Rigorous extension of the proof of zeta-function regularization

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(Received 3 January 1989)

The proof of ζ-function regularization of high-temperature expansions, a technique which provides correct results for many field-theoretical quantities of interest, is known to fail, however, in the case of "Epstein-type" expressions such as $\sum_{n_1,\ldots,n_N=1}^{\infty}(\sum_{j=1}^{N}a_j n_j^{\alpha})^{-s}$, $\alpha=2,4,\ldots$. After showing where precisely the existing demonstration breaks down, we provide a new proof of this regularization valid for a wider range of the parameter α. The extra terms are calculated explicitly for any value of $\alpha\leq 2$. As an application, we provide the finite results corresponding to the ζ-function regularization of expressions associated with field theories evaluated in partially compactified, toroidal spacetimes of the form $\mathbf{T}^p\times\mathbf{R}^{q+1}$.

I. INTRODUCTION

Few physicists would nowadays argue against the statement that the ζ-function regularization procedure has proven to be a very powerful and elegant technique. Its range of applicability is quickly expanding. Let us briefly mention Actor's program of associating ζ functions with Feynman loop diagrams having any number of external lines, corresponding to field theories formulated on spacetimes with partial toroidal compactification.[1] The power of the diagram ζ function lies in its ability to provide an exact power-series expansion in mass and external momentum of the corresponding regularized diagram. Worthy of being commented is also its relevance to summation methods in issues concerning the problem of mass generation for the fermionic Gross-Neveu model.[2]

However, it must be recognized that the ζ-function regularization procedure has important limitations. We will here concentrate only on the ones associated with the non-naive interchangeability of the order of the summations of infinite series. These series appear, for instance, when performing the high-temperature expansion of different field-theoretical quantities, and the interchange of the order of the summations is necessary if one wants to collect common powers of the temperature and to obtain, finally, a series expansion in it. The same would happen in the aforementioned case of expansions in terms of the mass and the internal momentum or when trying to express the energy density of the Casimir effect (obtained by direct summation of the zero modes) in powers of plate separation and field mass. This only to mention a few examples where the technique is clearly useful.

A most interesting result concerning the interchange of order of the summations of infinite series appearing in ζ-function regularization is due to Weldon.[3] His investigation originated in some difficulties which appeared in a paper by Actor,[4] when he tried to obtain the value of the thermodynamical potential corresponding to a relativistic Bose gas by using the ζ-function regularization procedure.

Weldon's proof of ζ-function regularization of high-temperature expansions has subsequently been used in several interesting examples,[5,6] and has been redone in Ref. 6 in great detail. Unfortunately, Weldon's proof has its own limitations, and some statements in Ref. 3 concerning the extent of its validity are actually not right. This is not difficult to check in some particular cases, and has been stressed in Ref. 6.

The purpose of this work is to extend the proof of ζ-function regularization, in a sense that will later be described, by finding the new counterterms that were absent in Ref. 3, and also to provide applications of the extended proof. The paper is organized as follows. In Sec. II we summarize the proof in Ref. 3, pointing out its limitations. In Sec. III we extend the proof to more general infinite series and explicitly provide the additional counterterms in some cases. Finally, in Sec. IV we apply our results to the case of the "Epstein-type" ζ functions. We perform, in particular, the direct calculation of the Casimir effect corresponding to a massless scalar field in $\mathbf{T}^p\times\mathbf{R}^{q+1}$ spacetime by summing first over the zero modes and ζ regularizing the multiple series.

II. THE PROOF OF ζ-FUNCTION REGULARIZATION AND ITS LIMITATIONS

We shall here briefly summarize the proof due to Weldon of the validity of the ζ-function regularization procedure[3] and point out its limitations. Using the same notation as in Ref. 3, let us consider the series

$$S_F=\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m^{s+1}}\sum_{a=0}^{\infty}m^a f(a)\ , \tag{2.1}$$

$$S_B=\sum_{m=1}^{\infty}\frac{1}{m^{s+1}}\sum_{a=0}^{\infty}m^a f(a)\ , \tag{2.2}$$

$$S_{AF}=\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m^{s+1}}\sum_{a=0}^{\infty}(-1)^a m^a f(a)\ , \tag{2.3}$$

$$S_{AB}=\sum_{m=1}^{\infty}\frac{1}{m^{s+1}}\sum_{a=0}^{\infty}(-1)^a m^a f(a)\ , \tag{2.4}$$

