Gaussian analysis of the Gross-Neveu model

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The Gross-Neveu model is solved within a simple Gaussian approach. The ground state is correctly reproduced. We find that two phases escape the triviality problem. Bound states and the analogy with $\lambda\phi^4$ theory are also considered.

I. INTRODUCTION

A relevant problem in quantum field theory (QFT) is triviality. Such an uneasy feature was augured a long time ago for $\lambda\phi^4$ theory (see Ref. 1 for details) and states that any behavior of λ_B (B = bare) leads to either an unbounded theory or a trivial one, i.e., $\lambda_R = 0$ (R = renormalized). The usual perturbative analysis seems to be a mirage of the actual theory, the subtle but far-reaching origin of the discrepancy lying on the noncommutativity $[\lim_{\lambda\to\infty}, \lim_{\lambda_R\to 0}] \neq 0$ (λ stands for an UV cutoff).[2]

Recently, Stevenson has undertaken the study of $\lambda\phi^4$ theory by means of a Gaussian approach with amazing results. Our aim in this work is twofold. On the one hand, we would like to test the Gaussian method in a different theory. We think that the Gross-Neveu (GN) model[3] fits this purpose since it is exactly solvable in the $N\to\infty$ limit (N = number of fermionic fields) and neat qualitative and quantitative physics, namely, dynamic mass generation, show up. On the other hand, we think that the special features of the nontrivial phase of $\lambda\phi^4$ theory should be contrasted with some other theory. GN argued a definite sign for g^2 in order to obtain a well-behaved theory. Therefore the Gaussian approximation should provide the correct solution to this problem.

We have organized the content in the following way. In Sec. II we comment on the Gaussian approximation and apply it to the GN model. Section III is concerned with renormalization and triviality analysis. Dynamical chiral symmetry breaking and bound states are studied in Sec. IV. Conclusions are summarized in the last section.

II. THE GAUSSIAN APPROXIMATION

The Gaussian approach was introduced long ago in QFT (Ref. 4) as a tool borrowed from quantum mechanics (for an excellent review see Ref. 5). Only recently Stevenson[2] made use of it in order to investigate the ground state of $\lambda\phi^4$. Thereupon several problems have been dealt with in these trends: $\lambda\phi^4$ at $T>0$ (Ref. 6), $\lambda\phi^4$ in a static Robertson-Walker background spacetime,[7] the same in a time-dependent space-time[8] and scalar quantum electrodynamics (SQED) (Ref. 9).

Let us consider the GN model defined by the Lagrangian density in two dimensions,

$$\mathscr{L} = i\bar{\psi}^a \partial\!\!\!/ \psi^a - \tfrac{1}{2} g_B{}^2 (\bar{\psi}^a \psi^a)^2 . \tag{1}$$

Notice that the $g_B{}^2$ sign corresponds to the naive one, and is opposite to GN's selection. The Gaussian approach—if powerful enough—will tell us which $g_B{}^2$ makes sense.

The Dirac algebra is defined by

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}, \quad g_{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{2}$$

and a possible realization is given by

$$\gamma_0 = \sigma_3, \quad \gamma_1 = i\sigma_1, \quad \gamma_5 = -\sigma_2 . \tag{3}$$

At the classical level a mass term in (1) is forbidden by the chiral symmetry $\psi \to \gamma_5 \psi$, $\bar{\psi}\psi \to -\bar{\psi}\psi$. However, dynamical symmetry breaking will generate a mass for the fermion field.

Barnes and Ghandour made a careful analysis of the free fermionic Fock space in the Schrödinger picture.[4] They found that the vacuum functional in the field basis is given by a delta (rather than a Gaussian) functional, i.e.,

$$\langle \psi \mid 0 \rangle_m = \delta[\psi_+^a(x^1)] = \prod_a \prod_{x^1} \psi_+^a(x^1) , \tag{4}$$

where

$$\begin{aligned} \psi_+^a(x^1) &\equiv P_+ \psi^a(x^1) \\ &\equiv \frac{1}{2}\left[1 + \frac{-i\gamma^0\gamma^1\partial_1 - \gamma^0 m}{(m^2 - \partial_1\partial_1)^{1/2}}\right]\psi^0(x^1) . \end{aligned} \tag{5}$$

P_+ selects the positive-energy part of a spinor and m stands for the mass of the free particle. The second equality in (4) holds because of the Grassmann nature of $\psi^0(x^1)$. They also gave the creation and annihilation operators which generate the Fock space.

