & \text{if } 0 < a \end{cases}$$

  The vertical asymptotes become jump discontinuities for $a = -1$ and $a = 0$. The three solutions are $a_1 = -1+m$, $a_2 = -1-m$, and $a_3 = 1-m$, and thus, the corresponding equilibrium points are then:

$$L_{-1} = (-1+m, 0), \quad L_0 = (m, 0),$$
$$L_1 = (-1+2m, 0), \quad L_2 = (-1, 0), \quad L_3 = (1, 0)$$

- **Case** $\alpha \approx 1$
  For $\alpha = 1$, $F_{-1,m}(a)$ becomes identically 0. However, the continued colinear equilibrium equation for $\alpha \approx 1$ can be computed taking into account the expansions of $F_{\alpha,m}$ for $\alpha = 1+\varepsilon$, for a small value of ε.

$$F_{1+\varepsilon,m}(a) = \begin{cases} a+m+(1-m)(-a)^{1+\varepsilon}+m(a+1)^{1+\varepsilon}, & \text{if } a < -1, \\ a+m+(1-m)(-a)^{1+\varepsilon}-m(a+1)^{1+\varepsilon}, & \text{if } -1 < a < 0, \\ a+m-(1-m)a^{1+\varepsilon}-m(a+1)^{1+\varepsilon}, & \text{if } 0 < a. \end{cases}$$

  Note that we can rewrite $F_{1+\varepsilon,m}(a)$ as

$$F_{1+\varepsilon,m}(a) = \begin{cases} (1-m)((-a)^{1+\varepsilon}-(-a))+m((-a-1)^{1+\varepsilon}-(-a-1)), & \text{if } a < -1, \\ (1-m)((-a)^{1+\varepsilon}-(-a))-m((a+1)^{1+\varepsilon}-(a+1)), & \text{if } -1 < a < 0, \\ -(1-m)(a^{1+\varepsilon}-a)-m((a+1)^{1+\varepsilon}-(a+1)), & \text{if } 0 < a. \end{cases}$$

  which using that

$$a^{1+\varepsilon} - a = aa^{\varepsilon} - a = a(e^{\varepsilon \log a} - 1) \approx \varepsilon a \log a,$$

  the expansions at first order in ε of $F_{1+\varepsilon,m}$ are:

$$F_{1+\varepsilon,m}(a) \approx \begin{cases} \varepsilon((1-m)(-a)\log(-a)+m(-a-1)\log(-a-1)), & \text{if } a < -1, \\ \varepsilon((1-m)(-a)\log(-a)-m(a+1)\log(a+1)), & \text{if } -1 < a < 0, \\ \varepsilon(-(1-m)(a)\log(a)-m(a+1)\log(a+1)), & \text{if } 0 < a. \end{cases}$$

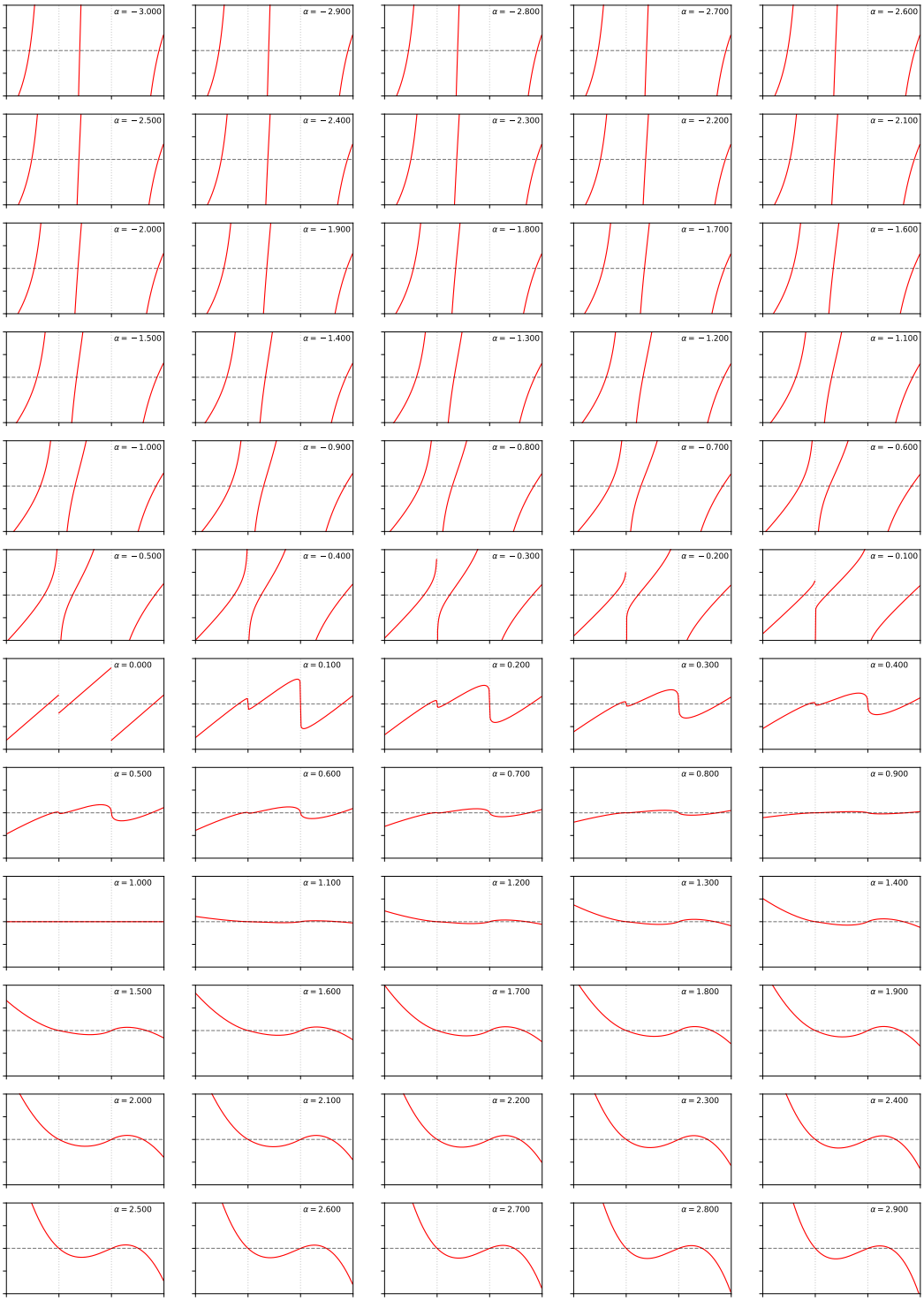

Figure 1.2: Colinear equation for $m = 0.2$ and different values of α

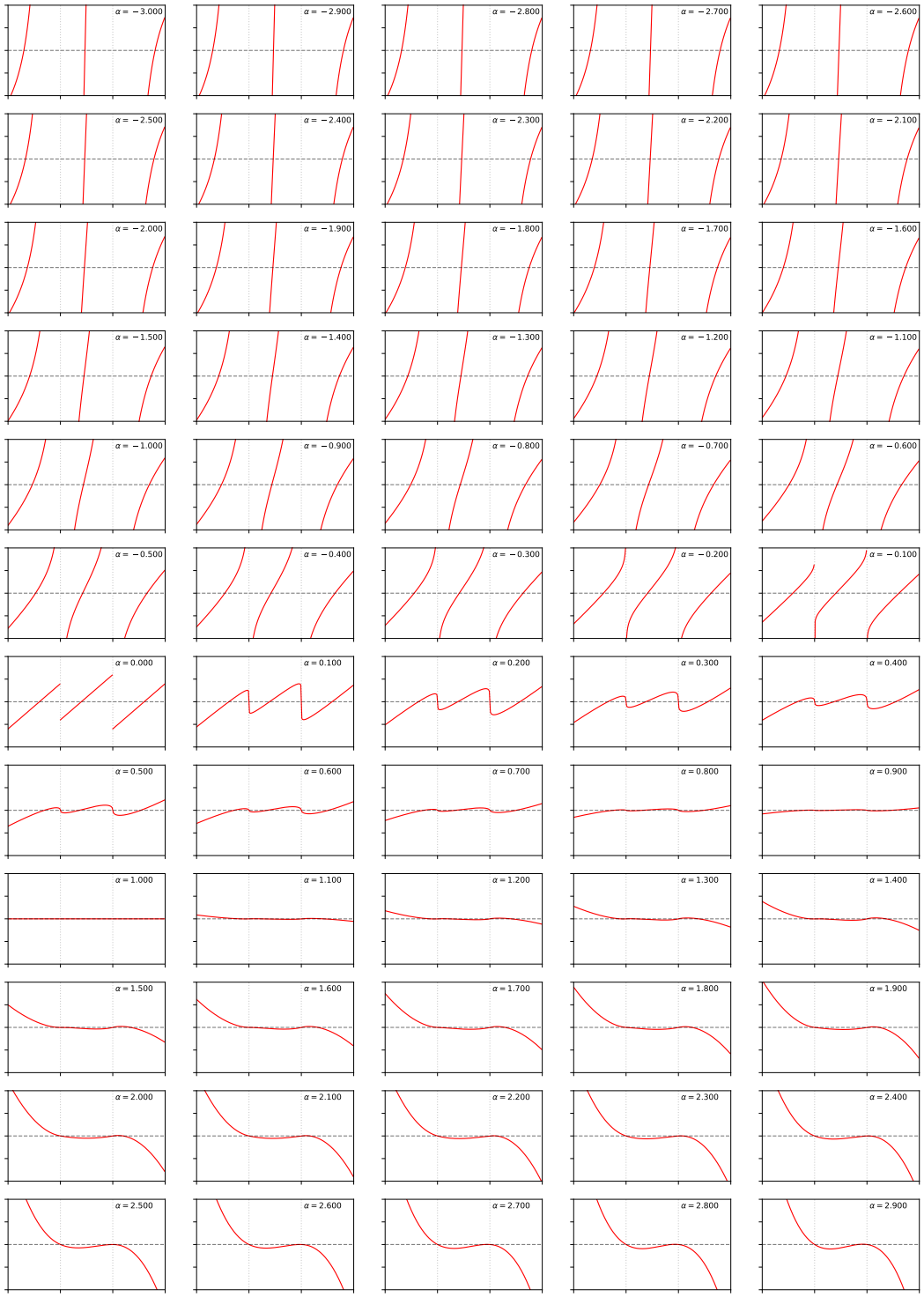

Figure 1.3: Colinear equation for $m = 0.4$ and different values of α

Chapter 2

Hill regions and zero velocity curves

This chapter is devoted to the study of the subset of the plane in which a particle with a given synodic energy is able to move.

This study will be built upon the Jacobi integral introduced in the first chapter. The constant C in the Jacobi integral will be related to the total energy of the body. Since the Jacobi integral relates the velocity (a magnitude for which is modulus is always a positive measure) of the body with with the Ω function that determines the movement of the body, there is only a subset of the plane for which movement is possible, i.e. the subset of the plane (x, y) in which the modulus of the velocity is a positive measure, that is, $\dot{x}^2 + \dot{y}^2$ is a positive measure.

These regions in which motion is possible are known as the *Hill regions*, and are delimited by the *zero velocity curves*, which as it name indicates, are the curves in which the velocity of the body becomes null.