where $f(a)\geq 0$ for positive integer a. They are assumed to be convergent, as they stand. The idea of the ζ-function regularization procedure begins with the interchange of the order of the summations of the two infinite series involved. It has been proven in Ref. 3 that, provided that $f(a)$ can be defined in the complex a plane and that it satisfies (1) $f(a)$ is regular for $\mathrm{Re}a \geq 0$; in the case of (2.1) and (2.2), (2a) $am^a f(a)\to 0$, as $|a|\to\infty$ for $\mathrm{Re}a\geq 0$ and fixed m; and, in the case of (2.3) and (2.4), (2b) $am^a f(a)e^{-\pi|\mathrm{Im}a|}\to 0$, as $|a|\to\infty$ for $\mathrm{Re}a\geq 0$ and fixed m, then it turns out that in the fermionic cases, (2.1) and (2.3), one can naively interchange the order of the summations to get

$$S_F=\sum_{a=0}^{\infty}\eta(s+1-a)f(a)\ , \qquad (2.5)$$

$$S_{AF}=\sum_{a=0}^{\infty}(-1)^a\eta(s+1-a)f(a)\ , \qquad (2.6)$$

while in the bosonic cases one obtains the additional contributions

$$S_B=\sum_{a=0}^{\infty}\zeta(s+1-a)f(a)-\pi\cot(\pi s)f(s),\quad s\notin\mathbf{N}\ , \qquad (2.7)$$

$$S_B=\sum_{\substack{a=0\\a\neq s}}^{\infty}\zeta(s+1-a)f(a)+\gamma f(s)-f'(s),\quad s\in\mathbf{N}\ , \qquad (2.8)$$

and

$$S_{AB}=\sum_{a=0}^{\infty}(-1)^a\zeta(s+1-a)f(a)$$
$$-\pi\csc(\pi s)f(s),\quad s\notin\mathbf{N}\ , \qquad (2.9)$$

$$S_{AB}=\sum_{\substack{a=0\\a\neq s}}^{\infty}(-1)^a\zeta(s+1-a)f(a)$$
$$+(-1)^s[\gamma f(s)-f'(s)],\quad s\in\mathbf{N}\ , \qquad (2.10)$$

respectively.

Here $\zeta(s)$ and $\eta(s)$ are the Riemann ordinary and alternating ζ functions

$$\zeta(s)=\sum_{m=1}^{\infty}m^{-s},\quad \mathrm{Re}s>1\ , \qquad (2.11)$$

and

$$\eta(s)=\sum_{m=1}^{\infty}(-1)^{-s}m^{-s},\quad \mathrm{Re}s>0\ , \qquad (2.12)$$

respectively. They are related by

$$\eta(s)=(1-2^{1-s})\zeta(s)\ . \qquad (2.13)$$

On the other hand, γ is the Euler-Mascheroni constant and $f'(s)$ means the derivative of f with respect to s.

The proof of the preceding theorem proceeds by integration in the complex a plane. One writes (2.1)–(2.4) in the form of contour integrals

$$S_F=\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m^{s+1}}\oint_C\frac{da}{2i}m^a f(a)\cot(\pi a)\ , \qquad (2.14)$$

$$S_B=\sum_{m=1}^{\infty}\frac{1}{m^{s+1}}\oint_C\frac{da}{2i}m^a f(a)\cot(\pi a)\ , \qquad (2.15)$$

$$S_{AF}=\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m^{s+1}}\oint_C\frac{da}{2i}m^a f(a)\csc(\pi a)\ , \qquad (2.16)$$

$$S_{AB}=\sum_{m=1}^{\infty}\frac{1}{m^{s+1}}\oint_C\frac{da}{2i}m^a f(a)\csc(\pi a)\ , \qquad (2.17)$$

where C is the closed contour defined by the straight line $\mathrm{Re}a=-a_0$, for fixed a_0 such that $0<a_0<1$, and by the semicircumference at infinity on the right (Fig. 1).

The contribution from the semicircumference is zero in every case, due to the asymptotic behavior of $f(a)$ and, as long as $\mathrm{Re}s>-1$, the integral extended to the line $\mathrm{Re}a=-a_0$ can be interchanged with the remaining sum over m. The final step is to close the contour C again with the semicircumference at infinity. In cases (2.15) and (2.17) there comes then an additional contribution from the pole of the ζ function $\zeta(s+1-a)$ at $a=s$. On the contrary, in cases (2.14) and (2.16) the alternating ζ function $\eta(s+1-a)$ has no pole in the region enclosed by C. All the steps in this procedure are right and one thus obtains (2.5)–(2.10).

