Natural inflation with a periodic non-minimal coupling to gravity

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Abstract: Natural Inflation is a model for inflation where the potential of the inflaton is $V(\phi) = \Lambda^4[1 + \cos(\phi/f)]$ and thus exhibits a discrete shift symmetry $\phi \to \phi + 2\pi f$. Successful inflation can be achieved if $f \gtrsim$ few M_{Pl} and $\Lambda \sim m_{GUT} \sim 10^{16}$ GeV. However, the latest observational constraints put Natural Inflation in tension with data. We propose in this letter the introduction of a non-minimal coupling to gravity $\gamma^2(\phi)R$ that respects the characteristic shift symmetry and the periodicity of the potential. We show that the agreement with cosmological data can be substantially improved, obtaining a scale $\Lambda \sim 10^{15} - 10^{16}$ GeV, and furthermore, in certain cases, successful inflation can be achieved even for a periodicity scale smaller than the Planck scale.

I. INTRODUCTION

Inflation is a model of quasiexponential expansion of the Universe proposed by Alan Guth in 1981 [1] thought to last from 10^{-36} to 10^{-32} s after the initial singularity. This model gives an explanation to some cosmological conundra that the conventional Big Bang scenario could not account for: the flatness of the observable Universe, the so-called horizon problem, the abundance of cosmological relics and the mechanism to produce density perturbations that would result in the current large-scale structure of the Universe. The standard inflationary scenario relies on the presence of an scalar field at those times, the inflaton. Its origin and nature has given rise to a lot of different proposals and models. The idea shared by most of them is that the potential energy of the field would propel the Universe to a quasi-de Sitter stage of expansion, required to solve the aforementioned issues.

The flatness of the Universe can be explained by inflation since it would make unappreciable a non-zero curvature: the size of the observable Universe after inflation would be small compared to the radius of curvature, making the local patch of the Universe appear flat or almost flat. The horizon problem, which refers to the fact that regions of the cosmic microwave background (CMB) that were supposedly causally disconnected are in thermal equilibrium, can also be solved: these regions were initially in causal contact, but inflation brought them beyond the horizon; thus, when they reenter the horizon we see them in thermal equilibrium. Inflation can also explain why we had not been able to detect exotic particles, such as magnetic monopoles, predicted by Grand Unification Theories (GUTs) to have formed during the initial high temperatures (relics): the expansion would have diluted its numerical density, making it improbable to find one of such particles inside the observable Universe. Finally, the quantum fluctuations experienced by the inflaton generate density perturbations and, consequently, gravitational instabilities that give rise to the large-scale structure of the Universe.

One of the models for inflation is called Natural Inflation (NI), proposed by Freese et al. [2]. The inflaton in this case has a potential $V(\phi) = \Lambda^4[1 + \cos(\phi/f)]$. In this letter we consider a minimal extension of the original model of NI by considering the simplest non-minimal coupling in the action of the inflaton to gravity $\gamma^2(\phi)R$, where R is the Ricci scalar and γ is a function of the inflaton, which preserves the characteristic discrete shift symmetry $\phi \to \phi + 2\pi f$ of the original model and gives Einstein gravity, i.e. $\gamma^2 = 1$, at the minimum of the potential. The motivation to consider this extension is two-folded. First, and on a more practical level, the predictions of natural inflation are in tension with the latest Planck results [3]. Therefore, it is interesting to understand what kind of extensions could alleviate the tension. Second, in NI in order to have successful inflation f needs to be super-Planckian, what could be problematic. As we will show, this simple and well motivated extension comes with a set of interesting predictions which can address both issues at the same time.

II. NON-MINIMAL COUPLING TO GRAVITY

In order to consider the Natural Inflation scalar field ϕ non-minimally coupled to gravity, we start from the following action:

$$S = \int d^4x\sqrt{-g}\Big(\frac{1}{2}M_P^2\gamma^2(\phi)R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\Big) \tag{1}$$

with $V(\phi) = \Lambda^4\Big(1 + \cos(\phi/f)\Big)$, and

$$\gamma(\phi)^2 \equiv 1 + \alpha\Big(1 + \cos\frac{\phi}{f}\Big) \tag{2}$$

where R is the Ricci scalar, Λ is normally taken to be of the order of the GUT scale, as we will obtain in this letter, f is a mass scale for the inflaton potential which normally has to take trans-Planckian values to achieve successful inflation, and α is a dimensionless parameter. Notice that $\alpha > -\frac{1}{2}$ for the Planck scale $M_P\gamma$ to be well-defined, corresponding the value $\alpha = 0$ to the conventional case of Natural Inflation with a minimal coupling to gravity.