The vacuum functional can be changed in the following way:

$$\begin{aligned} \langle \psi \mid 0 \rangle_{m,\psi_0} &= \delta[(\psi^a(x^1) - \psi_0^a)_+] \\ &= \prod_a \prod_{x^1} [\psi^a(x^1) - \psi_0^a]_+ \end{aligned} \tag{6}$$

in order to introduce a constant background field (which could be induced by an external source). ψ_0^a is a constant Grassmann spinor and in our representation (3) we have

$$\begin{aligned} \psi_{0+}^a &= P_+ \psi_0^a = P_+ \begin{bmatrix} \psi_{01}^a \\ \psi_{02}^a \end{bmatrix} = \begin{bmatrix} \psi_{01}^a \\ 0 \end{bmatrix} , \\ \bar{\psi}_{0+}^a \psi_{0+}^a &= \psi_{01}^{a\dagger} \psi_{01}^a \geq 0 . \end{aligned} \tag{7}$$

It is still possible to define creation and annihilation operators which are the same as in the free case but for a straightforward modification of the particle ones (a and $a^\dagger$). Therefore we are allowed to write the fermionic field in terms of ψ_{0+}^a and the creation and annihilation operators.

The idea of our approach is to use

$$\langle \psi | 0 \rangle_{\Omega,\psi_0} \tag{8}$$

as an ansatz for the vacuum. Ω plays the role of a trial mass function depending on ψ_{0+} to be fixed by minimization of the effective potential. Nonperturbative effects are included through this variable.

This ansatz enables us to formally work with the algebra

$$\begin{aligned} \psi^a(x) = \psi_{0+}^a + \int \frac{dk^1}{2\pi 2\omega_\Omega(k^1)} [U_\Omega(k^1) a_\Omega^a(k^1) e^{-ikx} \\ + V_\Omega(k^1) b_\Omega^{\dagger a}(k^1) e^{ikx}] , \\ \omega_\Omega(k^1) = [(k^1)^2 + \Omega^2]^{1/2}, \quad kx = \omega_\Omega(k^1) x^a - k^1 x^1 , \\ a = 1, \ldots, N , \end{aligned} \tag{9}$$

where the Fock space is defined by

$$\begin{aligned} & a_\Omega^a(k^1) | 0 \rangle_{\Omega,\psi_0} = b_\Omega^a(k^1) | 0 \rangle_{\Omega,\psi_0} = 0 , \\ & {}_{\Omega,\psi_0}\langle 0 | 0 \rangle_{\Omega,\psi_0} = 1 , \\ & \{ a_\Omega^a(k^1), a_\Omega^{b\dagger}(q^1) \} = 2\pi 2\omega_\Omega(k^1) \delta^{ab} \delta(k^1 - q^1) , \\ & \{ b_\Omega^a(k^1), b_\Omega^{b\dagger}(q^1) \} = 2\pi 2\omega_\Omega(k^1) \delta^{ab} \delta(k^1 - q^1) . \end{aligned} \tag{10}$$

Although the fermionic vacuum is represented by a delta rather than a Gaussian functional, the previous algebra is similar to the usual bosonic one.[2] For this reason we are going to stick to the word "Gaussian" as other authors have already done.[10]

The calculation of the Hamiltonian density is quite simple and yields the result

$$\begin{aligned} \frac{1}{N} {}_{\Omega,\psi_0}\langle 0 | \mathscr{H} | 0 \rangle_{\Omega,\psi_0} = & \, 2[-I_1(\Omega) + \Omega^2 I_0(\Omega)] \\ & + \frac{\lambda_B}{2} \left[\alpha^2 + \frac{2}{N}(1-2N)\Omega I_0(\Omega)\alpha + \frac{2}{N}(2N-1)\Omega^2 I_0^2(\Omega) + \frac{2}{N} I_{1/2}{}^2 \right] , \end{aligned} \tag{11}$$

where $\alpha \equiv \bar{\psi}_{0+}^a \psi_{0+}^a \geq 0$, $\lambda_B \equiv g_B{}^2 N$ and

$$I_n(\Omega) \equiv \int \frac{dk^1}{2\pi 2\omega_\Omega(k^1)} [\omega_\Omega{}^2(k^1)]^n . \tag{12}$$

