Renormalization of gauge-invariant operators and the axial anomaly

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The renormalization properties of gauge-invariant composite operators that vanish when the classical equations of motion are used class IIa operators and which lead to diagrams where the Adler-Bell-Jackiw anomaly occurs are discussed. It is shown that gauge-invariant operators of this kind do not, in general, nonvanishing gauge-invariant (class I) counterterms.

I. INTRODUCTION

Composite operators have long been used in field theories, mainly in deep-inelastic analysis, through Wilson’s operator-product expansion.1 With the appearance of the Shifman-Vainshtein-Zakharov sum rules2 as a useful way to extend the domain of applicability of quantum chromodynamics to low-energy phenomenology, the renormalization properties of gauge-independent composite operators have been the object of renewed interest.3,4

Composite operators can be divided into two classes. Class I contains those operators which are invariant under classical gauge transformations and which do not vanish when one uses the covariant equations of motion (we will qualify this later). The operators that, being formally gauge invariant, vanish by virtue of the equations of motion form class IIa. Finally, class IIb groups the non-gauge-invariant operators.5

In general, bare operators cannot be made finite by means of multiplicative renormalization, but rather a composite operator can mix with others of the same dimension and Lorentz structure along the renormalization procedure. The problem of mixing among composite operators has been dealt with by several authors5–7 and, in the light of their work, the following properties are known.

(i) In ordinary covariant gauges class-I operators mix not only among themselves, but also with classes IIa and IIb. On the contrary, in the background-field gauge,8 where gauge invariance is retained in the external (classical) field, class-I operators do not mix with class IIb.5

(ii) The submatrix concerning the renormalization of class-I operators among themselves, ZII, is gauge independent. However, even in the background-field gauge, the contribution to a renormalized class-I operator coming from class-II operators, ZI, is gauge dependent.

\[
W[J,\bar{J},\phi] = \int [dA][d\phi][d\bar{\phi}][d\psi][d\bar{\psi}] \exp \left[ i \int d^4x (\mathcal{L} + \phi J + J^a_\mu A^a_\mu + \bar{\psi} \gamma^\alpha \psi + \bar{\psi} \gamma^\alpha \phi) \right].
\]

The action

\[
S = \int d^4x \mathcal{L}(x) = \tilde{S} - \frac{1}{2a} \int d^4x [\partial^\mu A^a_\mu(x)]^2
\]

satisfies

\[
\frac{\delta \tilde{S}}{\delta A^a_\mu} + \frac{\delta \tilde{S}}{\delta \phi^a} + \frac{\delta \tilde{S}}{\delta \bar{\phi}^a} + \frac{\delta \tilde{S}}{\delta \psi^a} + \frac{\delta \tilde{S}}{\delta \bar{\psi}^a} + \frac{\delta \tilde{S}}{\delta \phi^a} \frac{\delta \tilde{S}}{\delta \bar{\phi}^a} = 0.
\]
By using the appropriate functional derivatives and then setting all sources equal to zero, we obtain the desired Green's functions with an insertion of a composite operator.

Beyond the tree level appropriate counterterms have to be added to $S$ in order to keep all the Green's functions finite. Let us now suppose that we are, for the time being, concerned with the renormalization of gauge-invariant operators like $F^{ab}_\mu F_{\mu \nu}$ or $\bar{\psi} \gamma^\nu \psi$. These operators obey an equation of the type (5),

$$
\left[ -D_\mu \bar{\psi}_b \frac{\delta}{\delta A^a_\mu} + \frac{1}{2} g f^{abc} \bar{\psi}_b \gamma^\nu \psi_c \frac{\delta}{\delta A^c_\nu} - 3 g \frac{\delta}{\delta \bar{\psi}} - \frac{1}{2} g \gamma^\nu (\bar{\psi} T^a \psi)_a \frac{\delta}{\delta \bar{\psi}^a} - \frac{1}{2} g \gamma^\nu (\bar{\psi} T^a \psi)_a \frac{\delta}{\delta \bar{\psi}^a} \right] \phi_i O_i = W_{\phi_i O_i} = 0 .
$$

We have introduced a shorthand notation $W$ for the functional-differential operator. Clearly $W^2 = 0$; i.e., $W$ is nilpotent. Equation (6) is not only reflecting the invariance of the theory under BRS transformations, and thus the fulfillment of Ward-Slavnov identities, but also is providing restrictions on the form of the permitted counterterms. When renormalizing a gauge-invariant operator, one adds gauge-invariant counterterms, but in general, one needs (because of the breakdown of gauge invariance by the gauge-fixing term) non-gauge-invariant counterterms as well, $\phi_i N_i$, provided that they conspire to build a quantity still satisfying $W_{\phi_i N_i} = 0$. The simplest possibility is obviously to take $\phi_i N_i = W_{\phi_i F_n}$, with no restriction on $F_n$, provided they have suitable dimension and Lorentz and color structure. In fact, it can be shown that this is the general solution.