The usual way to get rid of the non-minimal coupling from the action is to go to the so-called Einstein frame by means of a conformal transformation of the metric [4]:

$$\tilde{g}_{\mu\nu} = \gamma^2 g_{\mu\nu} \tag{3}$$

We obtain the transformed integration measure $\sqrt{-\tilde{g}} = \gamma^4\sqrt{-g}$ and the inverse metric $\tilde{g}^{\mu\nu} = \gamma^{-2}g^{\mu\nu}$. Then, using the fact that the original Ricci scalar can be written as $R = \gamma^2\left[\tilde{R}+6\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\ln\gamma-6\tilde{g}^{\mu\nu}(\partial_\mu\ln\gamma)(\partial_\nu\ln\gamma)\right]$ [4], the action in terms of the quantities computed with the new metric becomes:

$$S = \int d^4x\sqrt{-\tilde{g}}\bigg(\frac{1}{2}M_P^2\tilde{R} - \frac{1}{2}\gamma^{-2}\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \\ -\frac{V(\phi)}{\gamma^4} + 3M_P^2\left(\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\ln\gamma - \tilde{g}^{\mu\nu}(\partial_\mu\ln\gamma)(\partial_\nu\ln\gamma)\right)\bigg) \tag{4}$$

The term $\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\ln\gamma = \tilde{\nabla}_\mu\tilde{\nabla}^\mu\ln\gamma = \tilde{\nabla}_\mu(\partial^\mu\ln\gamma)$ is a 4-divergence, giving only a boundary contribution that does not affect the equations of motion. Dropping this term, taking into account that $(\partial_\mu\ln\gamma)(\partial_\nu\ln\gamma) = \gamma^{-2}\partial_\mu\gamma\partial_\nu\gamma$ and considering our definition of γ^2:

$$\partial_\mu\gamma = \frac{1}{2\gamma}\frac{\partial\gamma^2}{\partial\phi}\partial_\mu\phi = -\frac{1}{2\gamma}\frac{\alpha}{f}\sin\frac{\phi}{f}\partial_\mu\phi, \tag{5}$$

we see that the conformal transformation leads to a non-canonical contribution to the kinetic term and a modification of the potential. Explicitly, the resulting action in the Einstein frame reads:

$$S = \int d^4x\sqrt{-\tilde{g}}\bigg(\frac{1}{2}M_P^2\tilde{R} \\ -\frac{1}{2}\bigg(\frac{2\gamma^2f^2 + 3M_P^2\alpha^2\sin^2\frac{\phi}{f}}{2\gamma^4f^2}\bigg)\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{V(\phi)}{\gamma^4}\bigg) \tag{6}$$

By redefining the scalar field using the transformation

$$\frac{d\chi}{d\phi} = \sqrt{\frac{2\gamma^2f^2 + 3M_P^2\alpha^2\sin^2\frac{\phi}{f}}{2\gamma^4f^2}} \tag{7}$$

we obtain the action in the Einstein frame in terms of the canonical field χ and an effective potential $U(\chi) \equiv V(\phi(\chi))/\gamma(\phi(\chi))^4$:

$$S = \int d^4x\sqrt{-\tilde{g}}\left(\frac{1}{2}M_P^2\tilde{R} - \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\chi\partial_\nu\chi - U(\chi)\right) \tag{8}$$

Now, varying the action with respect to the metric and the field gives, respectively, Einstein field equations and the Klein-Gordon equation on a curved background:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{1}{M_P^2}T_{\mu\nu} \tag{9}$$

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\chi + \frac{dU}{d\chi} = 0 \tag{10}$$

where we have omitted the tildes since we will always work in the Einstein frame with the canonical field χ from now on, and $T_{\mu\nu}$ and $R_{\mu\nu}$ are the stress-energy tensor associated to the scalar field and the Ricci tensor, respectively. These equations dictate the dynamics of spacetime and the scalar field and, by making appropriate considerations described in the next section, it is possible to achieve an inflationary scenario giving rise to the current observed properties of the universe.