These integrals satisfy

$$\frac{d}{d\Omega} I_n(\Omega) = (2n-1)\Omega I_{n-1}(\Omega) . \tag{13}$$

Notice that $I_1(\Omega)$, $I_{1/2}$, and $I_0(\Omega)$ are quadratically, linearly, and logarithmically divergent, respectively. Therefore we introduce a cutofflike regularization

$$\begin{aligned} I_1(\Omega) &= \frac{1}{4\pi} \left[\Lambda^2 + \frac{\Omega^2}{2} + \frac{\Omega^2}{2} \ln \frac{4\Lambda^2}{\Omega^2} \right] + O\left[\frac{\Omega^4}{\Lambda^2} \right] , \\ I_{1/2} &= \frac{\Lambda}{2\pi} , \\ I_0(\Omega) &= \frac{1}{4\pi} \ln \frac{4\Lambda^2}{\Omega^2} + O\left[\frac{\Omega^2}{\Lambda^2} \right] , \\ I_{-1}(\Omega) &= \frac{1}{2\pi} \frac{1}{\Omega^2} . \end{aligned} \tag{14}$$

The last convergent integral has been displayed for further needs. One could avoid fixing a regularization scheme[2] but we prefer to work with (14) just for the sake of simplicity.

III. PHASES OF THE GN MODEL

A. Renormalization

The renormalizability of the GN model guarantees that the bare parameters of the Lagrangian density can absorb any ultraviolet divergency. Actually, λ_B reparametrization provides a finite theory in our approach but for the zero-point energy which obviously must be substracted. In order to easily compare with perturbation theory we normalized the vacuum energy as

$$\begin{aligned} \mathscr{E}(\alpha,\Omega) &\equiv \frac{1}{N} [V(\alpha,\Omega) - \lim_{\tilde{\Omega}\to 0} V(0,\tilde{\Omega})] \\ &= \frac{\Omega^2}{4\pi} \left[\ln \frac{4\Lambda^2}{\Omega^2} - 1 \right] + \frac{\lambda_B}{2} \left[\alpha^2 - \frac{2}{N}(2N-1) \frac{\Omega}{4\pi} \ln \frac{4\Lambda^2}{\Omega^2} \alpha + \frac{\Omega^2}{8\pi^2} \frac{2N-1}{N} \ln^2 \frac{4\Lambda^2}{\Omega^2} \right] , \\ V(\alpha,\Omega) &\equiv {}_{\Omega,\psi_0}\langle 0 | \mathscr{H} | 0 \rangle_{\Omega,\psi_0} \end{aligned} \tag{15}$$

(the limiting procedure avoids dealing with intermediate infrared divergences). Notice that quadratic and linear divergences have disappeared, only logarithmic ones survive. The meaningful phases of the theory are those that render (15) finite.

We need to fix Ω. This is done by minimization of $\mathscr{E}(\alpha,\Omega)$ with respect to Ω for each α. Therefore the trial mass Ω is a function of α.

The set of fundamental equations needed in our analysis includes the usual renormalization prescription[4] for λ_R

$$\lambda_R \equiv \frac{d^2}{d\alpha^2} V(\alpha,\overline{\Omega}(\alpha))\bigg|_{\alpha=\alpha_0}, \tag{16}$$

where α_0 is the substraction point. For the sake of simplicity we choose $\alpha_0=0$ so that