However, it was further explicitly stated by Weldon in Ref. 3 that the results for the alternating fermionic and for the alternating bosonic cases S_{AF} and S_{AB}, respectively, could be naively extended to the following types of series:

$$S_{AF}^{(N)}=\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m^{s+1}}\sum_{a=0}^{\infty}(-1)^a m^{Na}f(a) \qquad (2.18)$$

and

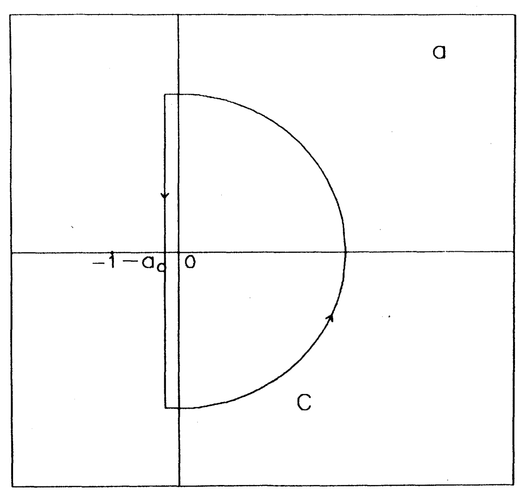

FIG. 1. The closed contour C consists of the straight line $\mathrm{Re}a=-a_0$, $0<a_0<1$, and of the semicircumference at infinity on the right of it.

$$S_{AB}^{(N)}=\sum_{m=1}^{\infty}\frac{1}{m^{s+1}}\sum_{a=0}^{\infty}(-1)^a m^{Na} f(a)\ , \qquad (2.19)$$

with N *any* positive integer. By going over the same proof once more, he just obtained the trivial modification of the results (2.6), (2.9), and (2.10). In particular, for the alternating bosonic case he got

$$S_{AB}^{(N)}=\sum_{a=0}^{\infty}(-1)^a\zeta(s+1-Na)f(a) - \frac{\pi}{N}\csc\left[\frac{\pi s}{N}\right]f\left[\frac{s}{N}\right],\quad \frac{s}{N}\notin \mathbf{N}\ , \qquad (2.20)$$

$$S_{AB}^{(N)}=\sum_{\substack{a=0\\ a\neq s/N}}^{\infty}(-1)^a\zeta(s+1-Na)f(a) +(-1)^{s/N}\left[\gamma f\left[\frac{s}{N}\right]-\frac{1}{N}f'\left[\frac{s}{N}\right]\right],\quad \frac{s}{N}\in\mathbf{N}\ . \qquad (2.21)$$

That this generalization of (2.9) and (2.10) for any positive integer N is not right is very easy to check. In particular, it has been noticed by Actor in Ref. 6. As an easy example let us consider the simplest case after the (only correct) one $N=1$ (explicitly considered in Ref. 3), i.e., $N=2$. Let

$$S\equiv\sum_{m=1}^{\infty}e^{-m^2}=\sum_{m=1}^{\infty}\frac{1}{m^{s+1}}\sum_{a=0}^{\infty}(-1)^a\frac{m^{2a}}{a!}\Bigg|_{s=-1}\ , \qquad (2.22)$$

where the last operation consists in making the analytic continuation of the resulting series to $s=-1$. The function $f(a)$ is here $f(a)=1/\Gamma(a+1)$ and all the hypotheses of the theorem are satisfied. Use of Weldon's formula (2.20) gives

$$S=\sum_{a=0}^{\infty}\frac{(-1)^a}{a!}\zeta(-2a)-\frac{(\pi/2)\csc(-\pi/2)}{\Gamma(1-\frac{1}{2})} =-\frac{1}{2}+\frac{\sqrt{\pi}}{2}\ , \qquad (2.23)$$

which is false, though numerically almost undetectable. In fact

$$S=0.386\,318\,6,\quad \frac{\sqrt{\pi}-1}{2}=0.386\,226\,9\ ,$$
$$\Delta=\frac{\sqrt{\pi}-1}{2}-S=-9.17\times10^{-5}\ . \qquad (2.24)$$

Going on to $N=2,3,4,\ldots$, it is not difficult to see that, if N is constrained to be a positive integer, Eq. (2.20) is true only for $N=1$ [i.e., Eq. (2.9)].