2.1 An analysis of the Ω function

As stated previously, taking into account the Jacobian integral

$$\dot{x}^2 + \dot{y}^2 = 2\Omega(x, y) - C \tag{2.1}$$

establishes regions in which motion is possible, since the expression on the left-hand side is always positive. If we compute the constant C for a given set of initial conditions, we may impose $\dot{x}^2 + \dot{y}^2 = 0$ and then

$$2\Omega(x, y) = C \tag{2.2}$$

determines the zero velocity curves that are the border of the subset of the plane in which motion can occur.

Since the study of these regions is of utmost importance in the applications of the study of the three-body problem, this subject will be discussed in detail in upcoming section.

In this section, however, we will introduce some properties of the Ω function that will simplify the treatment of the problem of finding the zero velocity curves.

List of properties of the Ω function

1. For $r_1 = r_2 = 1$ gives $\Omega(r_1, r_2) = \frac{1}{2}$. For $\alpha < 1$ then this is a relative minimum whereas for $\alpha > 1$ it is a relative maximum.

   Proof. With a simple study of the hessian of the Ω function it is easy to prove, for

$$\begin{aligned} \frac{\partial \Omega}{\partial r_1} &= (1-m)(r_1 - r_1^\alpha) = 0 \\ \frac{\partial \Omega}{\partial r_2} &= m(r_2 - r_2^\alpha) = 0, \end{aligned} \tag{2.3}$$

   which holds only for $r_1 = r_2 = 1$. The hessian matrix for $r_1 = 1$, $r_2 = 1$ is

$$\begin{pmatrix} (1-m)(1-\alpha) & 0 \\ 0 & m(1-\alpha) \end{pmatrix}. \tag{2.4}$$

   Since $1 - \alpha > 0$ for $\alpha < 1$ and $1 - \alpha < 0$ for $\alpha > 1$, it is clear that for $\alpha < 1$ we have a minimum and for $\alpha > 1$ we have a maximum. □

2. (a) For $\alpha < -1$,

$$\begin{aligned} \lim_{r_1 \to 0} \Omega(r_1, r_2) &= \lim_{r_2 \to 0} \Omega(r_1, r_2) = \infty \\ \lim_{r_1 \to \infty} \Omega(r_1, r_2) &= \lim_{r_2 \to \infty} \Omega(r_1, r_2) = \infty. \end{aligned} \tag{2.5}$$

   (b) For $-1 < \alpha < 1$,

$$\begin{aligned} \lim_{r_1 \to 0} \Omega(r_1, r_2) &= \frac{m}{2} + \frac{1-m}{\alpha+1} \\ \lim_{r_2 \to 0} \Omega(r_1, r_2) &= \frac{1-m}{2} + \frac{m}{\alpha+1} \\ \lim_{r_1 \to \infty} \Omega(r_1, r_2) &= \lim_{r_2 \to \infty} \Omega(r_1, r_2) = \infty. \end{aligned} \tag{2.6}$$

   (c) For $1 < \alpha$,

$$\begin{aligned} \lim_{r_1 \to 0} \Omega(r_1, r_2) &= \frac{m}{2} + \frac{1-m}{\alpha+1} \\ \lim_{r_2 \to 0} \Omega(r_1, r_2) &= \frac{1-m}{2} + \frac{m}{\alpha+1} \\ \lim_{r_1 \to \infty} \Omega(r_1, r_2) &= \lim_{r_2 \to \infty} \Omega(r_1, r_2) = -\infty \end{aligned} \tag{2.7}$$

   Proof. We will only prove the results for r_1, since proving the results for r_2 does not require any different procedure.

   First of all, observe that since the main two bodies are at distance 1, then as $r_1 \to 0$, $r_2 \to 1$ and viceversa. Therefore,

$$\begin{aligned} \lim_{r_1 \to 0} \Omega(r_1, r_2) &= \lim_{r_1 \to 0, r_2 \to 1} \Omega(r_1, r_2) = \\ &= \lim_{r_1 \to 0} (1-m) \left(-\frac{1}{\alpha+1} (r^{\alpha+1} - 1) \right) + \frac{1}{2} m, \end{aligned} \tag{2.8}$$

   and thus,

(a) For $\alpha < -1$, then $\lim_{r_1 \to 0} r^{\alpha+1} = -\infty$, and therefore,

$$\lim_{r_1 \to 0} \Omega(r_1, r_2) = \lim_{r_1 \to 0} -\frac{1-m}{\alpha+1}(r^{\alpha+1} - 1) + \frac{1}{2}m = \infty \tag{2.9}$$

(b) For $-1 < \alpha$, then $\lim_{r_1 \to 0} r^{\alpha+1} = 0$, and therefore,

$$\lim_{r_1 \to 0} \Omega(r_1, r_2) = \lim_{r_1 \to 0} = \frac{1-m}{\alpha+1} + \frac{1}{2}m. \tag{2.10}$$

Now, for the case of $r_1 \to \infty$, we have

$$\lim_{r_1 \to \infty} \Omega(r_1, r_2) = \lim_{r_1 \to \infty} (1-m)\left(\frac{r_1^2}{2} - \frac{1}{\alpha+1}(r_1^{\alpha+1} - 1)\right) \tag{2.11}$$

(a) For $\alpha < 1$, $\lim_{r_1 \to \infty} r_1^{\alpha+1} = 0$, and thus,

$$\lim_{r_1 \to \infty} \Omega(r_1, r_2) = \infty \tag{2.12}$$

(b) For $1 < \alpha$, $\lim_{r_1 \to \infty} r_1^{\alpha+1} = \infty$, and thus,

$$\lim_{r_1 \to \infty} \Omega(r_1, r_2) = -\infty \tag{2.13}$$

□

3. $\Omega(x, y) = \Omega(x, -y)$.

   Proof. It follows immediately taking into account that Ω may be written solely in terms of r_1 and r_1 and observing that $r_1 = ((x-m)^2+y^2)^{\frac{1}{2}}$ and $r_2 = ((x-m+1)^2+y^2)^{\frac{1}{2}}$. □

4. For $\alpha < 1$, the absolute minimum of the function occurs at the Lagrangian points L_4 and L_5. For $1 < \alpha$, they are absolute maximums.

   Proof. It follows immediately from items 1 and 2. □

2.2 Regions of motion

As mentioned in the introduction of this chapter, the curves $\Omega(x, y) =$ ctant represent regions in which motion is possible (the outside of the curve if $\alpha < 1$ and the inside if $1 < \alpha$), since this function is connected with an essentially positive magnitude, namely the square of the relative velocity by means of the Jacobian integral.

Let us consider first of all the trivial case in which $m = 0$, to provide some insight into the problem at hand. In this case, the function $\Omega(r_1, r_2)$ has the form

$$\Omega(r_1, r_2) = \frac{r_1^2}{2} - \frac{1}{\alpha+1}(r_1^{\alpha+1} - 1), \tag{2.14}$$

where $r_1 = r$ is the distance between the third body and the primary body, located at the origin of coordinates. Therefore, the curves $\Omega(x, y) =$ ctant are concentric circles around the origin.

The Jacobian integral $\dot{x}^2 + \dot{y}^2 = 2\Omega(x, y) + C$, or $v^2 = 2\Omega(x, y) + C$, where v is the velocity of the third particle, becomes

$$v^2 = r^2 - \frac{2}{\alpha + 1}(r^{\alpha+1} - 1) - C. \tag{2.15}$$

Considering initial conditions of velocity and radius $v = v_0$ and $r = r_0$, and taking into account that the Jacobian integral is a constant of the movement, equation 2.15 may be written as

$$C_0 = r_0^2 - \frac{2}{\alpha + 1}(r_0^{\alpha+1} - 1) - v_0^2. \tag{2.16}$$

The radius of the circle of zero velocity for the given constant C_0 may be found solving the equation

$$C_0 = r_z^2 - \frac{2}{\alpha + 1}(r_z^{\alpha+1} - 1), \tag{2.17}$$

or,

$$r_0^2 - \frac{2}{\alpha + 1}(r_0^{\alpha+1} - 1) - v_0^2 = r_z^2 - \frac{2}{\alpha + 1}(r_z^{\alpha+1} - 1). \tag{2.18}$$

At any given point in time, the velocity of the third body is related to the radius from the origin r by means of the Jacobian integral 2.15 by

$$v^2 = r^2 - \frac{2}{\alpha + 1}(r^{\alpha+1} - 1) - C_0 \tag{2.19}$$

or

$$v^2 = r^2 - \frac{2}{\alpha + 1}(r^{\alpha+1} - 1) - r_z^2 + \frac{2}{\alpha + 1}(r_z^{\alpha+1} - 1). \tag{2.20}$$

Since $v^2 \geq 0$, we have

$$r^2 - \frac{2}{\alpha + 1}(r^{\alpha+1} - 1) - r_z^2 + \frac{2}{\alpha + 1}(r_z^{\alpha+1} - 1) \geq 0, \tag{2.21}$$

where the equality holds for $r = r_z$ (i.e. $v = 0$), or in other words, when the particle is over the its own zero velocity curve.

In order to find this regions of movement, one may take the following equation and solve it for r_z, to find the values of the radius for which there arises a change in the regions for which movement is possible and not:

$$0 = r_z^2 - \frac{2}{\alpha + 1}(r_z^{\alpha+1} - 1) - C_0. \tag{2.22}$$

In the general case the zero velocity curves for a given particle such that its initial conditions give a value of the Jacobi constant of C_0, are defined by

$$\text{ZVC}(C_0) = \{(x, y) \in \mathbb{R}^2 \mid 2\Omega(x, y) = J_0\}. \tag{2.23}$$

The critical points of $\Omega(x, y)$ and their extended Jacobi constants play an important role in the analysis of these curves. In these points, the derivative of $\Omega(x, y)$ equals 0, i.e.

$$\begin{aligned} \frac{\partial \Omega(x,y)}{\partial x} &= x - (1-m)(x-m)r_1^{\alpha-1} - m(x-m+1)r_2^{\alpha-1} = 0 \\ \frac{\partial \Omega(x,y)}{\partial y} &= y - (1-m)yr_1^{\alpha-1} - myr_2^{\alpha-1} = 0 \end{aligned} \tag{2.24}$$

Note that these conditions are the same as for the Lagrangian equilibrium points. Hence, the Lagrangian equilibrium points are critical points of $\Omega(x, y)$.

For $\alpha > 0$ the collision points are equilibrium points and so they are critical points of $\Omega(x, y)$, and play a role in the ulterior analysis of the zero velocity curves. For $\alpha < 0$ they constitute singularities of the Hamiltonian field and they are no longer equilibrium points. However for $-1 < \alpha < 0$ they become cuspid points of $\Omega(x, y)$, i.e. points where the function has finite values but unbounded derivatives.

The critical points can also be classified taking into account the value of second derivative of $\Omega(x, y)$ over them.