$$\begin{aligned}\lambda_R = {} & \lambda_B+\Omega_0'\lambda_B\frac{2}{N}(1-2N)(I_0-\Omega_0{}^2I_{-1})+2\Omega_0'^2\left[I_0-2\Omega_0{}^2I_{-1}+\lambda_B\frac{2N-1}{N}[(I_0-\Omega_0{}^2I_{-1})^2-\Omega_0{}^2I_0I_{-1}]\right]\\ & +2\Omega_0''\left[I_0-\Omega_0{}^2I_{-1}+\lambda_B\frac{2N-1}{N}(I_0{}^2-\Omega_0{}^2I_0I_{-1})\right]\Omega_0\,,\\ & \Omega_0\equiv\overline{\Omega}(0),\quad \Omega_0'\equiv\frac{d\overline{\Omega}(\alpha)}{d\alpha}\bigg|_{\alpha=0},\quad \Omega_0''\equiv\frac{d^2\overline{\Omega}(\alpha)}{d\alpha^2}\bigg|_{\alpha=0},\quad I_n\equiv I_n(\Omega_0)\,.\end{aligned} \tag{17}$$

The next step consists of performing a detailed casuistic of our basic formulas.

B. Analysis

Let us first consider $\alpha=0$. It is obvious that the theory must make sense at this point at least. $\Omega_{\min}(\alpha=0)=\Omega_0$ is defined by

$$\begin{aligned}0 &= \frac{\partial V(0,\Omega_0)}{\partial\Omega_0}\\ &= \left[I_0-\frac{1}{2\pi}\right]2\Omega_0\left[1+\frac{2N-1}{N}\lambda_B I_0\right]\end{aligned} \tag{18}$$

or by the end points of the variational range of Ω_0, i.e., $\Omega_0=0$ or $\Omega_0\sim\Lambda$. In this model $\Omega_0=0$ is a particular solution of (18). We distinguish two possible cases: (i) $\Omega_0>0$; (ii) $\Omega_0=0$.

$\Omega_0>0$. Equation (18) can be used at will provided we confirm that Ω_0 corresponds to a minimum. Therefore a further restriction appears

$$0=\frac{\partial^2 V(0,\Omega_0)}{\partial\Omega_0{}^2}>0 \Longrightarrow \lambda_B<0 \tag{19}$$

and the point $\Omega_0\sim\Lambda$ becomes safe.

The general restriction

$$\begin{aligned}0 &= \frac{\partial V(\alpha,\Omega)}{\partial\Omega}\\ &= \left[I_0(\Omega)-\frac{1}{2\pi}\right]\left[2\Omega+\frac{2N-1}{N}\lambda_B[2\Omega I_0(\Omega)-\alpha]\right]\end{aligned} \tag{20}$$

allows the computation of Ω_0':

$$\Omega_0'=-\pi\,. \tag{21}$$

Furthermore the behavior of λ_B turns out to be fixed by (18)

$$\lambda_B=-\frac{N}{2N-1}\frac{1}{I_0}=-\frac{N}{2N-1}\frac{4\pi}{\ln\frac{4\lambda^2}{\Omega_0{}^2}} \tag{22}$$

leading to

$$\lambda_R=-\pi\,. \tag{23}$$

The energy comes to be

$$\mathscr{E}(\alpha,\overline{\Omega}(\alpha))=-\frac{\overline{\Omega}^2}{4\pi}+\overline{\Omega}\alpha+\frac{\overline{\Omega}^2}{4\pi}\ln\frac{\overline{\Omega}^2}{\Omega_0{}^2} \tag{24}$$

and the $\Omega=\Omega(\alpha)$ relation is given by

$$0=\frac{\partial\mathscr{E}(\alpha,\Omega)}{\partial\Omega}\bigg|_{\Omega=\overline{\Omega}}=\frac{\overline{\Omega}}{2\pi}\ln\frac{\overline{\Omega}^2}{\Omega_0{}^2}+\alpha \tag{25}$$

which could also be obtained from (20). α should be kept positive as we found in (7). Let us note that (25) has solution only in the range $\Omega_0/e<\Omega<\Omega_0$ and therefore (24) is bounded from below. Nevertheless the minimum of $\mathscr{E}(\alpha,\overline{\Omega}(\alpha))$ could be provided by the operative end points $\Omega=0$ or $\Omega=\infty$. From (24) we discard $\Omega=\infty$ but $\Omega=0$ leads to

$$\mathscr{E}(\alpha,0)=0\,. \tag{26}$$

We conclude that for $\lambda_B\sim-1/\ln\Lambda^2$ a meaningful theory is recovered and we postpone its analysis to the next section. Let us note that the original GN sign appears in an elegant way within the Gaussian approach.