Then the most general action satisfying Ward-Slavnov identities can be written as

$$
\int d^4 x \left[ \mathcal{L} + \phi_i O_i + \frac{\delta S}{\delta A^a_\mu} \bigg|_{\phi = 0} \phi_b C^a_{bk} + \phi_f C^a_{pa} \frac{\delta S}{\delta \bar{\psi}^a} \right. $$

$$
\left. + \phi_q \bar{C}^a_{q\alpha} \frac{\delta S}{\delta \bar{\psi}^a} + W_{\phi_i F_n} \right],
$$

where the separation in classes I, II, and III is clearly exhibited. Let us recall that any gauge-invariant composite operator vanishing when the classical equations of motion are used,

$$
D^{ab}_\mu F^\nu_b + g \bar{\psi} T^a \gamma^\nu \psi = \frac{\delta S}{\delta A^a_\mu} |_{\phi = 0} = 0 ,
$$

$$
i(\bar{\psi} \gamma^\mu \gamma^\nu \psi) = \frac{\delta S}{\delta \psi^a} = 0 ,
$$

or

$$
(D^{ab}_\mu F^\nu_b + g \bar{\psi} T^a \gamma^\nu \psi) C^a_{\nu}(A_i, \bar{\psi}, \gamma, \psi) ,
$$

with $C^a_\nu$, $\bar{C}^a_\nu$, $C^a$ satisfying

$$
W C^a_\nu = g f^{abc} \bar{\psi} \gamma^\alpha C^c_\nu ,
$$

$$
W \bar{C}^a_\alpha = - ig \bar{C}^a \gamma^\alpha T^a \psi ,
$$

$$
W C^a = - ig T^a \psi C^a .
$$

Using

$$
\frac{\delta S}{\delta A^a_\mu} = D^{ab}_\mu F^\nu_b - g f^{abc} \bar{\psi} \gamma^\mu \gamma^\nu \psi \gamma^a + g \bar{\psi} \gamma^\nu T^a \psi ,
$$

the generating functional (3) can be written as

$$
W_{J, \bar{J}} = \int [d A] [d \bar{A}] [d \psi] [d \bar{\psi}] [d \psi] [d \bar{\psi}] \exp \left[ i \int d^4 x \left[ \mathcal{L} + \phi_i O_i + W_{\phi_i F_n} + \phi_f \bar{C}^a_{k\nu} \frac{\delta S}{\delta \bar{\psi}^a} + \phi_p C^a \frac{\delta S}{\delta \psi^a} \right. 
$$

$$
\left. + \phi_q \bar{C}^a_{q\alpha} \frac{\delta S}{\delta \bar{\psi}^a} + W_{\phi_i F_n} \right] .
$$

Let us now perform the following change of variables:

$$
A^a_\mu = A^a_{\mu k} + \phi_k C^a_{mk} , \quad \psi = \psi + \phi_q C_{\psi} , \quad \bar{\psi} = \bar{\psi} + \phi_q \bar{C}^a_{\psi} .
$$

Equation (13) becomes (up to terms of higher order in the sources)

$$
W_{J, \bar{J}, \phi} = \int [d A] [d \bar{A}] [d \psi] [d \bar{\psi}] [d \psi] [d \bar{\psi}] \exp \left[ i \int d^4 x \left[ \mathcal{L} + \phi_i O_i + W_{\phi_i F_n} + g f^{abc} \bar{\psi} \gamma^\mu \gamma^\nu \psi \gamma^a \right. 
$$

$$
\left. + \phi_p C_{\mu k} + j^a \gamma^\mu \phi_k C^a_{mk} + J^a + \phi_q \bar{C}^a_{\psi} \left( \psi - \phi_q C_{\psi} \right) + \bar{\psi} \left( \bar{\psi} - \phi_q \bar{C}^a_{\bar{\psi}} \right) J^a \right] .
$$