It was Actor's conjecture that, even if we allowed N to be any positive real number, then also (2.20) and (2.21) would be true only for $N\leq 1$. In the next section we shall prove that this is not the case either. Another important question one would like to answer is "where is exactly the error in Weldon's proof?" This has been dealt with in Ref. 6, Appendix C. However, the essential point has been overlooked [see the statement between Eqs. (C8) and (C9) in Appendix C of Ref. 6]. In fact, the step which fails to be correct in Weldon's proof for general N is *the last one*, namely, even if the asymptotic behavior (2b) of the function $f(a)$ allows us to suppress the contribution from the curved contour in the second step, this will be no longer true when we try to close again the circuit C in the last step. There is in fact a contribution coming from the integral of $\zeta(s+1-Na)f(a)$ over the semicircumference at infinity (due to the asymptotic behavior of the ζ function). And this is so whatever the value we choose for s. The study of the asymptotic behavior of $\zeta(s+1-Na)$ immediately distinguishes the case $N\leq 1$ from $N>1$. It is, however, misleading in some sense, because the fact that the ζ function diverges for $N>1$ does *not* necessarily mean that the contour actually provides a nonzero contribution invalidating Weldon's proof (that was Actor's conjecture). Things must be done with greater care due to the presence of highly oscillating factors.

III. EXTENSION OF THE PROOF OF ζ-FUNCTION REGULARIZATION: ADDITIONAL CONTRIBUTIONS

We shall restrict ourselves to the case $f(a)=1/\Gamma(a+1)$. This will be enough for the application to the Casimir effect to be developed in Sec. IV. In this case a slight modification of Weldon's procedure is in order. The fact that the poles of Γ are the nonpositive integers, and suitable application of the ζ function reflection formula, will allow us to write the additional contribution as a contour integral over a curved path in the complex left half-plane. Besides, the use of the relation

$$\Gamma(z/2)\zeta(z)=\int_0^{\infty}dt\, t^{z/2-1}S_2(t),\quad \mathrm{Re}z>0\ , \qquad (3.1)$$

where $S_2(t)$ is defined through (3.6), will be crucial. Otherwise, for a general f, there will be no way of kicking out the $\csc(\pi a)$ factor. Whether or not the technique here developed might be appropriately modified so as to work in such cases is left for the reader as a matter of further thought which is beyond the scope of the present work. Owing to the behavior of the complex function $\Gamma(z)$ which has simple poles at $z=-n$ for $n=0,1,2,\ldots$, with residues

$$\mathrm{Res}_{z=-n}\Gamma(z)=\frac{(-1)^n}{n!}\ , \qquad (3.2)$$

we can write

$$S_{AB}^{(\alpha)}\equiv\sum_{m=1}^{\infty}\frac{1}{m^{s+1}}\sum_{a=0}^{\infty}(-1)^a\frac{m^{\alpha a}}{\Gamma(a+1)},\quad \alpha\in\mathbf{R}\ , \qquad (3.3)$$

as

$$S_{AB}^{(\alpha)}\equiv\sum_{m=1}^{\infty}\frac{1}{m^{s+1}}\oint_{\bar{C}}\frac{da}{2\pi i}m^{-\alpha a}\Gamma(a)\ , \qquad (3.4)$$

where now the contour $\bar{C}$ consists of the line $\mathrm{Re}a=a_0$, with a_0 fixed, $0<a_0<1$, and of the semicircumference at infinity on the left (Fig. 2). Given that the case of special

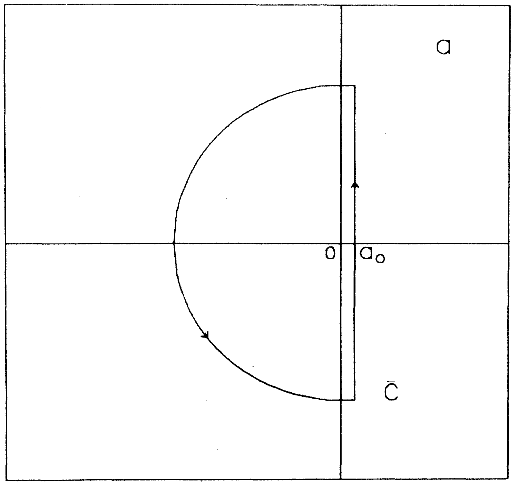

FIG. 2. The closed contour $\bar{C}$ consists of the straight line $\mathrm{Re}a=a_0$, $0<a_0<1$, and of the semicircumference at infinity on its left.