$\Omega_0=0$. In case the unique minimum corresponds to the extremum value $\Omega_0=0$, $V(0,\Omega_0)$ is a monotonous function. In order to have a bounded theory we need

$$\frac{\partial V(0,\Omega_0)}{\partial\Omega_0}>0\Longrightarrow\lambda_B>0$$

or

$$\frac{\partial V(0,\Omega_0)}{\partial\Omega_0}=0,\quad \frac{\partial^2 V(0,\Omega_0)}{\partial\Omega_0{}^2}>0\Longrightarrow\lambda_B>0\,. \tag{27}$$

The point $\Omega_0 \sim \Lambda$ is excluded by these restrictions. Working on (20) one finds Ω_0' and consequently λ_R:

$$\lambda_R = \lambda_B \frac{1 + \dfrac{2N-1}{2N^2}\lambda_B \left[I_0 - \dfrac{1}{2\pi} \right]}{1 + \dfrac{2N-1}{2N}\lambda_B \left[I_0 - \dfrac{1}{2\pi} \right]} . \tag{28}$$

Let us accept that $\lambda_B \sim a/\ln\lambda^2$ with $a>0$. In such a case

$$\lambda_R \sim \frac{1}{\ln\lambda^2} \tag{29}$$

which amounts to end with a trivial theory. In a similar way any other behavior of λ_B leads either to triviality or to a nonfinite theory but for $\lambda_B \sim$const. In this case we obtain

$$\lambda_R = \lambda_B/2N . \tag{30}$$

Ω is fixed by the variational criterion

$$0 = \frac{\partial \mathscr{E}(\alpha,\Omega)}{\partial \Omega} \Longrightarrow \overline{\Omega}(\alpha) = \frac{2\pi\alpha}{\ln \dfrac{4\lambda^2}{\overline{\Omega}^2(\alpha)}} + \frac{\text{const}}{\ln^2 \dfrac{4\lambda^2}{\overline{\Omega}^2(\alpha)}} \tag{31}$$

so that

$$\mathscr{E}(\alpha,\overline{\Omega}(\alpha)) = \frac{\lambda_B}{4N}\alpha^2 = \frac{\lambda_R}{2}\alpha^2 . \tag{32}$$

We have found a second relevant phase of the GN model. Here λ_R is finite and positive but

$$\lambda_R = \frac{\lambda_B}{2N} = \frac{{g_B}^2}{2} \tag{33}$$

so that ${g_B}^2 \sim_{N\to\infty} 1$ in order to obtain this nontrivial case. Had we taken ${g_B}^2 N \sim_{N\to\infty} 1$, λ_R would vanish in the $N\to\infty$ limit.

IV. PRECARIOUS PHASE

A. Precariousness

Let us go back to the first interacting case. Since λ_B satisfies Eq. (18), Ω_0 stands unrestricted and introduces a mass scale. This is the only free parameter of the theory. Once Ω_0 is fixed, all the remaining physical parameters take a definite value. Dimensionless quantities are then pure numbers, e.g., $\lambda_R = -\pi$. Such a feature was already found out by GN. We have plotted $\mathscr{E}(\alpha,\overline{\Omega}(\alpha))$ vs α in Fig. 1 where one can notice that at a certain point $\mathscr{E}(\alpha,\overline{\Omega}(\alpha))=0$ corresponds to the lower minimum of the potential.