III. SLOW-ROLL ANALYSIS

We will start assuming that the scalar field χ is spatially homogeneous, i.e. the spatial gradients are zero. This consideration gives a diagonal stress-energy tensor for the field which takes the form of a perfect fluid, with an energy density given by $T^{00} = \rho = \frac{1}{2}\dot{\chi}^2 + U(\chi)$ and a pressure given by $T^{ii} = p = \frac{1}{2}\dot{\chi}^2 - U(\chi)$ (a dot denotes differentiation with respect to time). Furthermore, demanding an expanding, homogeneous and isotropic universe yields the Friedmann-Robertson-Walker metric for the spacetime:

$$ds^2 = -dt^2 + a^2(t)(dr^2 + r^2d\Omega^2) \tag{11}$$

where a is the scale factor, $d\Omega^2$ is the metric on S^2 and we have also assumed flat spacelike hypersurfaces, since it is one of the fundamental consequences of inflation. We could have included a non-zero curvature, but its effects would be made negligible soon after inflation starts.

Using the FRW metric given by (11), Einstein field equations give the Friedmann equations:

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho}{3M_P^2} \tag{12}$$

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{1}{6M_P^2}(\rho + 3p) \tag{13}$$

where H is the Hubble parameter, and the Klein-Gordon equation becomes:

$$\ddot{\chi} + 3H\dot{\chi} = -\frac{dU}{d\chi} \tag{14}$$

We are interested in an accelerated expansion. The requirement $\ddot{a}/a > 0$ is equivalent to $\rho + 3p < 0$. This condition will be satisfied by the scalar field whenever the potential energy sufficiently dominates over the kinetic contribution, and it will take place if the potential of the field is very flat. In this case, the field will start rolling down the potential with a small velocity and acceleration (slow-roll) and $\rho \approx -p \approx U$. Then, the Friedmann equation and the equation of motion of the field become in the slow-roll approximation [5]:

$$3M_P^2H^2 = U \tag{15}$$

$$3H\dot{\chi} + \frac{dU}{d\chi} = 0 \quad (16)$$

In order to perform the slow-roll analysis, the slow-roll parameters ϵ and η need to be introduced [5]. They measure the slope and the curvature of the potential, respectively, and the necessary conditions for the slow-roll approximation to hold are $\epsilon \ll 1$ and $|\eta| \ll 1$. These parameters are given by:

$$\epsilon = \frac{M_P^2}{2}\frac{1}{U^2}\left(\frac{dU}{d\chi}\right)^2 \quad (17)$$

$$\eta = M_P^2 \frac{1}{U}\left(\frac{d^2U}{d\chi^2}\right) \quad (18)$$

On the other hand, slow-roll inflation will end when $\epsilon \simeq 1$. It is possible then to infer the value of the field at the end of inflation χ_{end}, as the value that fulfills the slow-roll end condition.

The number of e-foldings N experienced by the scale factor (an expansion of a factor e^N) during inflation is given by the following expression:

$$N \equiv \int_{a_0}^{a_{end}} d\ln a = -\frac{1}{M_P^2}\int_{\chi_0}^{\chi_{end}} U\left(\frac{dU}{d\chi}\right)^{-1} d\chi \quad (19)$$

where the last equality is reached making use of the equation of motion of the field in the slow-roll approximation. In order to solve the horizon problem, the observable CMB scales are considered to have exited the horizon $N \sim 60$ before the end of inflation. By imposing a desired number of e-foldings, the corresponding value for χ at that time, χ_0, can be found.

This can be used to obtain relevant results of the model that can be compared with observational data: the scalar spectral index n_s and the tensor-to-scalar ratio r. These two parameters are related to the primordial perturbations produced by inflation. The scalar spectral index informs about the variation of the amplitude of the density fluctuations when changing the scale in consideration. Bearing in mind that in cosmology length scales λ are given in terms of its corresponding wavenumber $k = 2\pi/\lambda$, the dependence of the amplitude of density fluctuations on the considered scale in terms of the spectral index n_s is written as [3]:

$$P_\zeta(k) = A^2 k^{n_s - 1} \quad (20)$$

where P_ζ is called the amplitude power spectrum and A^2 is the amplitude of the density fluctuations at a particular scale. We will use its observed value later on. One can see that the case $n_s = 1$ would correspond to a scale-invariant spectrum of density perturbations, and current observational data from Planck satellite place its value at 0.960 ± 0.007 [3]. On the other hand, the tensor-to-scalar ratio gives a comparison between the amplitude of gravitational-wave perturbations, also predicted by inflation, and the density perturbations. Taking into account that some cosmological measurements are given at a particular pivot scale k_*, Planck gives the measurement of r at the scale $k_* = 0.002$ Mpc^{-1}, denoting it by $r_{0.002}$, and its value has been constrained to be $r_{0.002} < 0.11$ at 95% CL by Planck [3]. These two observables can be written in terms of the slow-roll parameters ϵ and η as $n_s \approx 1 - 6\epsilon + 2\eta$ and $r \approx 16\epsilon$ [5]. We will evaluate them at the value χ_0 obtained by imposing a number of e-foldings N. Therefore, we will be implicitly saying that the scale $k_* = 0.002$ Mpc^{-1}, where observational data is obtained, exited the horizon N e-foldings before the end of inflation. In our results we will use $N = 55$, $N = 60$ and $N = 65$.

If we now take a look at the effective potential in the Einstein frame (Fig. 1), it presents a progressive flattening as α approaches 0.5, and when $\alpha > 0.5$ new extrema appear. Moreover, the periodicity of the potential in the Einstein frame as a function of the canonical field χ is different from the one corresponding to the original field. This issue will be addressed in the following section.

We will constrain the analysis to the interval of the potential shown in Fig. 1, but its shift symmetry allows to perform the analysis with respect to any other maximum.

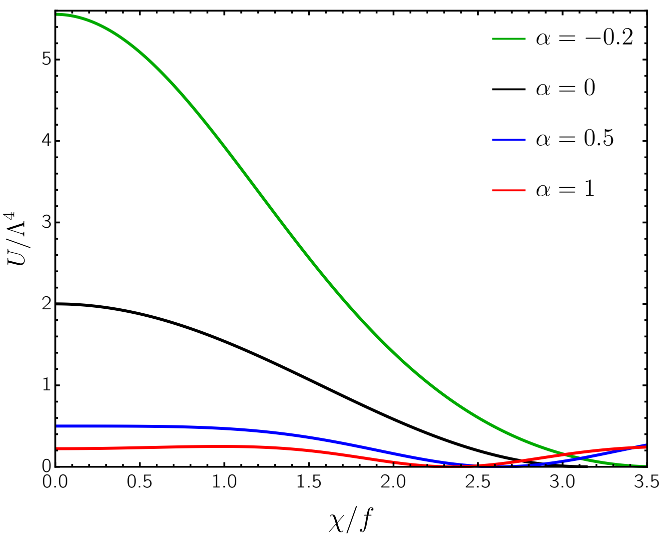

FIG. 1: Normalized effective potential U/Λ^4 in the Einstein frame as a function of χ/f. The symmetry of the potential is not shown.

IV. RESULTS

As it has been said, the potential in the Einstein frame as a function of the canonical field χ presents a different periodicity when considering different values for α. This can be interpreted as the appearance of an effective scale

$\tilde{f}$, giving a periodicity $2\pi\tilde{f}$ different from $2\pi f$, the periodicity before the frame transformation. We will refer all the results to this new scale from now on. If we define as L the increment in units of f of the field χ from $\chi = 0$ to the absolute minimum (see Fig. 1), the effective scale $\tilde{f}$ is related to the scale f through the following expression:

$$\frac{\tilde{f}}{M_P} = \frac{L}{\pi}\frac{f}{M_P} \tag{21}$$

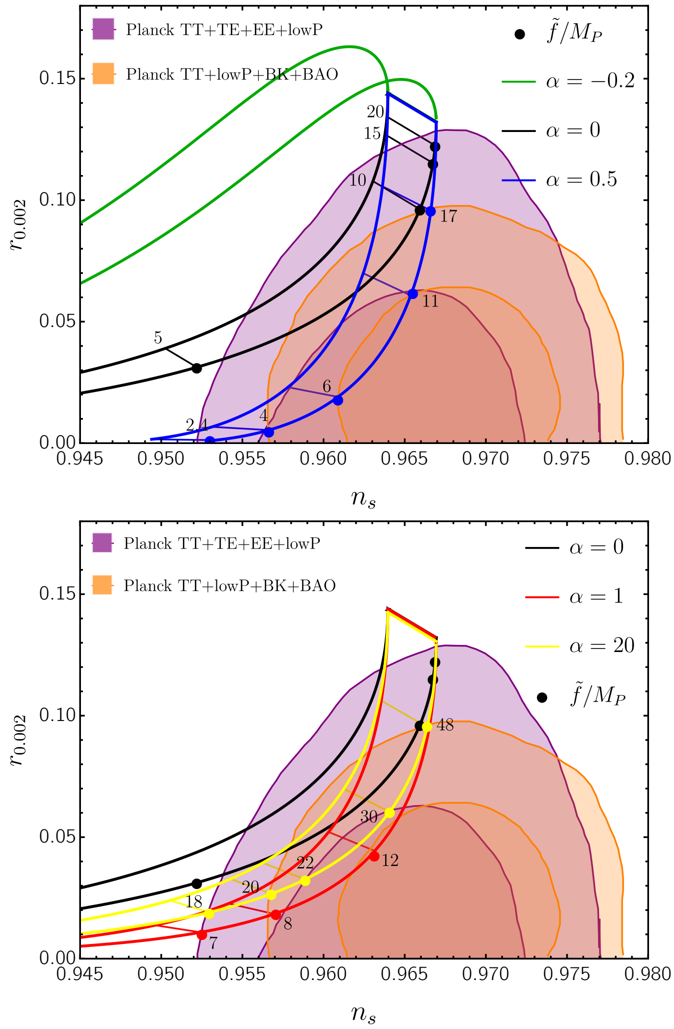

FIG. 2: Results obtained for the scalar spectral index n_s and the tensor-to-scalar ratio $r_{0.002}$. Each colored solid line corresponds to a different value of α and an increasing value of $\tilde{f}$. There are two lines per color corresponding to $N = 55$ (left) and $N = 60$ (right). Some representative values of $\tilde{f}$ are denoted with bullets, together with their corresponding value, with lines connecting $N = 55$ and $N = 60$ representing these constant $\tilde{f}$. The shaded purple regions are the observational constraints from Planck data [3] while the orange are the constraints from Planck combined with Bicep-Keck and BAO data sets [6]. The outer regions represent 95% CL and the inner ones 68% CL.

We have to ensure the model satisfies additional observational data. The Planck collaboration gave the observed value of the amplitude power spectrum at $k_* = 0.05$ Mpc^{-1} [3]:

$$A^2 = \left(\frac{H^2}{2\pi\dot{\chi}}\right)^2_{k_*} = 2.2 \times 10^{-9} \tag{22}$$

We will neglect the difference $\ln(0.05)/\ln(0.002) \approx 3$ in the number of e-foldings between $k = 0.002$ Mpc^{-1} and $k = 0.05$ Mpc^{-1}, and thus evaluate A^2 at the same field values χ_0 used to compute n_s and r. Now, using the Friedmann equation and the equation of motion of the field, both in the slow-roll approximation, allows us to put a constraint between the values of Λ, α and f. We then use this constraint to select the value of the scale Λ corresponding to given values of α and f.

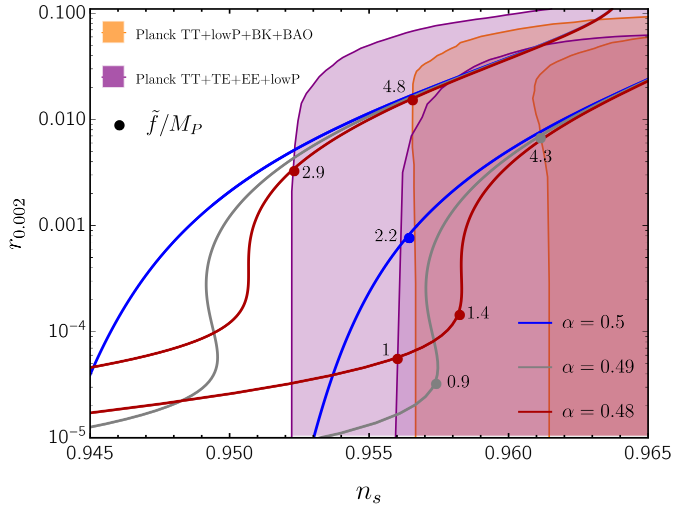

FIG. 3: Results obtained for n_s and $r_{0.002}$ for α near the threshold value $\alpha = 0.5$. Solid lines denote $N = 55$ (left) and $N = 65$ (right). The shaded purple regions are the observational constraints from Planck data [3] while the orange are the constraints from Planck combined with Bicep-Keck and BAO data sets [6]. The outer regions represent 95% CL and the inner ones 68% CL.