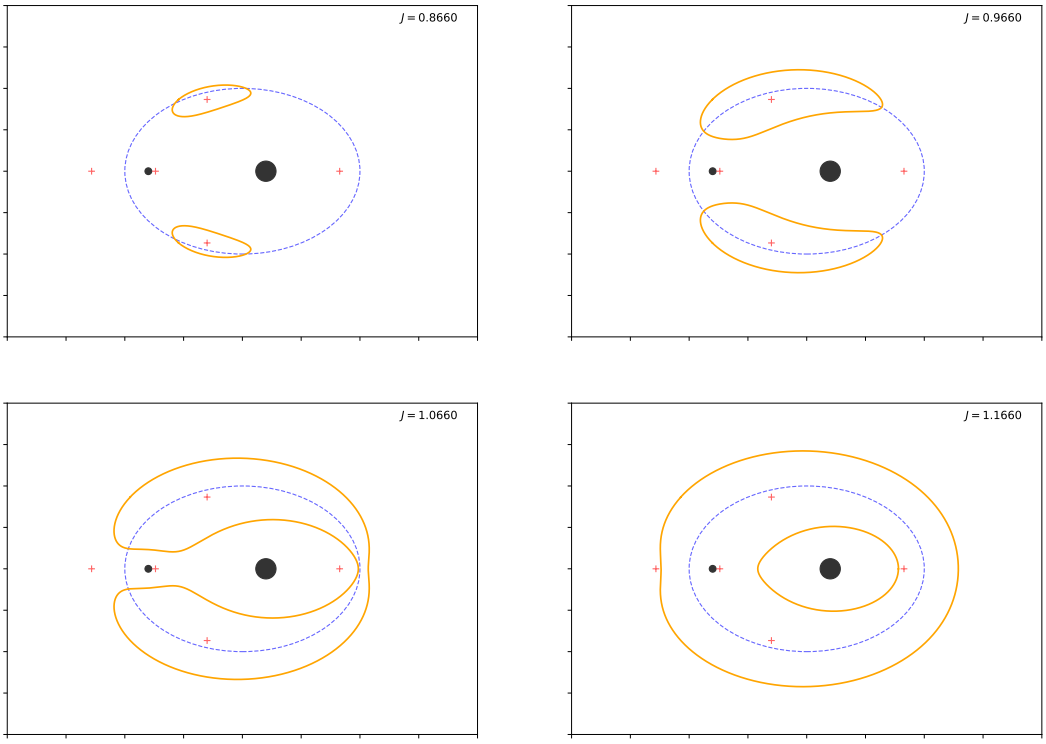

Figure 2.1: Zero velocity curves for $m = 0.2$ and $\alpha = -0.5$ for different values of the Jacobi constant J

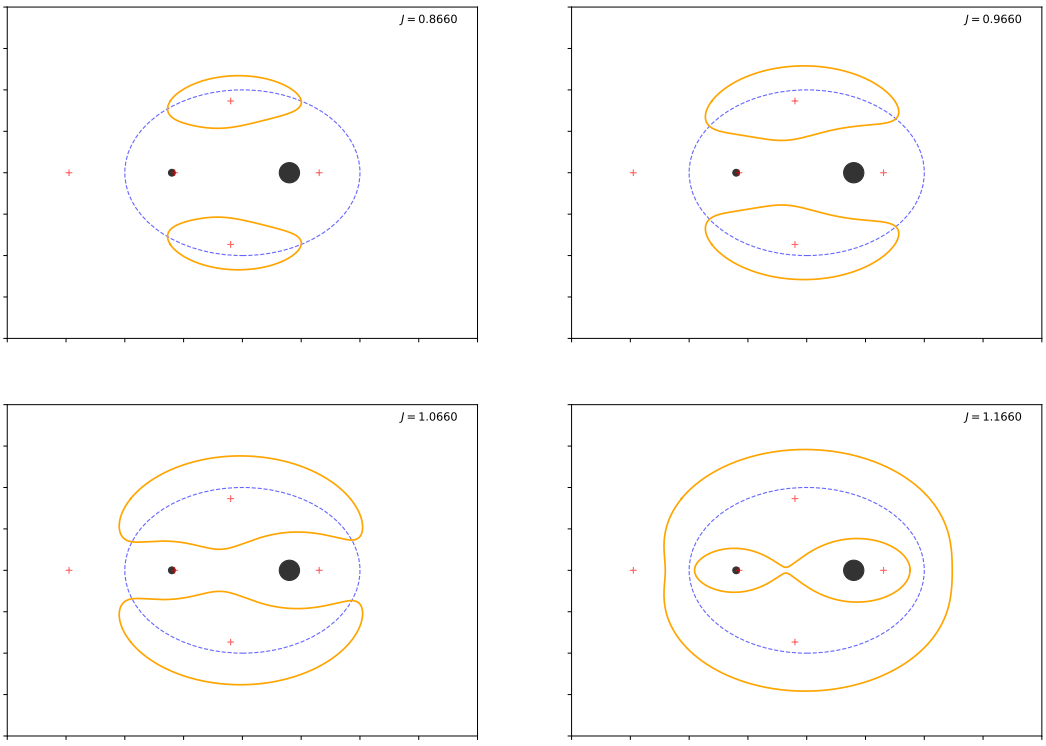

Figure 2.2: Zero velocity curves for $m = 0.4$ and $\alpha = -0.5$ for different values of the Jacobi constant J

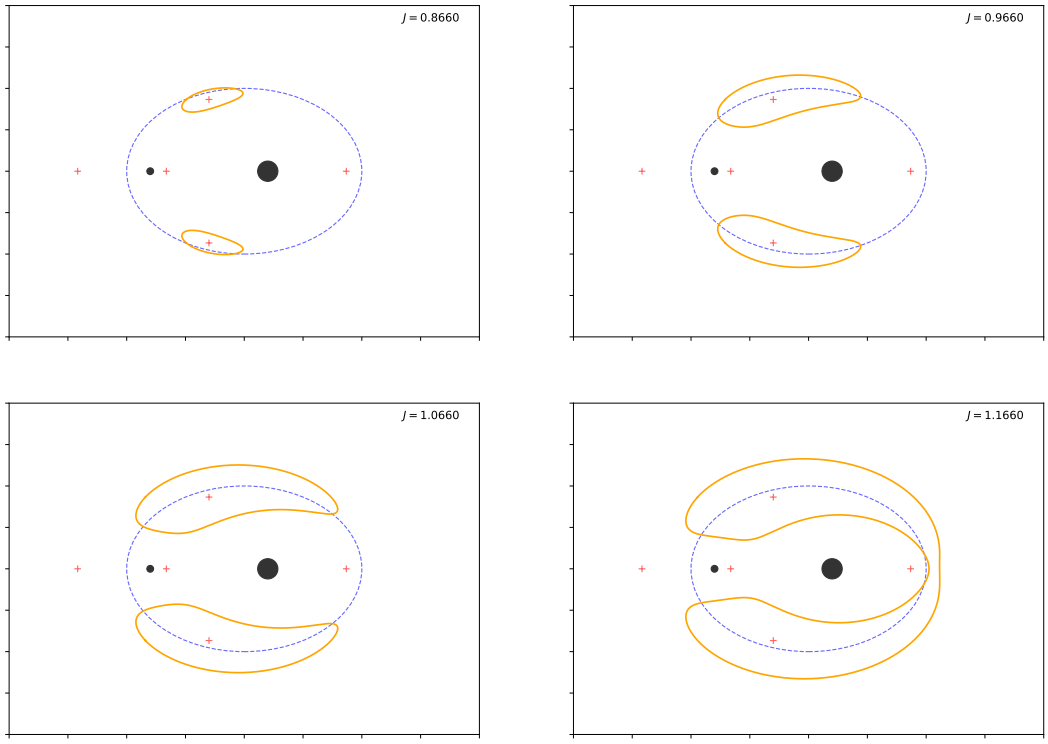

Figure 2.3: Zero velocity curves for $m = 0.2$ and $\alpha = -1.5$ for different values of the Jacobi constant J

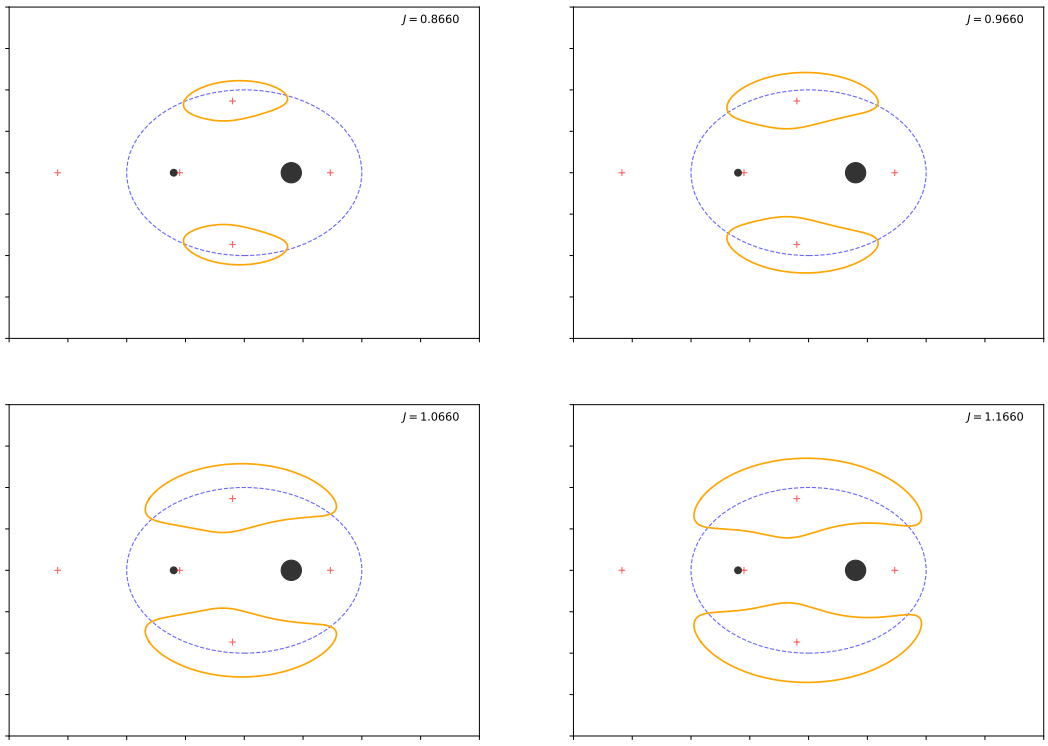

Figure 2.4: Zero velocity curves for $m = 0.4$ and $\alpha = -1.5$ for different values of the Jacobi constant J

Chapter 3

Regularisation of the extended, planar, circular, restricted three-body problem

As seen in chapter 1, for $\alpha < -1$, the system of differential equations corresponding to the restricted three-body problem presents singularities at the positions of the primary and the secondary body. Therefore, problems arise when trying to compute the motion of the third body near the two bodies.

However, such singularities do not present an essential character, an hence, can be eliminated with a proper transformation of time and space coordinates.

Eliminating the singularities by means of an adequate transformation has the following beneficial effects in the treatment of the problem

1. There exists solutions for any selection of the initial conditions.

2. Solutions approaching the singularities (and going through them, although this is impossible in the physical sense) may be studied analytically.

The main goal of this chapter is to find such a transformation that will allow us to regularise the extended problem. To reach this objective, we will begin by studying the case of the two-body problem and then proceed to regularise the equations of motion at each of the singularities separately.