Notice that the change (14) is easily done with field $A^a_\mu$, since it is a local change of variables it has unit Jacobian. However, this is not so simple with fermionic fields as we will see and we have introduced the Jacobian $\mathcal{D}$ of the transformation in Eq. (15).
A further manipulation allows us to write Eq. (15) as

\[
W[j, J, \phi] = \int [dA][d\varphi][d\bar{\varphi}][d\psi][d\bar{\psi}] \exp \left[ i \int d^4x \left( \mathcal{L} + \phi_i O_i + W \phi_a F_a + W \phi_b ( - \partial^2 \varphi^2 C_{\mu\nu}^b ) 
+ j^a_\mu ( A_{\mu}^a - \phi_k C_{\mu\nu}^b ) + \bar{J}^a ( \bar{\psi} - \phi_k C_{\mu}^b ) + ( \bar{\psi} - \phi_k C_{\rho}^b ) \partial^a \right) \right],
\]

(16)

where we have used Eq. (11). Notice the two terms in Eq. (15) which have been integrated as \(W \phi_a F_a\), of variables (14) is performed are exactly canceled by the (modified) propagators, which will contain the new insertions too.

Connected Green's functions with an insertion of a composite operator will be obtained from

\[
Z[j, J, \phi] = -i \ln W[j, J, \phi]
\]

(17)

while one-particle irreducible (1PI) Green's functions will be obtained after Legendre-transforming Eq. (17). When computing 1PI Green's functions, the insertions in the external legs which are generated by the new terms in (16) coupled to the field sources which appear once the change of variables (14) is performed are exactly canceled by the (modified) propagators, which will contain the new insertions too.

Clearly, had we neglected the Jacobian we would have entirely reabsorbed class-II\(^a\) operators in the terms \(W \phi_a F_a\), so that operators which vanish by virtue of the classical equations of motion are, in spite of their gauge invariance, generated from a gauge-dependent composite operator. When this is the case both class-II\(^a\) and class-II\(^b\) operators can be expressed in the form \(W \phi_a F_a\) and thus the results of Kluberg-Stern and Zuber indeed hold. Let us summarize, for the sake of completeness, their argument.

Consider an insertion of \(WF_a\) in a Green's function. It is given by the functional derivatives of

\[
\int [dA][d\varphi][d\bar{\varphi}][d\psi][d\bar{\psi}] W F_a \exp \left[ i \int d^4x \left( \mathcal{L} + j^a_\mu A_{\mu}^a + \bar{J}^a \phi_a + \bar{\psi} \psi + C_{a} \partial^a \right) \right].
\]

(18)

Using the invariance of the action under BRS transformations, Eq. (18) can be written as

\[
\int [dA][d\varphi][d\bar{\varphi}][d\psi][d\bar{\psi}] \left\{ j^a_\mu \frac{\delta}{\delta j^a_\mu} + \bar{C}^a \frac{\delta}{\delta \bar{C}^a} + \frac{1}{a} \partial \varphi^2 \partial^a \frac{\delta}{\delta J^a} - \bar{J}^a \frac{\delta}{\delta \bar{J}^a} \right\}
\]

\[\times F_a \exp \left[ i \int d^4x \left( \mathcal{L} + j^a_\mu D^a_{\mu} \phi_a + K_b \frac{1}{2} f^{abc} \phi_b \phi_c + \bar{K}_a (ig T^a \psi^2 \psi^a) + (ig \bar{\psi}^2 T_{\alpha}^a \psi^a) K_a + j^a_\mu A_{\mu}^a + \bar{J}^a \phi_a 
+ \bar{\psi} \psi + C_{a} \partial^a \right) \right].
\]

(19)

The operator

\[
\Omega = j^a_\mu \frac{\delta}{\delta j^a_\mu} + \bar{C}^a \frac{\delta}{\delta \bar{C}^a} + \frac{1}{a} \partial \varphi^2 \partial^a + \frac{\delta}{\delta J^a} - \bar{J}^a \frac{\delta}{\delta \bar{J}^a} + \bar{J}^a \frac{\delta}{\delta \bar{J}^a},
\]

(20)

being independent of the fields, can be taken out of the functional integral in (19). Notice that the auxiliary sources \(J^a, K_b, \bar{K}_a\), which are to be set equal to zero, are necessary to write the operator (20) in a linearized form.