interest will be the one associated with $s=-1$, it is appropriate to rewrite (3.4) in the following manner:

$$S_{AB}^{(\alpha)}(s=-1)=\sum_{m=1}^{\infty}\sum_{a=0}^{\infty}(-1)^a\frac{m^{\alpha a}}{a!}=\sum_{m=1}^{\infty}e^{-m^{\alpha}}=S_{\alpha}(1)\ , \tag{3.5}$$

where

$$S_{\alpha}(t)\equiv\sum_{m=1}^{\infty}e^{-m^{\alpha}t}\ . \tag{3.6}$$

Let us go through the same steps as in Weldon's proof (see Sec. II). By correctly making the last step, we end up with

$$S_{AB}^{(\alpha)}=\sum_{a=0}^{\infty}\frac{(-1)^a}{a!}\zeta(s+1-\alpha a)+\frac{1}{\alpha}\Gamma\left[-\frac{s}{\alpha}\right]-\Delta_{AB}^{(\alpha)},\quad \frac{s}{\alpha}\notin\mathbf{N}\ , \tag{3.7}$$

$$S_{AB}^{(\alpha)}=\sum_{\substack{a=0\\ a\neq s/a}}^{\infty}\frac{(-1)^a}{a!}\zeta(s+1-\alpha a)+(-1)^{s/\alpha}\left[\frac{\gamma}{\Gamma(s/\alpha+1)}-\frac{1}{\alpha\Gamma'(s/\alpha+1)}\right]-\Delta_{AB}^{(\alpha)},\quad \frac{s}{\alpha}\in\mathbf{N}\ , \tag{3.8}$$

where $\Delta_{AB}^{(\alpha)}$ is the contribution of the curved part of the contour $\bar{C}$:

$$\Delta_{AB}^{(\alpha)}\equiv\int_C\frac{da}{2\pi i}\zeta(s+1+\alpha a)\Gamma(a)\ . \tag{3.9}$$

This contribution is *not* zero for any value of s. We are going to see that it actually provides the term missing from (2.23).

Before proceeding to the actual calculation of (3.9), one can, as an illustrating exercise, reclose the contour on the right instead of the left, and verify that the same series is obtained.

Let us now come back to Eq. (3.9) and proceed similarly for $s=-1$ and $\alpha=2$. Now, we cannot employ (3.1) unless we make use first of the reflection formula[7]

$$\Gamma(z/2)\zeta(z)=\pi^{z-1/2}\Gamma\left[\frac{1-z}{2}\right]\zeta(1-z)\ . \tag{3.10}$$

We thus find

$$\begin{aligned}\Delta_{AB}^{(2)}(s=-1)&=\int_C\frac{da}{2i\sqrt{\pi}}\int_0^{\infty}dt\,t^{-a-1/2}S_2(\pi^2t)=-\frac{1}{\sqrt{\pi}}\lim_{R\to\infty}\int_0^{\infty}dt\,t^{-a_0-1/2}S_2(\pi^2t)\frac{\sin(R\ln t)}{\ln t}\\&=-\frac{1}{\sqrt{\pi}}\mathrm{Im}\lim_{R\to\infty}\int_{-\infty}^{\infty}du\,e^{-(a_0-1/2)u/R}S_2(\pi^2e^{u/R})\frac{e^{iu}}{u}\\&=-\sqrt{\pi}S_2(\pi^2)\ .\end{aligned} \tag{3.11}$$

Another way of writing this is

$$S_2(1)=-\frac{1}{2}+\frac{\sqrt{\pi}}{2}+\sqrt{\pi}S_2(\pi^2)\ . \tag{3.12}$$

Surprising as it may look, this result happens to be nothing but a particular case of the famous θ-function identity

$$\theta(z,\tau)=\tau^{-1/2}e^{\pi z^2/\tau}\theta\left[\frac{z}{i\tau},\frac{1}{\tau}\right]\ , \tag{3.13}$$

with θ being the elliptic function

$$\theta(z,\tau)=\sum_{n=-\infty}^{\infty}e^{-\pi n^2\tau+2\pi nz},\quad z\in\mathbf{C},\quad \tau\in\mathbf{R}^+ \tag{3.14}$$

($\mathbf{R}^+$ denotes the positive real numbers). This can easily be checked by noticing that $S_2(\pi t)=\frac{1}{2}[\theta(0,t)-1]$. Equation (3.11) is an exact value. Once more, we observe that the contribution of the contour provides, in fact, the missing term in (2.20), (2.21), (3.7), and (3.8).