3.1 Regularization in the problem of two bodies

Consider Ω as defined in 1.27. We will begin by studying a restricted case of the problem, where $m = 0$ and $1 - m = 1$, which will provide some insight that will be of use when developing the general theory. In this case, the corresponding function ω becomes

$$\omega = \frac{1}{2}r^2 - \frac{1}{\alpha+1}(r^{\alpha+1} - 1), \tag{3.1}$$

where $r^2 = r_1^2 = x^2 + y^2$. The equations of motion thus become,

$$\begin{aligned} \ddot{x} - 2\dot{y} &= x(1 - r^{\alpha-1}) \\ \ddot{y} - 2\dot{x} &= y(1 - r^{\alpha-1}), \end{aligned} \tag{3.2}$$

and the Jacobian

$$\dot{x}^2 + \dot{y}^2 = r^2 - \frac{2}{\alpha+1}(r^{\alpha+1} - 1) + C. \tag{3.3}$$

Observe that this situation correspond to a simplified restricted problem of three bodies, in which the mass of the secondary body is zero. However, this is in a synodical system, and therefore the equations become somewhat complicated. The equations corresponding to a fixed system are

$$\begin{aligned} \ddot{\xi} &= -\xi\rho^{\alpha-1} \\ \ddot{\eta} &= -\eta\rho^{\alpha-1}, \end{aligned} \tag{3.4}$$

where $\rho^2 = r^2 = x^2 + y^2 = \xi^2 + \eta^2$. The Jacobi integral

$$\dot{\xi}^2 + \dot{\eta}^2 = -\frac{2}{\alpha+1}(\rho^{\alpha+1} - 1) - C. \tag{3.5}$$

Let us consider a specific situation: at $t_0 = 0$, consider $\xi = \xi_0$, $\dot{\xi} = 0$, $\eta = 0$, $\dot{\eta} = 0$. In this case, $\rho = |\xi|^2$, and therefore, equation 3.5 becomes

$$\dot{\xi}^2 = -\frac{2}{\alpha+1}(|\xi|^{\alpha+1} - 1) - C, \tag{3.6}$$

and thus,

$$\dot{\xi} = \left(-\frac{2}{\alpha+1}(|\xi|^{\alpha+1} - 1) - C\right)^{\frac{1}{2}}. \tag{3.7}$$

For the sake of the discussion, assume $\alpha < -1$. In this case, as the third body approaches the origin (i.e. $\xi \to 0$), velocity increases in absolute value ($\dot{\xi} \to \infty$). Hence it is clear that there exists a singularity at the origin. We shall now eliminate this singularity with an adequate transformation, which shall also be valid for the case $\alpha > -1$, although in this case there is no such singularity.

In order to eliminate this singularity, new independent and dependent variables must be introduced, such that

$$\xi = f(u), \tag{3.8}$$

and

$$\tau = \int_{t_0}^{t} \frac{\mathrm{d}t}{g(u)}, \tag{3.9}$$

which may also be written as

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = g(u). \tag{3.10}$$

The goal now is to find explicit expressions for $f(u)$ and $g(u)$. Once these are found, equation 3.8 gives u as a function of t, since $\xi = \xi(t)$, and equation 3.9 gives the relation between t and τ.

Let $u' = \frac{\mathrm{d}u}{\mathrm{d}\tau}$ and $f' = \frac{\mathrm{d}f}{\mathrm{d}u}$. Since $\xi = f(u)$, $\dot{\xi} = \frac{\mathrm{d}\xi}{\mathrm{d}t}$ may be written as

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = \frac{\mathrm{d}f}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t}, \tag{3.11}$$

or

$$\dot{\xi} = u'\frac{f'}{g}. \tag{3.12}$$

The new velocity in this system is therefore

$$u' = \frac{g}{f'}\dot{\xi}, \tag{3.13}$$

and hence, in order to have a finite value of this new velocity at the origin, it must be that $\frac{g}{f'} \to 0$ as $\dot{\xi} \to \infty$. Assuming that $\xi > 0$, we may rewrite the Jacobi integral 3.6 as

$$\dot{\xi}^2 = -\frac{2}{\alpha+1}(\xi^{\alpha+1} - 1) - C = 2U \tag{3.14}$$

In terms of u, f and g, from 3.13 and 3.14:

$$(u')^2 = \frac{g^2}{(f')^2}\left(-\frac{2}{\alpha+1}(f^{\alpha+1} - 1) - C\right) = \frac{g^2}{(f')^2}\left(-\frac{2}{\alpha+1}f^{\alpha+1} - C'\right) = \frac{g^2}{(f')^2}2U. \tag{3.15}$$

Since $U \to \infty$ as $\xi \to 0$ and since u' must be finite at collision, it must be the case that $\frac{g^2}{(f')^2}U$ is finite as $\xi \to 0$.

Since close to the collision $2U = -\frac{2}{\alpha+1}\xi^{\alpha+1} = -\frac{2}{\alpha+1}f^{\alpha+1}$, the requirement for finite velocity u in the system (u, τ) is that

$$\frac{g^2}{(f')^2}f^{\alpha+1} \tag{3.16}$$

is finite as ξ approaches 0, or also that

$$\frac{g}{f'}f^{\frac{\alpha+1}{2}} \tag{3.17}$$

is finite as f tends to 0.

Consider now $\frac{g}{f'}$ expressed as a power series in $f^{-\frac{\alpha+1}{2}}$,

$$\frac{g}{f'} = A_0 + A_1 f^{-\frac{\alpha+1}{2}} + A_2 f^{\alpha+1} + A_3 f^{-\frac{3(\alpha+1)}{2}} + \dots. \tag{3.18}$$

Therefore,

$$\frac{g}{f'}f^{\frac{\alpha+1}{2}} = A_0 f^{\frac{\alpha+1}{2}} + A_1 + A_2 f^{-\frac{\alpha+1}{2}} + A_3 f^{-(\alpha+1)} + \dots. \tag{3.19}$$

Since $\frac{g}{f'}f^{\frac{\alpha+1}{2}}$ must be finite as $f \to 0$, it implies that $A_0 = 0$. Consequently,

$$\frac{g}{f'}f^{\frac{\alpha+1}{2}} = A_1 + A_2 f^{-\frac{\alpha+1}{2}} + A_3 f^{-(\alpha+1)} + \dots \tag{3.20}$$

and thus, as $f \to 0$, $\frac{g}{f'} f^{\frac{\alpha+1}{2}} \to A_1$. Now, from equation 3.13 and considering that near the singularity $C' = 0$, we find that, near the singularity,

$$(u')^2 = -\frac{2}{\alpha+1}\frac{g}{f'}f^{\alpha+1}, \tag{3.21}$$

and so,

$$u' = \frac{g}{f'}f^{\frac{\alpha+1}{2}}\left(-\frac{2}{\alpha+1}\right)^{\frac{1}{2}} = A_1\left(-\frac{2}{\alpha+1}\right)^{\frac{1}{2}}. \tag{3.22}$$

For instance, assume we choose $\xi = f(u) = u^n$. In that case, we have $u' = nu^{n-1} = A_1\left(-\frac{2}{\alpha+1}\right)^{\frac{1}{2}}$, and therefore,

$$g = A_1 f' f^{-\frac{\alpha+1}{2}} = A_1 n u^{\frac{n(1-\alpha)}{2}-1}. \tag{3.23}$$

3.1.1 The equations of motion

We shall now analyse how the equations of motion in regard to singularities. Deriving equation 3.12 we obtain

$$\ddot{\xi} = f'u'\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{g}\right) + (f'u'' + f''u'^2)\frac{1}{g^2}, \tag{3.24}$$

and thus,

$$\ddot{\xi} = -(u')^2\frac{f'g'}{g^3} + \frac{f'u'' + f''(u')^2}{g^2} = \frac{f'u''}{g^2} + (u')^2\left(\frac{f''}{g^2} - \frac{f'g'}{g^3}\right) \tag{3.25}$$

Since we are considering that $\rho = |\xi|$, considering $\xi > 0$ and from equations 3.4 we find that $\ddot{\xi} = -\xi^\alpha$, and hence,

$$-f^\alpha = \frac{f'u''}{g^2} + (u')^2\left(\frac{f''}{g^2} - \frac{f'g'}{g^3}\right). \tag{3.26}$$

We may rewrite equation 3.26 in terms of $\frac{\mathrm{d}U}{\mathrm{d}u} = -f^\alpha f'$ as

$$\frac{\mathrm{d}U}{\mathrm{d}u}\frac{1}{f'} = \frac{f'u''}{g^2} + (u')^2\left(\frac{f''}{g^2} - \frac{f'g'}{g^3}\right), \tag{3.27}$$

or equivalently,

$$\frac{g^2}{(f')^2}\frac{\mathrm{d}U}{\mathrm{d}u} = u'' + (u')^2\frac{gf'' - f'g'}{f'g}. \tag{3.28}$$

Observe now that

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{g^2}{(f')^2}U\right) = \frac{g^2}{(f')^2}\frac{\mathrm{d}U}{\mathrm{d}u} + \frac{(u')^2}{f'g}(g'f' - gf''), \tag{3.29}$$

which substituted into equation 3.28 gives us the expression

$$u'' = \frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{g^2}{(f')^2}U\right). \tag{3.30}$$

3.1.2 An example of functions f and g

Let us now provide an example of these functions f and g building upon the example presented previously for $f(u) = u^n$. As we will see shortly, it would be an interesting property that both, f and g to be functions of the same power of u. Following from 3.23, that means that

$$n = \frac{n(1-\alpha)}{2} - 1, \tag{3.31}$$

which holds for $n = -\frac{2}{\alpha+1}$.

Therefore, let

$$f(u) = u^{-\frac{2}{\alpha+1}}, \tag{3.32}$$

and as stated before and following from 3.23,

$$g(u) = Bu^{-\frac{2}{\alpha+1}}, \tag{3.33}$$

where B is an arbitrary constant.

As seen in equation 3.15,

$$\begin{aligned}(u')^2 &= \frac{g^2}{(f')^2}\left(-\frac{2}{\alpha+1}f^{\alpha+1} - C'\right) \\ &= \frac{B^2u^{-\frac{4}{\alpha+1}}}{\left(-\frac{2}{\alpha+1}\right)^2 u^{-\frac{4}{\alpha+1}-2}}\left(-\frac{2}{\alpha+1}u^{-2} - C'\right) \\ &= B^2\left(-\frac{1}{\frac{2}{\alpha+1}} - \frac{C'u^2}{\frac{4}{(\alpha+1)^2}}\right)\end{aligned} \tag{3.34}$$

Since B is an arbitrary constant, we may choose $B = \frac{2}{\alpha+1}$, and then, the previous equation simplify to

$$(u')^2 = -\frac{2}{\alpha+1} - C'u^2, \tag{3.35}$$

or

$$u' = \left(-\frac{2}{\alpha+1} - C'u^2\right)^{\frac{1}{2}}. \tag{3.36}$$

The equation of motion can be found by differentiating equation 3.36:

$$u'' = -\left(-\frac{2}{\alpha+1} - C'u^2\right)^{-\frac{1}{2}} C'uu', \tag{3.37}$$

and since $u' = \left(-\frac{2}{\alpha+1} - C'u^2\right)^{\frac{1}{2}}$, it simplifies to

$$u'' + C'u = 0. \tag{3.38}$$

Solving the differential equation 3.36 or 3.38, we may find the equation of motion in the regularised system of coordinates:

$$u(\tau) = \left(-\frac{2}{\alpha+1}\frac{1}{C'}\right)^{\frac{1}{2}} \sin((C')^{\frac{1}{2}}\tau), \tag{3.39}$$

where we have applied the initial conditions $\tau_0 = 0$, $u_0 = \left(-\frac{2}{\alpha+1}\frac{1}{C'}\right)^{\frac{1}{2}}$ and $u_0' = 0$.