After Legendre-transforming both Eqs. (18) and (19),

Then the generating functional with the suitable counterterms, which at the one-loop level takes the form

\[
\int [dA][d\varphi][d\bar{\varphi}][d\psi][d\bar{\psi}] \exp \left[ i \int d^4x ( \mathcal{L} + \Delta \mathcal{L} + \phi_i O_i + \Delta \phi_i O_i + W \phi_a F_a + \Delta W \phi_a F_a + \text{field sources} ) \right],
\]

(22)

can be written, introducing the bare fields and parameters, in the form

\[
\int [dA][d\varphi][d\bar{\varphi}][d\psi][d\bar{\psi}] \exp \left[ i \int d^4x ( \mathcal{L}^{\phi} + \phi_i O_i^{\phi} + W \phi_a F_a^{\phi} + \text{field sources} ) \right],
\]

(23)

where \(\phi_i^{\phi}\) and \(\phi_a^{\phi}\) stand for

\[
\phi_i^{\phi} = \phi_i + \sum_j \delta_{ij} \phi_j,
\]

(24)

and \(\phi_i^{\phi}\) and \(\phi_a^{\phi}\) denote the operators written in terms of bare fields. (Indices \(i, j\) run over the gauge-independent operators and \(m, n\) over the gauge-dependent operators.) Notice that Eq. (24) is indeed providing the sort of mixing which is expected from the results of Kluberg-Stern.
and Zuber. What is relevant to us is that Eq. (23) takes exactly the same form as Eq. (16), so that one can extend the previous argument to any order.\footnote{Eq. (23) takes the form of Eq. (16) because...}

What we want to do now is to study whether things are altered when the change of Eq. (14) does not have a trivial Jacobian and thus cannot be dropped out of the generating functional (16).

III. EVALUATION THE JACOBIAN

By performing the suitable Wick rotation in Eq. (16), we will calculate in Euclidean space-time. The Euclidean spinors \( \psi \) and \( \bar{\psi} \) can be expanded according to

\[
\psi(x) = \sum_n b_n \varphi_n(x),
\]

\[
\bar{\psi}(x) = \sum \bar{a}_n \varphi_n^*(x).
\]

\( a_n \) and \( \bar{b}_n \) are elements of the Grassmann algebra. \( \varphi_n \) are chosen to be the eigenfunctions of the equation

\[
i\partial_k \varphi_n = \lambda_n \varphi_n.
\]

\( i\partial_k \) being now a Hermitian operator, has real eigenvalues. The solutions to Eq. (26) are taken to satisfy

\[
\int d^4x \varphi_n^*(x) \varphi_m(x) = \delta_{nm}.
\]

The path-integral fermionic measure, which is properly defined by

\[
\prod_n d a_n d \bar{b}_n,
\]

under the change

\[
da_n = C_{nm} d a_m,
\]

\[
\bar{b}_n = \bar{b}_m C_{nm}^{-1},
\]

transforms as

\[
\prod_n d a_n d \bar{b}_n \rightarrow \prod_n d a'_n d \bar{b}_n \det[C C^{-1}].
\]

To be definite consider, for instance, that we are interested in the pseudoscalar operators of dimension 4. Class II\( ^c \) contains the operators \( i \bar{\psi} \gamma^\nu \bar{D} \psi \) and \( -i \bar{\psi} \bar{D} \gamma^\nu \psi \), so that \( C_\gamma = \bar{\gamma} \gamma \) and \( C_\rho = \gamma^\rho \gamma^\rho \). (Our \( \gamma^\nu \) is defined as \( \gamma^\nu = \gamma^\nu \gamma^\rho \gamma^\rho \). In Euclidean space-time \( \gamma^\nu = \gamma^\nu \gamma^\rho \gamma^\rho \) with \( \gamma^\nu = i \gamma^\nu \). Then

\[
C_{nm} = \delta_{nm} + \int d^4x \varphi_n^*(x) \varphi_m(x) \gamma^\nu \varphi_m(x),
\]

\[
\bar{C}_{nm} = \delta_{nm} + \int d^4x \varphi_n^*(x) \varphi_m(x) \gamma^\nu \varphi_m(x).
\]

Since we are ultimately going to take a derivative and set \( \phi = 0 \), we will retain terms at most linear in \( \phi \). So, using

\[
\det[1 + \mathcal{L}] = \exp[\text{Tr} \ln(1 + \mathcal{L})] = \exp[\text{Tr} \mathcal{L}],
\]

we find

\[
\mathcal{L} = \det[C C^{-1}]^{-1} = \exp \left[ -\sum_n \int d^4x \varphi_n^*(x) (\phi_\rho + \phi_\nu) \gamma^\rho \varphi_n(x) \right].
\]

What is left is to evaluate

\[
D(x) = \sum_n \varphi_n^*(x) \gamma^\rho \varphi_n(x).
\]