Let us now again consider (3.2) for *general* α, and, we must insist, $s=-1$. Equations (3.1) and (3.6) read, in this case,

$$\Gamma(z)\zeta(\alpha z)=\int_0^{\infty} dt\, t^{z-1} S_\alpha(t) , \qquad (3.15)$$

with $S_\alpha(t)$ being the function given in (3.6). However, no simple reflection formula similar to (3.10) exists for $\alpha\neq 2$. We have, instead,

$$\Gamma(z)=\frac{\pi \csc(\pi z)}{\Gamma(1-z)} , \qquad (3.16)$$

$$\zeta(\alpha z)=\frac{2\Gamma(1-\alpha z)}{(2\pi)^{1-\alpha z}}\sin\left[\frac{\pi\alpha z}{2}\right]\zeta(1-\alpha z) . \qquad (3.17)$$

Consider (3.7) and (3.8) for $s=-1$:

$$S_\alpha \equiv S_\alpha(1)=\sum_{m=1}^{\infty} e^{-m^\alpha} = \sum_{a=0}^{\infty}\frac{(-1)^a}{a!}\zeta(-\alpha a)+\frac{1}{\alpha}\Gamma\left[\frac{1}{\alpha}\right]-\Delta_\alpha , \qquad (3.18)$$

being the contribution of the contour

$$\Delta_\alpha=\int_{\subset}\frac{da}{2\pi i}\zeta(\alpha a)\Gamma(a) . \qquad (3.19)$$

Using (3.16), (3.17), and (3.15) we can write, after some work,

$$\Delta_\alpha=\int_{\subset}\frac{da}{i\sqrt{2\pi\alpha}}\phi_\alpha(a)\int_0^{\infty} dt\, t^{(1/\alpha)-a-1} S_\alpha\left[\left[\frac{2\pi}{\alpha}\right]^\alpha t\right] , \qquad (3.20)$$

where the function $\phi_\alpha(a)$ comes from the asymptotic behavior of the integrand (3.19) [see (3.16) and (3.17)] for $|a|\to\infty$:

$$\phi_\alpha(a)=\exp\left\{\left[(2-\alpha)+\left[\frac{1}{2}-\frac{1}{\alpha}\right]\right]\ln(-a) +(\alpha-2)a+\left[\frac{\alpha}{2}-1\right]\pi|\mathrm{Im}a| +\mathrm{sgn}(\mathrm{Im}a)i\pi\left[\frac{\alpha}{2}-1\right]|\mathrm{Re}a|\right\} . \qquad (3.21)$$

It is immediate that, for $\alpha=2$,

$$\phi_2(a)=1 , \qquad (3.22)$$

so (3.20) is in agreement with (3.11). Note also that for $\alpha<2$ we have

$$\phi_\alpha(a)\to 0, \quad |a|\to\infty . \qquad (3.23)$$

For the sake of completeness, we quote the following result obtained as a by-product of this instructive exercise. When, putting $\phi_\alpha(a)=1$ in (3.20), the remaining integral is *finite* and yields (see the proof for $\alpha=2$ above)

$$\tilde{\Delta}_\alpha\equiv\Delta_\alpha(\phi_\alpha=1)=-\left[\frac{2\pi}{\alpha}\right]^{1/2} S_\alpha\left[\left[\frac{2\pi}{\alpha}\right]^\alpha\right] . \qquad (3.24)$$

It is also important to notice that, in (3.20) the factors of quickest oscillation in $\phi_\alpha(a)$ and in the function of a defined by the t integral have an analogous functional dependence on $R\equiv|a|$, namely, $\exp(icR\ln R\sin\theta)$, where $\theta\equiv\mathrm{Arg}a$ and c is a constant.

Collecting everything together, it becomes clear that we have proven the following.