Since we have the relation $\frac{\mathrm{d}t}{\mathrm{d}\tau} = g(u)$, we can obtain the relationship between the time τ in the regularised system and the old time t,

$$t = \int_0^\tau g(\tau)\mathrm{d}\tau = \int_0^\tau \frac{2}{\alpha+1} u(\tau)^{-\frac{2}{\alpha+1}} \mathrm{d}\tau. \tag{3.40}$$

Note that it is not possible to find an analytic expression for this integral for arbitrary values of α, and as such we will have to fallback to a numerical analysis when performing the computation of the trajectory of the third body in upcoming chapters.

3.2 Generalisation to the general two-body problem

The preceding section performed a series of simplifications to introduce the matter of regularisation, namely assume that the body was at $\eta = 0$ and thus, $\rho = |\xi|$. To study the general case, we may introduce

$$\zeta = \xi + i\eta, \tag{3.41}$$

and thus $\rho = |\zeta|$, and rewrite the equations of motion 3.4 in terms of this new variable:

$$\ddot{\zeta} = -\zeta|\zeta|^{\alpha-1}. \tag{3.42}$$

As before, the singularity is also at the origin, and as such, mathematically, it is only interesting to consider the degenerate case in which the trajectory degenerates into a straight line going through the collision. Note however, that when performing numerical simulations, for trajectories sufficiently close to the origin, it is also necessary to apply this regularisation, since accuracy problems arise.

The corresponding generalisation in the general case is given again by two functions f and g, such that

$$\zeta = f(\omega) \tag{3.43}$$

and

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = g(\omega), \tag{3.44}$$

where ω is now a complex variable, $\omega = u + iv$, and $g(\omega)$ is a real function of the complex variable ω, so that the new time τ is also a real variable.

The Jacobi integral becomes

$$|\dot{\zeta}|^2 = -\frac{2}{\alpha+1}(|\zeta|^{\alpha+1} - 1) - C = -\frac{2}{\alpha+1}|\zeta|^{\alpha+1} - C' = 2U, \tag{3.45}$$

also in accordance with the one dimensional case. The computation of ζ also follows the pattern established previously:

$$\dot{\zeta} = \frac{\mathrm{d}\zeta}{\mathrm{d}\omega}\frac{\mathrm{d}\omega}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t}, \tag{3.46}$$

or in terms of the functions f and g,

$$|\dot{\zeta}| = \frac{|f'(\omega)|^2|\omega'|^2}{g^2}. \tag{3.47}$$

Combining this with the energy integral equation, we may rewrite the energy integral only in terms of the variables in this new system of coordinates:

$$|\omega'|^2 = \frac{g^2}{|f'|^2}\left(-\frac{2}{\alpha+1}(|f|^{\alpha+1}-1)-C\right) = \frac{g^2}{|f'|^2}\left(-\frac{2}{\alpha+1}|f|^{\alpha+1}-C'\right), \tag{3.48}$$

where

$$|\omega'|^2 = \left(\frac{\mathrm{d}u}{\mathrm{d}\tau}\right)^2 + \left(\frac{\mathrm{d}v}{\mathrm{d}\tau}\right)^2 \tag{3.49}$$

is the square of the new velocity.

Observe that we may rewrite equation 3.42 as

$$\ddot{\zeta} = \mathrm{grad}_\zeta -\frac{|\zeta|^{\alpha+1}}{\alpha+1}, \tag{3.50}$$

where the gradient of a complex function $\mathrm{grad}_\zeta F(\zeta)$ is defined as

$$\mathrm{grad}\, F(\zeta) = \frac{\partial F}{\partial \xi} + i\frac{\partial F}{\partial \eta}. \tag{3.51}$$

Since $U = -\frac{|\zeta|^{\alpha+1}}{\alpha+1} - \frac{C'}{2}$, we have

$$\ddot{\zeta} = \mathrm{grad}_\zeta U. \tag{3.52}$$

Let

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \dot{\tau}, \quad \frac{\mathrm{d}f}{\mathrm{d}\omega} = f'(\omega), \quad \frac{\mathrm{d}\omega}{\mathrm{d}\tau} = \omega'(\tau), \tag{3.53}$$

such that we can rewrite equation 3.46 as

$$\dot{\zeta} = f'\omega'\dot{\tau}. \tag{3.54}$$

Its second derivative becomes

$$\ddot{\zeta} = f'\omega'\ddot{\tau} + (f''(\omega')^2 + f'\omega'')\dot{\tau}^2. \tag{3.55}$$

Let us transform the gradient operator of U in terms of the gradient of ω. Observe first of all that

$$\mathrm{grad}_\omega U(\omega) = U_u + iU_v. \tag{3.56}$$

Since $\omega = u + iv$, we have

$$\operatorname{grad}_\omega U = U_\xi \xi_u + U_\eta \eta_u + i(U_\xi \xi_v + U_\eta \eta_v). \tag{3.57}$$

Taking into account the Cauchy-Riemman equations, we find that

$$U_\xi \xi_v + U_\eta \eta_v = -U_\xi \eta_u + U_\eta \xi_u, \tag{3.58}$$

and thus,

$$\operatorname{grad}_\omega U = (\xi_u - i\eta_u)(U_\xi + iU_\eta). \tag{3.59}$$

Therefore, we arrive at

$$\operatorname{grad}_\omega U = \bar{f}' \operatorname{grad}_\zeta U. \tag{3.60}$$

Taking into account equations 3.55 and 3.60, equation 3.52 can be transformed into the following expression:

$$\omega'' + \omega' \frac{\ddot{\tau}}{\dot{\tau}^2} + (\omega')^2 \frac{f''}{f'} = \frac{\operatorname{grad}_\omega U}{\dot{\tau}^2 |f'|^2} \tag{3.61}$$

Taking into account equation 3.44, we have that $\dot{\tau} = \frac{1}{g}$, and therefore,

$$\ddot{\tau} = -\frac{1}{g^2}\dot{g} = -\dot{\tau}^2 \dot{g}, \tag{3.62}$$

and hence,

$$\frac{\ddot{\tau}}{\dot{\tau}^2} = -\dot{g}. \tag{3.63}$$

Since $g(\omega)$ is a real function over a complex variable, we may rewrite it in terms of a new differentiable complex function $h(\omega)$ as $g(\omega) = h(\omega)\bar{h}(\omega) = |h|^2$. Therefore,

$$\frac{\mathrm{d}g}{\mathrm{d}t} = \left(h \frac{\mathrm{d}\bar{h}}{\mathrm{d}\omega} \frac{\mathrm{d}\bar{\omega}}{\mathrm{d}\tau} + \bar{h} \frac{\mathrm{d}h}{\mathrm{d}\omega} \frac{\mathrm{d}\omega}{\mathrm{d}\tau} \right) \dot{\tau}. \tag{3.64}$$

Let $\frac{\mathrm{d}h}{\mathrm{d}\omega} = h'$. Since

$$(\bar{h})' = \frac{\mathrm{d}\bar{h}}{\mathrm{d}\omega} = \frac{\mathrm{d}\bar{h}}{\omega} = \bar{h}', \tag{3.65}$$

hence,

$$\frac{\mathrm{d}\bar{h}}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}\omega} \frac{\mathrm{d}\omega}{\mathrm{d}\tau} \frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{\mathrm{d}\bar{h}}{\mathrm{d}\omega} \frac{\mathrm{d}\bar{\omega}}{\mathrm{d}\tau} \dot{\tau} = \frac{\bar{h}'\bar{\omega}'}{\bar{h}h}. \tag{3.66}$$

Consequently,

$$\frac{\ddot{\tau}}{\dot{\tau}^2} = -\dot{g} = -\left(\frac{\bar{h}'\bar{\omega}'}{h} + \frac{h'\omega'}{h} \right), \tag{3.67}$$

and

$$\omega'' - \frac{|\omega'|^2}{\bar{h}} \frac{\mathrm{d}\bar{h}}{\mathrm{d}\omega} + (\omega')^2 \left(\frac{f''}{f'} - \frac{h'}{h} \right) = \frac{|h|^4}{|f'|^2} \operatorname{grad}_\omega U. \tag{3.68}$$

Since

$$|\omega'|^2 = \frac{g^2}{|f'|^2} 2U, \tag{3.69}$$

and thus the equation of motion becomes

$$\omega'' + (\omega')^2 \frac{\mathrm{d}}{\mathrm{d}\omega} \left(\log \frac{f'}{h} \right) = \frac{|h|^4}{|f'|^2} \left(2U \frac{\mathrm{d}\,\bar{\log}\, h}{\mathrm{d}\omega} + \operatorname{grad}_\omega U \right). \tag{3.70}$$

Taking into account the following properties of the gradient

1. $\operatorname{grad}_\omega g_1(\omega) g_2(\omega) = g_1 \operatorname{grad}_\omega g_2 + g_2 \operatorname{grad}_\omega g_1$, and that
2. $\operatorname{grad}_\omega |G(\omega)|^2 = 2G \frac{\mathrm{d}\bar{G}}{\mathrm{d}\omega}$,

where g_1 and g_2 are real functions over the complex variable ω and G is a complex function over the complex variable ω, we may rewrite equation 3.70 as

$$\omega'' = \operatorname{grad}_\omega \left| \frac{h^2}{f'} \right|^2 U - 2i\omega' \operatorname{Im} \left(\omega' \frac{\mathrm{d}}{\mathrm{d}\omega} \log \frac{f'}{h} \right). \tag{3.71}$$

Note that if $f' = h$, the nonlinear part of the equation 3.71 becomes null, and therefore, the equation of motion simplifies to

$$\omega'' = \operatorname{grad}_\omega |f'|^2 U, \tag{3.72}$$

and $g = |f'|^2$, and whence,

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = |f'|^2. \tag{3.73}$$

3.3 Local regularisation of the three-body problem

This section will deal with the general formulation of the regularisation of the three-body problem. We introduce again the two transformations similarly to the introduced in the preceding section,

$$z = f(\omega) \tag{3.74}$$

and

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = g(\omega) = |h(\omega)|^2. \tag{3.75}$$

Note that now, we do not deal with transformation in a fixed system of coordinates, but we with the nondimensional synodical one, thus the selection of the variable $z(\omega)$ instead of $\zeta(\omega)$.

The equation of motion in complex form of the restricted problem is simply

$$\ddot{z} + 2i\dot{z} = \mathrm{grad}_z\, U, \tag{3.76}$$

which after applying the transformation similarly as in the preceding section, transforms into

$$\omega'' + 2ig(\omega)\omega' = \mathrm{grad}_z \left|\frac{h^2}{f'}\right|^2 U - 2i\omega' \,\mathrm{Im}\left(\omega' \frac{\mathrm{d}}{\mathrm{d}\omega} \log \frac{f'}{h}\right), \tag{3.77}$$

with

$$U = \Omega - \frac{C}{2}. \tag{3.78}$$

Once again, if $f' = h$ (or $g(\omega) = |f'|^2$), the equation of motion simplifies into

$$\omega'' + 2i|f'|^2\omega' = \mathrm{grad}_z\, |f'|^2 U. \tag{3.79}$$

The Jacobian integral is

$$|\dot{z}|^2 = 2U, \tag{3.80}$$

and in the transformed system,

$$\left|\frac{\mathrm{d}\omega}{\mathrm{d}\tau}\right|^2 = 2|f'|^2 U. \tag{3.81}$$

According to there equations, if $\frac{\mathrm{d}t}{\mathrm{d}\tau} = g(\omega) = |h(\omega)|^2 = |f'(w)|^2$, then the regularised equation is a linear equation with respect to ω.