The ground state is provided by $\alpha=0$ and $\Omega=\Omega_0$; then

$$\mathscr{E}_{\text{vac}} = \mathscr{E}(0,\Omega_0) = -\frac{{\Omega_0}^2}{4\pi} \tag{34}$$

which is lower than the perturbative vacuum. The effective potential for small α is given by

$$\mathscr{E}(\alpha,\overline{\Omega}(\alpha)) = -\frac{{\Omega_0}^2}{4\pi} + \Omega_0\alpha - \frac{\pi}{2}\alpha^2 - \frac{\pi^2}{6}\frac{\alpha^3}{\Omega_0} + \cdots . \tag{35}$$

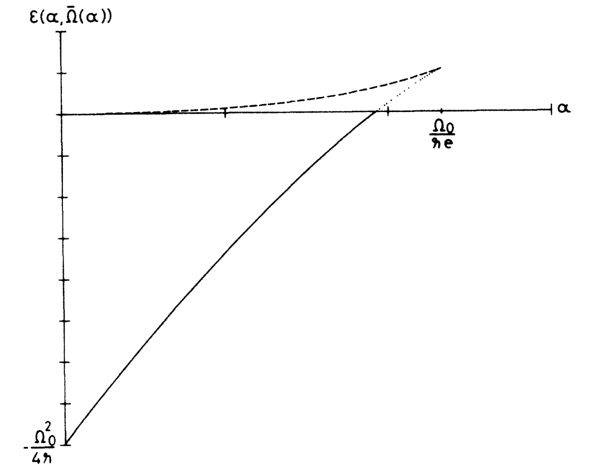

FIG. 1. Effective potential in terms of α. The dashed line corresponds to Ω in the range $0<\Omega<\Omega_0/e$ (unphysical). The solid line describes the Gaussian effective potential. Notice that beyond a certain α the Ω given by (25) is not the absolute minimum any more (dotted line).

Clearly Ω_0 corresponds to a dynamically generated mass for the fermion field. As a consequence chiral symmetry is dynamically broken. Let us consider

$$\langle g^2\overline{\psi}^a\psi^a\rangle_{\text{vac}} - \langle g^2\overline{\psi}^a\psi^a\rangle_{\text{pert}} = \frac{2N}{2N-1}\Omega_0 \tag{36}$$

which corresponds to the nonperturbative part of $\langle g^2\overline{\psi}\psi\rangle$. Let us stress that $\alpha=0$ does not imply a vanishing result since (36) picks up quantum contributions. The dimensionless quantity obtained from (35) and (36) is

$$\mathscr{E}_{\text{vac}}/\langle g^2\overline{\psi}\psi\rangle^2 = -\frac{(2N-1)^2}{16\pi N^2} \underset{N\to\infty}{\longrightarrow} -\frac{1}{4\pi} \tag{37}$$

in agreement with the GN result. Further comparisons with GN are extremely difficult since we have worked in a rather different way. They worked out the effective potential for the composite operator $\langle g\overline{\psi}\psi\rangle$ but sorrowfully it is not simply related to our calculations. Moreover our renormalization scheme is quite different. Only dimensionless, renormalization invariant quantities are susceptible to simple checking.

Asymptotic freedom appears in a rather deficient way. Beyond a certain α value the Gaussian effective potential becomes flat. There is no interaction.

Anybody familiar with this approach should realize the strong parallelism between $\lambda\phi^4$ in $D=4$ and the GN model. Both of them make sense in the $\lambda_B \sim -1/\ln\Lambda$ phase. An infinitesimal, negative bare coupling constant leads to a well-behaved λ_R. Because of the minus sign these phases remain unstudied within the current use of computers.[11] There is a subtle point in passing from (15) to (24). As far as the regularized theory is considered the $\lambda_B\alpha^2/2$ term has the form

$$-\frac{1}{\ln\Lambda^2}\alpha^2 . \tag{38}$$

Therefore large enough α leads to an unbounded theory. Nevertheless there is a syllogism. A regularized theory requires a cutoff for the fields which, by the way, is never introduced in the lattice approach. When the cutoff Λ is sent to infinity before any other limit is performed (even $N\to\infty$) the phase makes sense. What seems so unusual in $\lambda\phi^4$ turns out to be common wisdom for the GN model.

We think that this point supports widely the precarious phase discovered by Stevenson in $\lambda\phi^4$. The Gaussian approach seems to provide a very adequate tool for the analysis of the ground state of a QFT.