Theorem. (i) For $-\infty<\alpha<2$, the contribution of the semicircumference at infinity is zero, i.e.,

$$\Delta_\alpha=0, \quad \alpha<2 . \qquad (3.25)$$

(ii) For $\alpha=2$, the contribution of the semicircumference at infinity is given by

$$\Delta_2=-\sqrt{\pi}S_2(\pi^2) . \qquad (3.26)$$

The result for $\alpha\leq 1$ was already known and constitutes Weldon's proof of ζ-function regularization. The result for $\alpha=2$ shows, on the contrary, that the statements in Ref. 3 about the validity of the proof for any positive integer α were false, the (very simple) reason being that the semicircumference at infinity does not have a zero contribution. It was precisely the last step of the proof in Ref. 3 that was wrong. This had not been clearly appreciated in Ref. 6, on the contrary, the same false statement as in Ref. 3 was repeated (see Appendix C). Finally, the result of our theorem for $1<\alpha<2$ shows that the conjecture by Actor that the validity of Weldon's proof would be restricted to $\alpha\leq 1$ does not hold. Notice, by the way, that the fact that, for $\alpha>2$, $\phi_\alpha(a)$ diverges as $|a|\to\infty$ does *not* mean that Δ_α is going to blow up for these values of α. In fact, the strong oscillation [imaginary exponential in $\phi_\alpha(a)$] of the values of $\phi_\alpha(a)$ for $|a|\to\infty$ yields a finite value for the integral (3.20). However, it is not easy to obtain such a simple result as (3.26) for any value of α.

The fact that the numerical value of Δ_α is so small [it can be thought of as an infinitesimal correction, see (3.26)] as compared with the rest of the terms in Eqs. (3.7) and (3.8) gives sense to the whole procedure of ζ-function regularization.

However, this is strictly true only for small α. In fact, by working on Eq. (3.20) it is not difficult to extract an accurate estimation for the additional terms Δ_α, for large α. It is given by

$$\Delta_\alpha=-\left[1-\frac{2}{e}\right]\frac{\pi}{\alpha}S_\alpha\left[\left[\frac{2\pi}{\alpha}\right]^\alpha\right] . \qquad (3.27)$$

We observe that, for large α, Δ_α ceases to be an infinitesimal contribution. Actually,

$$\Delta_\alpha=0, \quad \alpha<2 ,$$
$$\Delta_2=9.17\times 10^{-5}, \quad \Delta_4=0.04, \quad \Delta_6=0.07 , \qquad (3.28)$$
$$\Delta_\alpha\to 0.13, \quad \alpha\to\infty ,$$

which represent, respectively, contributions of the 0%, 0.02%, 11%, 19%, and 36% on the whole value of $S_\alpha(1)$.

IV. THE CASIMIR EFFECT FOR A MASSLESS SCALAR FIELD IN $\mathbf{T}^p\times\mathbf{R}^{q+1}$

As a direct application of the ζ-function regularization theorem we have just proven, let us obtain the Casimir effect through direct summation of the zero modes. For simplicity, we shall only consider the case of a massless scalar field in a partially compactified $(p+q+1)$-dimensional spacetime $\mathbf{T}^p\times\mathbf{R}^{q+1}$ ($\mathbf{T}^p$ is a p-dimensional torus). It will be immediate, however, that our method can be generalized to very different massless or massive fields in different spacetimes with different kinds of boundary conditions.[8,9] There are general expressions which relate the different cases.[8]

As is known, the vacuum-energy density corresponding to a massless scalar field $\phi(t;x_1,\dots,x_{p+q})$ satisfying the periodic boundary conditions

$$\phi(t;x_1,\dots,x_{p+q})=\phi(t;x_1+L_1,\dots,x_p+L_p,x_{p+1},\dots,x_{p+q}) \quad (4.1)$$

can be obtained[8] from the (more direct to deal with) Dirichlet case

$$\epsilon_0=\tfrac{1}{2}(2\pi)^{-q}\left[\prod_{j=1}^{p}L_j\right]^{-1}\times\int d^q\mathbf{k}\sum_{n_1,\dots,n_p=1}^{\infty}\left[\mathbf{k}^2+\sum_{j=1}^{p}\left[\frac{\pi n_j}{L_j}\right]^2\right]^{1/2}. \quad (4.2)$$