3.4 Transformation of the Hamiltonian

Let H be the extended Hamiltonian function. The equivalent Hamiltonian $\mathcal{H}$ in the transformed coordinate system at every level $\bar{H} = -\frac{C}{2}$ is found by transforming the coordinates (x, y) into (ξ, η) and applying the time transformation $\frac{\mathrm{d}t}{\mathrm{d}\tau}$:

$$\mathcal{H} = \frac{\mathrm{d}t}{\mathrm{d}\tau}(H - \bar{H}). \tag{3.82}$$

$\bar{H}$ is treated as the conjugated momentum of variable t and the canonical transformation results.

In our case, we have found that a change of coordinates to regularise to Cartesian primarycentric coordinates of the form

$$x + iy = (\xi + i\eta)^2 = (\xi^2 - \eta^2) + i(2\xi\eta) \tag{3.83}$$

suffices. This transformation leads to the following transformations for the momenta:

$$p_x = \frac{1}{2(\xi^2 + \eta^2)}(\xi p_\xi - \eta p_\eta), \qquad p_y = \frac{1}{2(\xi^2 + \eta^2)}(\eta p_\xi + \xi p_\eta), \tag{3.84}$$

$$p_\xi = 2(\xi p_x + \eta p_y), \qquad p_\eta = 2(-\eta p_x + \xi p_y), \tag{3.85}$$

and the Hamiltonian

$$H(\xi,\eta,p_\xi,p_\eta) = \frac{1}{8(\xi^2+\eta^2)}(p_\xi^2+p_\eta^2) - \frac{1}{2}(\xi p_\eta - \eta p_\xi) - m\frac{1}{2(\xi^2+\eta^2)}(\eta p_\xi + \xi p_\eta) + \\ (1-m)\frac{1}{\alpha+1}(\xi^2+\eta^2)^{\alpha+1} + m\frac{1}{\alpha+1}((\xi^2+\eta^2)^2+2(\xi^2-\eta^2)+1)^{\frac{\alpha+1}{2}}. \tag{3.86}$$

Observe that in this case, the Hamiltonian contains two irregular terms, namely $(\xi^2+\eta^2)^{-1}$ and $(\xi^2+\eta^2)^{\alpha+1}$. The second term becomes regular for values of $\alpha > -1$.

We will regularise the equations using a time transformation of the form

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = 4(\xi^2+\eta^2)^\gamma, \tag{3.87}$$

with an adequate value of γ depending on the value of α. With this transformation, the extended Hamiltoninan in each level $\bar{H}$ becomes

$$\mathcal{H} = 4(\xi^2+\eta^2)^\gamma(H-\bar{H}). \tag{3.88}$$

We will now show the adequate transformations to regularise the equations near the primary body for any value of α. The regularisation near the secondary body is obtained in a similar manner.

3.4.1 Case $-2 < \alpha < 1$

In this case, it suffices to consider $\gamma = 1$, and thus, consider the time transformation

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = 4(\xi^2+\eta^2). \tag{3.89}$$

Applying this regularisation leads to the transformed Hamiltonian for a given level $\bar{H}$,

$$\mathcal{H}(\xi,\eta,t;p_\xi,p_\eta,\bar{H}) = \frac{1}{2}(\xi^2+\eta^2) - 4(\xi^2+\eta^2)(\bar{H}+\frac{1}{2}(\xi p_\eta - \eta p_\xi)) - 2m((\eta p_\xi + \xi p_\eta)) \\ + 4(1-m)\frac{1}{\alpha+1}(\xi^2+\eta^2)^{\alpha+2} \\ + 4m\frac{1}{\alpha+1}(\xi^2+\eta^2)(((\xi^2+\eta^2)^2+2(\xi^2-\eta^2)+1)^{\frac{\alpha+1}{2}} \tag{3.90}$$

The regularised Hamiltonian equations with respect to the regularised time τ have the

following expressions:

$$
\begin{aligned}
\frac{\mathrm{d}\xi}{\mathrm{d}\tau} &= p_\xi + 2(\xi^2+\eta^2-m)\eta, \\
\frac{\mathrm{d}\eta}{\mathrm{d}\tau} &= p_\eta - 2(\xi^2+\eta^2+m)\xi, \\
\frac{\mathrm{d}p_\xi}{\mathrm{d}\tau} &= 8\xi(\bar{H}+\frac{1}{2}(\xi p_\eta - \eta p_\xi)) + 2(\xi^2+\eta^2+m)p_\eta - 8(1-m)\frac{\alpha+2}{\alpha+1}\xi(\xi^2+\eta^2)^{\alpha+1} \\
&\quad + 8m\xi(\frac{1}{\alpha+1}(r_2^2)^{\frac{\alpha+1}{2}} + (\xi^2+\eta^2)(\xi^2+\eta^2-1)(r_2^2)^{\frac{\alpha-1}{2}}), \\
\frac{\mathrm{d}p_\eta}{\mathrm{d}\tau} &= 8\eta(\bar{H}+\frac{1}{2}(\xi p_\eta - \eta p_\xi)) + 2(\xi^2+\eta^2-m)p_\xi - 8(1-m)\frac{\alpha+2}{\alpha+1}\eta(\xi^2+\eta^2)^{\alpha+1} \\
&\quad + 8m\eta(\frac{1}{\alpha+1}(r_2^2)^{\frac{\alpha+1}{2}} + (\xi^2+\eta^2)(\xi^2+\eta^2-1)(r_2^2)^{\frac{\alpha-1}{2}}),
\end{aligned}
\tag{3.91}
$$

where $r_2^2 = (\xi^2+\eta^2)^2 + 2(\xi^2-\eta^2)+1$.

3.4.2 Case $-3 < \alpha < -2$

In this case, taking $\gamma = 1$ is not enough to perform the regularisation, and thus we need a stronger transformation, namely, $\gamma = -\alpha$. Therefore,

$$
\frac{\mathrm{d}t}{\mathrm{d}\tau} = 4(\xi^2+\eta^2)^{-\alpha}. \tag{3.92}
$$

Applying this regularisation leads to the transformed Hamiltonian for a given level $\bar{H}$,

$$
\begin{aligned}
\mathcal{H}(\xi,\eta,t;p_\xi,p_\eta,\bar{H}) &= \frac{1}{2}(\xi^2+\eta^2)^{-(\alpha+1)}(p_\xi^2+p_\eta^2) - 4(\xi^2+\eta^2)^{-\alpha}(\bar{H}+\frac{1}{2}(\xi p_\eta - \eta p_\xi)) \\
&\quad - 2m(\xi^2+\eta^2)^{-(\alpha+1)}((\eta p_\xi + \xi p_\eta)) + 4(1-m)\frac{1}{\alpha+1}(\xi^2+\eta^2) \\
&\quad + 4m\frac{1}{\alpha+1}(\xi^2+\eta^2)^{-\alpha}((\xi^2+\eta^2)^2+2(\xi^2-\eta^2)+1)^{\frac{\alpha+1}{2}}
\end{aligned}
\tag{3.93}
$$

The regularised Hamiltonian equations with respect to the regularised time τ have the

following expressions:

$$
\begin{aligned}
\frac{\mathrm{d}\xi}{\mathrm{d}\tau} &= (\xi^2+\eta^2)^{-(\alpha+1)}(p_\xi + 2(\xi^2+\eta^2-m)\eta),\\
\frac{\mathrm{d}\eta}{\mathrm{d}\tau} &= (\xi^2+\eta^2)^{-(\alpha+1)}(p_\eta - 2(\xi^2+\eta^2+m)\xi),\\
\frac{\mathrm{d}p_\xi}{\mathrm{d}\tau} &= (\alpha+1)(\xi^2+\eta^2)^{-(\alpha+2)}\xi(p_\xi^2+p_\eta^2)\\
&\quad - 4\alpha(\xi^2+\eta^2)^{-(\alpha+1)}\xi(\bar{H}+\frac{1}{2}(\xi p_\eta - \eta p_\xi)) + 2(\xi^2+\eta^2)^{-\alpha}p_\eta\\
&\quad - 4m(\alpha+1)(\xi^2+\eta^2)^{-(\alpha+2)}\xi(\eta p_\xi + \xi p_\eta) + 2m(\xi^2+\eta^2)^{-(\alpha+1)}p_\eta + 8m\frac{1}{\alpha+1}\xi\\
&\quad + 8m(\xi^2+\eta^2)^{-(\alpha+1)}\xi(\frac{\alpha}{\alpha+1}(r_2^2)^{\frac{\alpha+1}{2}} + (\xi^2+\eta^2)(\xi^2+\eta^2-1)(r_2^2)^{\frac{\alpha-1}{2}}),\\
\frac{\mathrm{d}p_\eta}{\mathrm{d}\tau} &= (\alpha+1)(\xi^2+\eta^2)^{-(\alpha+2)}\eta(p_\xi^2+p_\eta^2)\\
&\quad - 4\alpha(\xi^2+\eta^2)^{-(\alpha+1)}\eta(\bar{H}+\frac{1}{2}(\xi p_\eta - \eta p_\xi)) - 2(\xi^2+\eta^2+m)p_\xi\\
&\quad - 4m(\alpha+1)(\xi^2+\eta^2)^{-(\alpha+2)}\eta(\eta p_\xi + \xi p_\eta) + 2m(\xi^2+\eta^2)^{-(\alpha+1)}p_\xi + 8m\frac{1}{\alpha+1}\eta\\
&\quad + 8m(\xi^2+\eta^2)^{-(\alpha+1)}\eta(\frac{\alpha}{\alpha+1}(r_2^2)^{\frac{\alpha+1}{2}} + (\xi^2+\eta^2)(\xi^2+\eta^2-1)(r_2^2)^{\frac{\alpha-1}{2}}),
\end{aligned}
\tag{3.94}
$$

where $r_2^2 = (\xi^2+\eta^2)^2 + 2(\xi^2-\eta^2)+1$.