This is easily done choosing a plane-wave basis for the \( \varphi_n(x) \) (see Ref. 13). The result is

\[
D(x) = \frac{g^2}{32\pi^2} N_f \frac{1}{2} e^{i m_0} F_{\mu\nu}(x) F^{\mu\nu}(x)
\]

so that

\[
\mathcal{L} = \exp \left[ -\sum_n \int d^4x (\phi_\rho + \phi_\nu) \frac{g^2}{32\pi^2} N_f F_{\mu\nu} F^{\mu\nu} \right]
\]

or, in Minkowski space,

\[
\mathcal{L} = \exp \left[ i \int d^4x (\phi_\rho + \phi_\nu) \frac{g^2}{32\pi^2} N_f F_{\mu\nu} F^{\mu\nu} \right].
\]

Equation (16) now reads for pseudoscalar operators of dimension 4

\[
W[J,I,J,\phi] = \int \left[ (dA)(d\varphi)(d\bar{\varphi})(d\psi)(d\bar{\psi}) \right] \exp \left[ i \int d^4x \left\{ \mathcal{L} + \phi_\rho O_\rho + (\phi_\rho + \phi_\nu) \frac{g^2}{32\pi^2} N_f F_{\mu\nu} F^{\mu\nu} + W\phi_\rho F_{\mu\nu} \right\} + \text{field sources} \right],
\]

where index \( i \) runs over gauge-invariant operators and index \( n \) labels gauge-dependent operators. \( O_\rho \) and \( F_{\mu\nu} \) are understood to have suitable quantum numbers. Of course, \( F_{\mu\nu} \) was already included in the set of class-I operators \( O_\rho \), but this does not affect our argument. The proof of Kluberg-Stern and Zuber we have sketched in the preceding section still implies the lack of mixing of \( W\phi_{\alpha} \) operators to class-I operators but now, and this is the crucial point, class-II\( ^c \) operators cannot be expressed as \( W\phi_{\alpha} \). On the contrary, they get a class-I contribution from the very beginning.

The renormalization argument previously used works in the same way at any order in perturbation theory, i.e., class-I operators mix among themselves and with \( W\phi_{\alpha} \)-type operators along the renormalization procedure, whereas operators of the form \( W\phi_{\alpha} \) mix only among themselves. Much as before, one can write the equivalent to Eq. (23) as

\[
\int \left[ (dA)(d\varphi)(d\bar{\varphi})(d\psi)(d\bar{\psi}) \right] \exp \left[ i \int d^4x \left\{ \mathcal{L}^0 + \phi_\rho^0 O_\rho^0 + (\phi_\rho^0 + \phi_\nu^0) \frac{g^2}{32\pi^2} N_f F_{\mu\nu} F^{\mu\nu}^0 + W\phi_\rho^0 F_{\mu\nu}^0 + \text{field sources} \right\} \right],
\]
with
\[
\phi_i^0 = \phi_i + \sum_j \Delta_{ij} \phi_j + \Delta_{ip} \phi_p + \Delta_{iq} \phi_q, \\
\phi_p^0 = \phi_p + \sum_j \Delta_{pj} \phi_j, \quad \phi_q^0 = \phi_q + \sum_j \Delta_{qj} \phi_j, \\
\phi_m^0 = \phi_m + \sum_j \Delta_{mj} \phi_j + \Delta_{mp} \phi_p + \Delta_{mq} \phi_q + \sum_m \Delta_{mn} \phi_n.
\] (40)

As usual, indices \(i, j\) label class-I operators, indices \(m, n\) class-II operators, and \(p\) and \(q\) stand for \(i\overline{\psi} \gamma^5 \psi\) and \(i\overline{\psi} \gamma^5 \overline{D} \psi\), respectively.

We have put \(\Delta_{pp} = \Delta_{qq} = \Delta_{pp} = \Delta_{qq} = 0\) as a consequence of the nonrenormalization of the axial anomaly, but nothing prevents \(FF\) from coupling to other gauge-invariant operators.4

Equations (39) and (40) are our final result. They indeed show the coupling of some class-II operators to class-I operators.

IV. CONCLUSION

\(\overline{\psi} \gamma^5 \overline{D} \psi\) and \(\overline{\psi} \gamma^5 \overline{D} \gamma^5 \psi\) do have a class-I contribution, as we have learned. In general, this can also happen with more sophisticated composite operators containing both \(\gamma^5\) and \(\overline{D}\). In other words, and remarkably enough, the claim about the independence of class-II operators on class-I operators seems to fail exactly by the same reasons that the axial anomaly occurs in field theory.

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