B. Bound states

In order to investigate the bound-state problem we have followed Refs. 4 and 2. The mass gap is given by

$$m_2=\frac{\langle 2|\mathcal{H}|2\rangle}{\langle 2|2\rangle}-{}_{\Omega_0}\langle 0|\mathcal{H}|0\rangle_{\Omega_0}, \tag{39}$$

where the two-particle scalar state is taken as

$$|2\rangle=\frac{1}{\sqrt{N}}\int\frac{dp^1}{2\pi 2\omega(p^1)}\sigma(p^1)a^{\dagger a}_{\Omega_0}(p^1)b^{\dagger a}_{\Omega_0}(-p^1)|0\rangle_{\Omega_0} \tag{40}$$

so that $m_2=m_2[\sigma]$ and its minimum value corresponds to the solution to

$$\frac{\delta m_2[\sigma]}{\delta\sigma(k)}=0\,. \tag{41}$$

A cumbersome calculation yields

$$m_2=\frac{2\pi}{\int dp\,\sigma^2(p)}\left[4\int\frac{dp}{2\pi 2\omega(p)}\omega^2(p)\sigma^2(p)+\frac{\lambda_B}{N}2\Omega_0{}^2\left[\int\frac{dp}{2\pi 2\omega(p)}\sigma(p)\right]^2\right.$$
$$\left.+\frac{2\lambda_B}{N}\left[\int\frac{dp}{2\pi 2\omega(p)}\sigma(p)\omega(p)\right]^2+\lambda_B\frac{2(2N-1)}{N}\left[\frac{dp}{2\pi 2\omega(p)}p\sigma(p)\right]^2\right]. \tag{42}$$

The variational equation is easily obtained

$$0=\frac{\delta m_2[\sigma]}{\delta\sigma(k)}\Longrightarrow\sigma(k)[2\omega(k)-m_2]+\frac{\Omega_0{}^2}{\omega(k)}C+B+\frac{k}{\omega(k)}A=0\,, \tag{43}$$

where

$$A\equiv\lambda_B\frac{2N-1}{N}\int\frac{dp}{2\pi 2\omega(p)}p\sigma(p),\quad B\equiv\frac{\lambda_B}{N}\int\frac{dp}{4\pi}\sigma(p),\quad C\equiv\frac{\lambda_B}{N}\int\frac{dp}{2\pi 2\omega(p)}\sigma(p)\,. \tag{44}$$

In the $N\to\infty$ limit only A survives and a possible ansatz for σ is

$$\sigma(k)=-\frac{k}{\omega(k)}\frac{A}{2\omega(k)-m_2}\,. \tag{45}$$

It is disappointing to check that divergencies in m_2 do not cancel as this solution is tried in (42) (a well-known factor 2 destroys the equality). The failure can be traced back to a consistency problem. Actually an absurd result is found when (45) is introduced in (44). Therefore the operative σ turns to be the "end function"

$$\sigma(k)=\text{const}\times\delta(2\omega(k)-m_2) \tag{46}$$

which leads to

$$m_2=2\Omega_0\,, \tag{47}$$

the correct result in the $N\to\infty$ limit.

When $N<\infty$ is considered the consistency problem remains and the solution is the same, in disagreement with other approximations.[12]

Similar features are found in $\lambda\phi^4$. One should accept that the description of bound states provided by the Gaussian approximation is poor. New ideas to build a systematic improvement are needed.

V. CONCLUSION

A powerful, variational Gaussian approach has been applied to the GN model. Two phases make sense. On the one hand, $\lambda_B\sim-1/\ln\Lambda^2$, $\lambda_R=-\pi$ leads to the usual theory. On the other hand, $\lambda_B=\text{const}>0$, $\lambda_R=\lambda_B/2N>0$ is also acceptable. This later phase disappears in the standard $N\to\infty$ limit.

Dynamical chiral-symmetry breaking is found in the former case in agreement with GN (we have noted that $\alpha=0$ is not equivalent to $\langle g^2\bar{\psi}\psi\rangle=0$). Nevertheless the Gaussian ansatz does not reproduce correctly bound states whereas asymptotic freedom is simulated by just freedom.

We argue that the precarious phase of $\lambda\phi^4$ corresponds exactly to the usual, widely accepted behavior of the GN model. Lattice experts could find a new interesting field of work within these lines.

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