3.4.3 Case $\alpha < -3$

Finally, for $\alpha < -3$, we have to strengthen yet another time our regularisation. In this case, taking $\gamma = -(\alpha+1)$. Hence,

$$
\frac{\mathrm{d}t}{\mathrm{d}\tau} = 4(\xi^2+\eta^2)^{-(\alpha+1)}. \tag{3.95}
$$

Applying this regularisation leads to the transformed Hamiltonian for a given level $\bar{H}$,

$$
\begin{aligned}
\mathcal{H}(\xi,\eta,t;p_\xi,p_\eta,\bar{H}) &= \frac{1}{2}(\xi^2+\eta^2)^{-(\alpha+2)}(p_\xi^2+p_\eta^2) - 4(\xi^2+\eta^2)^{-(\alpha+1)}(\bar{H}+\frac{1}{2}(\xi p_\eta - \eta p_\xi))\\
&\quad - 2m(\xi^2+\eta^2)^{-(\alpha+2)}((\eta p_\xi + \xi p_\eta)) + 4(1-m)\frac{1}{\alpha+1}\\
&\quad + 4m\frac{1}{\alpha+1}(\xi^2+\eta^2)^{-(\alpha+1)}((\xi^2+\eta^2)^2 + 2(\xi^2-\eta^2)+1)^{\frac{\alpha+1}{2}}
\end{aligned}
\tag{3.96}
$$

The regularised Hamiltonian equations with respect to the regularised time τ have the

following expressions:

$$
\begin{aligned}
\frac{\mathrm{d}\xi}{\mathrm{d}\tau} &= (\xi^2+\eta^2)^{-(\alpha+2)}(p_\xi + 2(\xi^2+\eta^2-m)\eta),\\
\frac{\mathrm{d}\eta}{\mathrm{d}\tau} &= (\xi^2+\eta^2)^{-(\alpha+2)}(p_\eta - 2(\xi^2+\eta^2+m)\xi),\\
\frac{\mathrm{d}p_\xi}{\mathrm{d}\tau} &= (\alpha+2)(\xi^2+\eta^2)^{-(\alpha+3)}\xi(p_\xi^2+p_\eta^2)\\
&\quad - 4(\alpha+1)(\xi^2+\eta^2)^{-(\alpha+2)}\xi(\bar{H}+\frac{1}{2}(\xi p_\eta - \eta p_\xi)) + 2(\xi^2+\eta^2)^{-(\alpha+1)}p_\eta\\
&\quad - 4m(\alpha+2)(\xi^2+\eta^2)^{-(\alpha+3)}\xi(\eta p_\xi + \xi p_\eta) + 2m(\xi^2+\eta^2)^{-(\alpha+2)}p_\eta\\
&\quad + 8m(\xi^2+\eta^2)^{-(\alpha+2)}\xi((r_2^2)^{\frac{\alpha+1}{2}} + (\xi^2+\eta^2)(\xi^2+\eta^2-1)(r_2^2)^{\frac{\alpha-1}{2}}),\\
\frac{\mathrm{d}p_\eta}{\mathrm{d}\tau} &= (\alpha+2)(\xi^2+\eta^2)^{-(\alpha+3)}\eta(p_\xi^2+p_\eta^2)\\
&\quad - 4(\alpha+1)(\xi^2+\eta^2)^{-(\alpha+2)}\eta(\bar{H}+\frac{1}{2}(\xi p_\eta - \eta p_\xi)) - 2(\xi^2+\eta^2)^{-(\alpha+1)}p_\xi\\
&\quad - 4m(\alpha+2)(\xi^2+\eta^2)^{-(\alpha+3)}\eta(\eta p_\xi + \xi p_\eta) + 2m(\xi^2+\eta^2)^{-(\alpha+2)}p_\xi\\
&\quad + 8m(\xi^2+\eta^2)^{-(\alpha+2)}\eta((r_2^2)^{\frac{\alpha+1}{2}} + (\xi^2+\eta^2)(\xi^2+\eta^2-1)(r_2^2)^{\frac{\alpha-1}{2}}),
\end{aligned}
\tag{3.97}
$$

where $r_2^2 = (\xi^2+\eta^2)^2 + 2(\xi^2-\eta^2) + 1$.

Chapter 4

Numerical simulations

This chapter is intended to be a short summary of how our program works internally, as well as discuss some problems that arose while trying to build it and how were they resolved.

4.1 The colinear equilibrium points

Our first problem was trying to determine which of these equilibrium points exists for any given value of m and α. This was necessary since otherwise, the algorithms used for finding them would return incorrect results if an equilibrium point is not found in a given interval. Therefore, we developed section 1.5 that let us to determine exactly which equilibrium points were present for any value of m and α.

Then it was straightforward to implement a Newton algorithm that could find an equilibrium point, provided that we had found a sufficiently close approximation (this was done by a simple bisect method, comparing values of the colinear equation at a certain distance and returning if one of them was positive and the other negative). The Newton algorithm is defined as follows:

Algorithm 1 Newton's algorithm

```
Procedure Newton F, x_0 {Finds a zero of F with starting point x_0}
i = 0, x_i = 0
while i ≤ MAX_ITERATIONS do
    x = F(x_i), d = F'(x_i)
    x_i = x_i − x/d
    if fabs(x) < TOLERANCE then
        return x_i
    end if
    i++
end while
en procedure
```

4.2 Runge-Kutta methods

The Runge-Kutta methods are a family of iterative methods used to approximate the solutions of ordinary differential equations, defined as follows:

Let us consider an initial value problem defined by

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}t} &= f(t, x(t)), \\ x(0) &= x_0. \end{aligned} \tag{4.1}$$

We are interested in finding a numerical approximation of the solution $x(t)$ of the initial value problem over a certain time interval $[a, b]$. To compute this solution, let us divide the interval $[a, b]$ into N equal subintervals and select arbitrary mesh points t_i such that,

$$t_i = a + ih, \quad j = 0, 1, \ldots, N, \quad h = \frac{b-a}{N}, \tag{4.2}$$

where h is the step size.

For every integer m, there is a family of Runge-Kutta methods that compute the numerical solution of $x(t)$ according to the relation:

$$x(t_{n+1}) = x_{n+1} = x_n + h \sum_{i+1}^{m} c_i k_i, \tag{4.3}$$

where

$$\begin{aligned} k_1 &= f(t_n, x_n), \\ k_2 &= f(t_n + a_2 h, x_n + h b_{21} k_1(t_n, x_n)), \\ k_3 &= f(t_n + a_3 h, x_n + h(b_{31} k_1 + b_{32} k_2(t_n, x_n)), \\ &\vdots \\ k_m &= f(t_n + a_m h, x_n + h \sum_{j_1} m - 1 b_{mj} k_j). \end{aligned} \tag{4.4}$$

A particular method is defined by fixing the number of steps m, as well as the coefficients a_i $(i = 2, 3, \ldots, m)$, b_{ij} $(1 \leq j < i \leq m)$, and c_i $(i = 1, 2, \ldots, m)$. These coefficients are usually displayed it what is called a *Butcher tableau*:

$$\begin{array}{c|ccccc} 0 & & & & & \\ a_2 & b_{21} & & & & \\ a_3 & b_{31} & b_{32} & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ a_m & b_{m1} & b_{m2} & \cdots & b_{m,m-1} & \\ \hline & c_1 & c_2 & \ldots & c_{m-1} & c_m \end{array} \tag{4.5}$$

4.2.1 RK4 Method

One of the most known and useful of these methods is known as the RK4 method, which represents one of the solutions corresponding to $m = 4$. In the most common case, the

coefficients of the Butcher Tableau are obtained by matching the coefficients with those of the Taylor series. In this case, the following system of equations is obtained:

$$
\begin{aligned}
1 &= c_1 + c_2 + c_3 + c_4, \\
a_2 &= b_{21}, \\
a_3 &= b_{31} + b_{32}, \\
\frac{1}{2} &= c_2 a_2 + c_3 a_3 + c_4 a_4, \\
\frac{1}{3} &= c_2 a_2^2 + c_3 a_3^2 + c_4 a_4^2, \\
\frac{1}{4} &= c_2 a_2^3 + c_3 a_3^3 + c_4 a_4^3, \\
\frac{1}{6} &= c_3 a_2 b_{32} + c_4 (a_2 b_{42} + a_3 b_{43}), \\
\frac{1}{8} &= c_3 a_2 a_3 b_{32} + c_4 a_4 (a_2 b_{42} + a_3 b_{43}), \\
\frac{1}{12} &= c_3 a_2^2 b_{32} + c_4 (a_2^2 b_{42} + a_3^2 b_{43}), \\
\frac{1}{24} &= c_4 a_2 b_{32} b_{43}
\end{aligned}
\tag{4.6}
$$

Note that there are thirteen variables and eleven equations, and thus, we need to provide two additional conditions. The most useful choices are

$$
a_2 = \frac{1}{2}, \qquad b_{31} = 0. \tag{4.7}
$$

Solving the equations yield its corresponding Butcher tableau:

$$
\begin{array}{c|cccc}
0 & & & & \\
\frac{1}{2} & \frac{1}{2} & & & \\
\frac{1}{2} & 0 & \frac{1}{2} & & \\
1 & 0 & 0 & 1 & \\
\hline
 & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6}
\end{array}
\tag{4.8}
$$

4.2.2 Adaptive Runge-Kutta methods

One way to improve the accuracy of these methods is to apply the same Runge-Kutta method twice with different values of the parameter m. For example, one with order p and

the other with order $p-1$. This is represented in a Butcher tableau of the form

$$\begin{array}{c|ccccc} 0 & & & & & \\ a_2 & b_{21} & & & & \\ a_3 & b_{31} & b_{32} & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ a_m & b_{m1} & b_{m2} & \cdots & b_{m,m-1} & \\ \hline & c_1 & c_2 & \dots & c_{m-1} & c_m \\ & c_1^\star & c_2^\star & \dots & c_{m-1}^\star & c_m^\star. \end{array} \tag{4.9}$$

We then compute

$$\begin{aligned} x_{n+1}^\star &= x_n + h\sum_{i=1}^{m} c_i^\star k_i \\ x_{n+1} &= x_n + h\sum_{i=1}^{m} c_i k_i, \end{aligned} \tag{4.10}$$

where the k_i's are defined as above, and $x_{n+1}^s tar$ is the solution given by the lower order method and x_{n+1} by the higher one.

The error estimate is then given by

$$\varepsilon = |x_{n+1} - x_{n+1}^\star| = h\sum_{i=1}^{m}(c_i - c_i^\star)k_i. \tag{4.11}$$

We may further increase the accuracy by modifying the value of the step size h at every iteration into the most optimal value of h. In this case,

$$h_{\text{opt}} = \beta h\left(\frac{\varepsilon_0}{\varepsilon}\right)^{0.2}, \tag{4.12}$$

where β is called a "safety" factor, $\beta \simeq 1$. If the error estimate given in equation 4.11 is greater than ε_0, the step is repeated with the newly found h_{opt}.