In Figs. 2 and 3 we show the results for n_s and r for different values of $\alpha, \tilde{f}$ and N after solving numerically eqs. (7) and (19). In the same figures we also show the 68% and 95%CL constraint contours coming from the Planck data [3], in orange, and from the combination of Planck with baryon acoustic oscillations (BAO) and Bicep-Keck collaboration data sets [6], in purple. We will take the later constraints as a benchmark for the rest of the analysis unless otherwise stated.

The results show that $\alpha > 0$ suppresses the amount of tensor modes (gravitational-wave perturbations) with respect to NI. This is expected since increasing α lowers and flattens the height of the potential, as can be seen from Fig. (1), being reflected on the value of ϵ. This suppression alleviates some of the current tension of NI with the observational constraints: the values for n_s and $r_{0.002}$ with a non-minimal coupling to gravity are found to be well within the 95% CL region for a wide range of parameters and, for $N = 60$ and $N = 65$, the values reach the 68% CL region. On the other hand, negative values of α worsen the compatibility with observations, compared to the ones predicted by NI, with predictions excluded from the 95% CL region. In all further analysis, this case will be omitted.

For $N = 60$ we find, for instance, that: $\alpha = 0.5$ gives results in the 68% CL region, when $4M_P \lesssim \tilde{f} \lesssim 11M_P$; for $\alpha = 1$ this happens for $8M_P \lesssim \tilde{f} \lesssim 14M_P$; and in the case of $\alpha = 20$ the region is reached when $22 \lesssim \tilde{f}/M_P \approx 30M_P$. Generically, and except for $\alpha \simeq 0.5$, in order to obtain predictions in agreement with Planck data, the scale $\tilde{f}$ needs to be increased when α is also increased.

When α gets close to the threshold value 0.5 both the scales $\tilde{f}$ and f can be slightly sub-Planckian. In Fig. 3 we show such cases, which are characterized by very small tensor modes, $10^{-5} \lesssim r \simeq 10^{-4}$. This range of tensor modes is unobservable by near future experiments.

When considering values for α and $\tilde{f}$ that give spectral indices n_s and tensor-to-scalar ratios r laying inside the 68% CL region of the Planck data (see Fig. 2 and 3), $\Lambda \sim 10^{15} - 10^{16}$ GeV is obtained in accordance to the expected value for the GUT scale. For example, for $\alpha = 0.4$ and $\tilde{f} \approx 8M_P$ we obtain $\Lambda \approx 1.8 \times 10^{16}$ GeV, and for $\alpha = 0.49$ and $\tilde{f} \approx 0.8M_P$, $\Lambda \approx 2.4 \times 10^{15}$.

V. CONCLUSIONS

We have considered in this letter a non-minimal coupling between the inflaton and gravity, in the context of Natural Inflation, that respects the symmetry $\phi \to \phi + 2\pi f$. Assuming a simple cosine form, with only one extra dimensionless parameter $\alpha \simeq \mathcal{O}(1)$, we have obtained a model that gives rise to predictions for the spectral index n_s and the tensor-to-scalar ratio r that lay within the 68% CL region of Planck data. This is an improvement over the predictions made by minimally coupled Natural Inflation, which are excluded from this region. The parameters that give rise to these results yield a scale $\Lambda \sim 10^{15} - 10^{16}$ GeV, consistent with the expected value of the GUT scale. Another interesting consequence of the non-minimal coupling is that inflation can be driven for smaller values of f. However, in the Einstein frame the new periodicity is $\tilde{f}$. In terms of this new scale we have shown that for α close to 0.5 we can have $\tilde{f}/M_P \lesssim 1$, within the 68% CL region. In this case, considerably smaller values of the tensor-to-scalar ratio $10^{-5} \lesssim r \lesssim 10^{-4}$ are obtained.

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