In our case, we will use a method named the Runge-Kutta-Fehlberg method of orders 7

and 8, or RKF78, whose Butcher tableau is

$$
\begin{array}{c|ccccccccccccc}
0 & & & & & & & & & & & & & \\
\frac{2}{27} & \frac{2}{27} & & & & & & & & & & & & \\
\frac{1}{9} & \frac{1}{36} & \frac{1}{12} & & & & & & & & & & & \\
\frac{1}{6} & \frac{1}{24} & 0 & \frac{1}{8} & & & & & & & & & & \\
\frac{5}{12} & \frac{5}{12} & 0 & \frac{-25}{16} & \frac{25}{16} & & & & & & & & & \\
\frac{1}{2} & \frac{1}{20} & 0 & 0 & \frac{1}{4} & \frac{1}{5} & & & & & & & & \\
\frac{5}{6} & \frac{-25}{108} & 0 & 0 & \frac{125}{108} & \frac{-65}{27} & \frac{125}{54} & & & & & & & \\
\frac{1}{6} & \frac{31}{300} & 0 & 0 & 0 & \frac{61}{225} & \frac{-2}{9} & \frac{13}{900} & & & & & & \\
\frac{2}{3} & 2 & 0 & 0 & \frac{-53}{6} & \frac{704}{45} & \frac{-107}{9} & \frac{67}{90} & 3 & & & & & \\
\frac{1}{3} & \frac{-91}{1080} & 0 & 0 & \frac{23}{108} & \frac{-976}{135} & \frac{311}{54} & \frac{-19}{60} & \frac{17}{6} & \frac{-1}{12} & & & & \\
1 & \frac{2383}{4100} & 0 & 0 & \frac{-341}{164} & \frac{4496}{1025} & \frac{-301}{82} & \frac{2133}{4100} & \frac{45}{82} & \frac{45}{164} & \frac{18}{41} & & & \\
0 & \frac{3}{205} & 0 & 0 & 0 & 0 & \frac{-6}{41} & \frac{-3}{205} & \frac{-3}{41} & \frac{3}{41} & \frac{6}{41} & 0 & & \\
1 & \frac{-1777}{4100} & 0 & 0 & \frac{-341}{164} & \frac{4496}{1025} & \frac{-289}{82} & \frac{2193}{4100} & \frac{51}{82} & \frac{33}{164} & \frac{12}{41} & 0 & 1 & \\
\hline
 & \frac{41}{840} & 0 & 0 & 0 & 0 & \frac{34}{105} & \frac{9}{35} & \frac{9}{35} & \frac{9}{280} & \frac{9}{280} & \frac{41}{840} & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0 & \frac{34}{105} & \frac{9}{35} & \frac{9}{35} & \frac{9}{280} & \frac{9}{280} & 0 & \frac{41}{840} & \frac{41}{840}
\end{array}
\tag{4.13}
$$

4.3 Zero velocity curves

Computing the zero velocity curves was pretty straightforward. Starting with the zero velocity curves equations given in 2.24, one can apply the Davidenko method to reduce the problem of finding the solution to this equation, to the integration of a system of ordinary differential equations, which can then be solved numerically by applying the Runge-Kutta method. The resulting system of equations is

$$x' = \frac{\Omega_y}{\sqrt{\Omega_x^2 + \Omega_y^2}} \tag{4.14}$$

$$y' = \frac{\Omega_x}{\sqrt{\Omega_x^2 + \Omega_y^2}}. \tag{4.15}$$

Of course, it is still necessary to find an initial point (x_0, y_0). To do so, we take into account the study of the Ω function developed in section 2.1. Since the L_4 and L_5 are both minimums or maximums of the omega function, it is clear that any zero velocity curve will surround them. Therefore, we can always find an initial point by beginning in one of these two points and move along the x axis until we find a point close enough to the zero velocity curve (i.e., a point whose velocity is small enough) applying again Newton's algorithm.

Note that since the Ω function is symmetrical with respect to the x axis, we may study only one of the two points.

Once this initial point is found, we can find the zero velocity curves by applying a Runge-Kutta method. As mentioned in the preceding section, we will apply an RKF78 adaptive Runge-Kutta method.

4.4 Third-body trajectory

While computing the third-body trajectory sufficiently away from the primary and the secondary body is straightforward, applying the Runge-Kutta method to the differential equation 1.5 with an arbitrary initial point and velocity. However, near the bodies, it is necessary to perform a regularisation to avoid the singularities, as discussed in chapter 3.

Moreover, once the regularisation is applied, for values $\alpha \lesssim -2.5$, the step h due to the adaptive nature of the method we are using, becomes too small to perform the computations in real time while near the primary and the secondary body, and therefore, if we want to represent the trajectory for such cases we must first of all compute the points and then represent the trajectory while reading these points.

We will now show some examples of trajectories in different scenarios that present interesting properties.

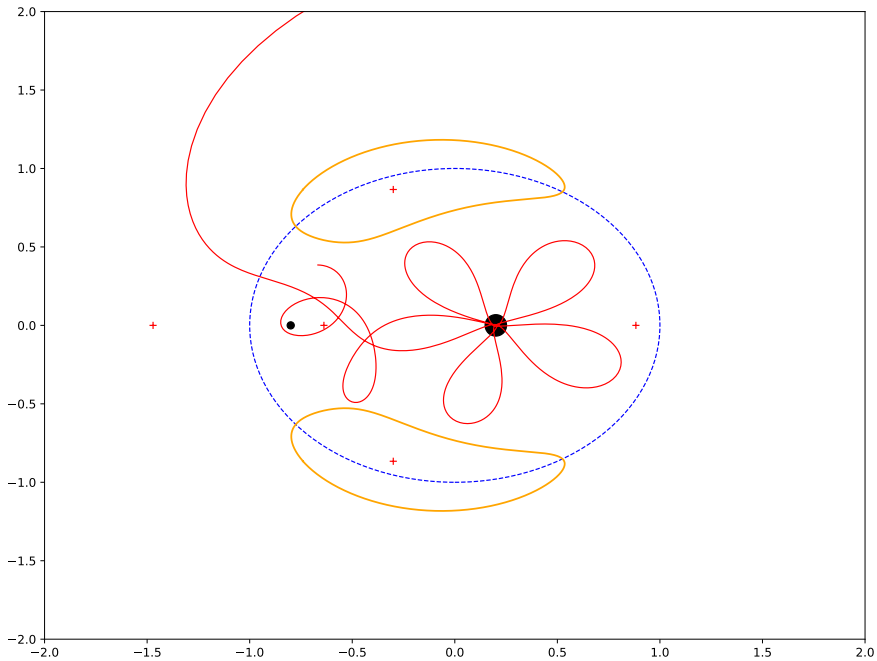

Figure 4.1: Example of a trajectory with $m = 0.2$, $\alpha = -2$ and $J = 1.01$

In this first case in figure 4.1, corresponding to the gravitational case, we can clearly the chaotic path of the third body. It begins orbiting the secondary body, switches quickly to the primary and then bounces off far away to the centre of the system.

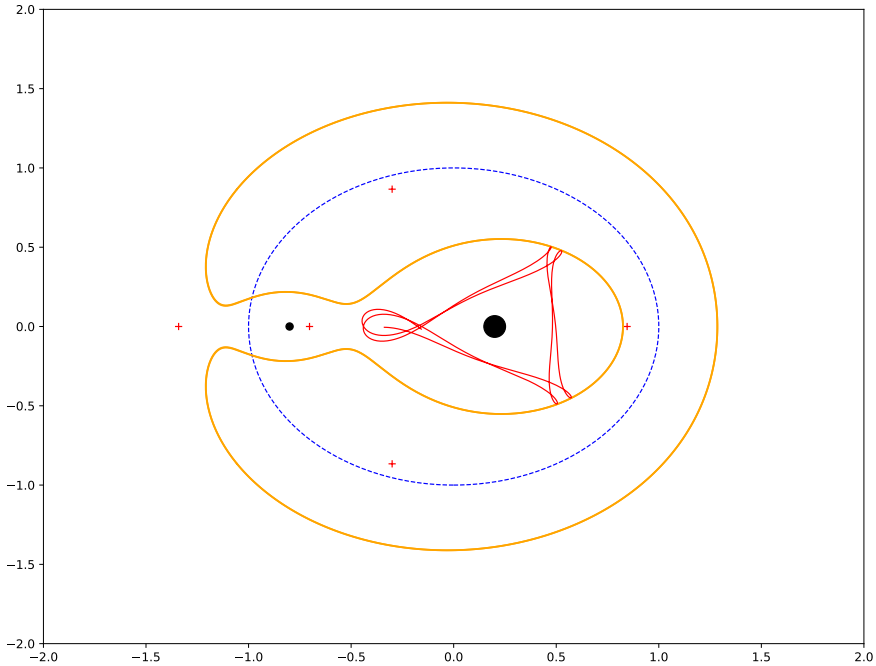

Figure 4.2: Example of a trajectory with $m = 0.2$, $\alpha = -0.9$ and $J = 1.216$

The second figure, figure 4.2, shows how the trajectory does not break through into the inside of the zero velocity curves, for it rapidly changes its trajectory once it is too close to the boundary.

Finally, figure 4.3 shows the change in the behaviour of the zero velocity curves for $\alpha > 1$, in which the inside of the region delimited by the zero velocity curves is now the region in which movement is possible.

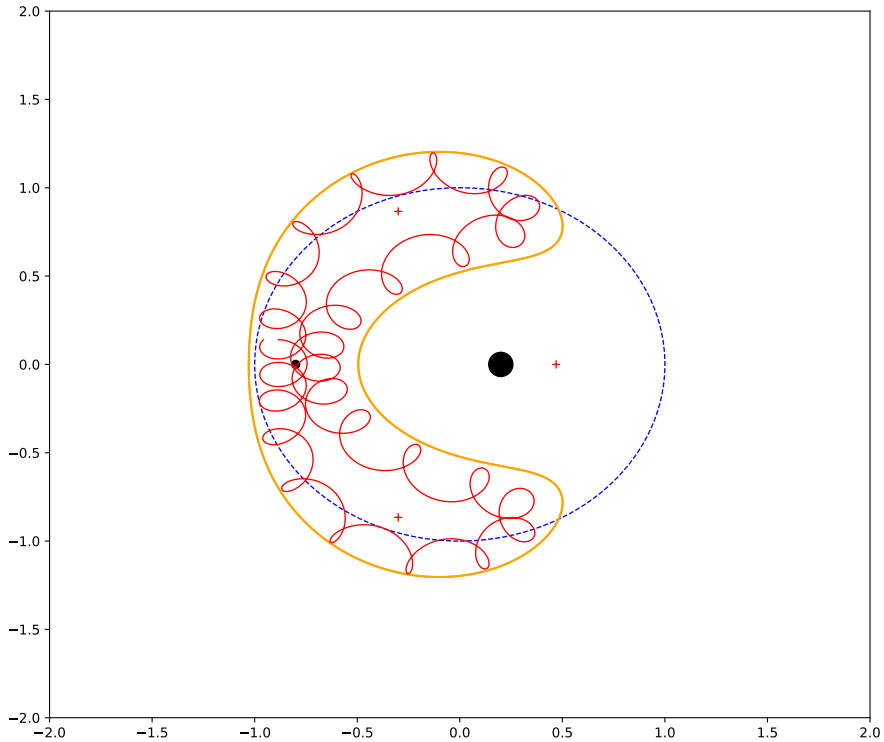

Figure 4.3: Example of a trajectory with $m = 0.2$, $\alpha = 2.7$ and $J = 0.672$

Chapter 5

Conclusion

This thesis has provided some insight into the planar, circular, extended restricted three-body problem.

We have given a precise mathematical definition of the problem, as well as generalise the main physical formulae needed for its study, such as the Hamiltonian, the potential and the Jacobi integral. We delved into the study of the colinear equilibrium points, which cannot be determined analytically, but a rigorous qualitative analysis has been performed on the existence of its solutions.

We performed a brief analysis on some properties of the Ω function. With its results, we provided some insight into how the zero velocity curves behave for different values of α, specifically for which values qualitative changes take place.

A generalisation of the regularisation method of the movement equations near the primary and the secondary bodies has been developed. A generalisation of the Levi-Civita regularisation has been applied to regularise the Hamiltonian and the equations of motion in the general case, with arbitrary values of α.

This theoretical work has been put into use to create a small computer program that can compute the trajectory of the third body with great accuracy. Some theoretical background on the numerical methods used internally has been provided, as well has some insights into the problems that arose during the development process, and how they were solved.

On a more personal note, I would like conclude by mentioning that this thesis has been a real challenge. It has allowed me to put into practice much of the knowledge I have gathered throughout my degree. I have certainly learned a lot and experienced how challenging it is to develop a work in mathematics.

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