In terms of the general multiple series

$$M(s;a_1,\dots,a_N;\alpha_1,\dots,\alpha_N)\equiv\sum_{n_1,\dots,n_N=1}^{\infty}\left[\sum_{j=1}^{n}a_jn_j^{\alpha_j}\right]^{-s}, \quad (4.3)$$

and defining

$$\epsilon_0(s)=\tfrac{1}{2}(2\pi)^{-q}\left[\prod_{j=1}^{p}L_j\right]^{-1}\frac{2\pi^{q/2}}{\Gamma(q/2)}\times\int_0^\infty dk\,k^{q-1}\times\sum_{n_1,\dots,n_p=1}^{\infty}\left[k^2+\sum_{j=1}^{p}\left[\frac{\pi n_j}{L_j}\right]^2\right]^{-s/2}, \quad (4.4)$$

after some calculations we obtain

$$\epsilon_0(s)=2^{-q-1}\pi^{q/2-s}\left[\prod_{j=1}^{p}L_j\right]^{-1}\frac{\Gamma(-(q-s)/2)}{\Gamma(s/2)}M\times\left[\frac{s-q}{2};L_1^{-2},\dots,L_p^{-2};2,\dots,2\right]. \quad (4.5)$$

Now, by representing (4.3) in the integral form

$$M(s;a_1,\dots,a_N;\alpha_1,\dots,\alpha_N)=\frac{1}{\Gamma(s)}\sum_{n_1,\dots,n_N=1}^{\infty}\int_0^\infty dt\,t^{s-1}\exp\left[-t\sum_{j=1}^{N}a_jn_j^{\alpha_j}\right], \quad (4.6)$$

and by making repeated use of the ζ-function regularization theorem from Sec. III, in the manner

$$S_2(t)=\sum_{n=1}^{\infty}e^{-n^2t}=\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\frac{(-n^2t)^k}{k!}=\sum_{k=0}^{\infty}\zeta(-2k)\frac{(-t)^k}{k!}+\frac{1}{2\sqrt{t}}\Gamma(\tfrac{1}{2})+\left[\frac{\pi}{t}\right]^{1/2}S_2\left[\frac{\pi^2}{t}\right]$$
$$=\frac{1}{2}\left[\left[\frac{\pi}{t}\right]^{1/2}-1\right]+\left[\frac{\pi}{t}\right]^{1/2}S_2\left[\frac{\pi^2}{t}\right] \quad (4.7)$$

(recall that the last term is a very small correction to the others), we obtain for

$$M_N(s;a_1,\dots,a_N)\equiv M(s;a_1,\dots,a_N;2,\dots,2), \quad (4.8)$$

the recurrence

$$M_N(s;a_1,\dots,a_N)=-\tfrac{1}{2}M_{N-1}(s;a_2,\dots,a_N)+\frac{1}{2}\left[\frac{\pi}{a_1}\right]^{1/2}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}M_{N-1}(s-\tfrac{1}{2};a_2,\dots,a_N),$$
$$+\left[\frac{\pi}{a_1}\right]^{1/2}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\Delta_N^{(1)}(s-\tfrac{1}{2};\pi^2/a_1;a_2,\dots,a_N), \quad (4.9)$$

where the third term $\Delta_N^{(1)}$ is a small correction to the first two. It is given, in general, by the expression

$$\Delta_N^{(j)}(s;a_1,\ldots,a_j;a_{j+1},\ldots,a_N)=\sum_{n_1,\ldots,n_N=1}\frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\exp\left[-\left(\frac{a_1n_1^2}{t}+\cdots+\frac{a_jn_j^2}{t}+a_{j+1}n_{j+1}^2t+\cdots+a_Nn_N^2t\right)\right]. \quad (4.10)$$

A recurrence for $\Delta_N^{(j)}$ is also easy to obtain

$$\begin{aligned}\Delta_N^{(j)}(s;a_1,\ldots,a_j;a_{j+1},\ldots,a_N)=&-\tfrac{1}{2}\Delta_{N-1}^{(j)}(s;a_1,\ldots,a_j;a_{j+2},\ldots,a_N)\\&+\frac{1}{2}\left[\frac{\pi}{a_{j+1}}\right]^{1/2}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\Delta_{N-1}^{(j)}(s-\tfrac{1}{2};a_1,\ldots,a_j;a_{j+2},\ldots,a_N)\\&+\left[\frac{\pi}{a_{j+1}}\right]^{1/2}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\Delta_N^{(j+1)}\left[s-\tfrac{1}{2};a_1,\ldots,a_j,\frac{\pi^2}{a_{j+1}};a_{j+2},\ldots,a_N\right]. \quad (4.11)\end{aligned}$$

We have, in particular,

$$\Delta_N^{(N)}(s;a_1,\ldots,a_N)=\frac{\Gamma(-s)}{\Gamma(s)}M_N(-s;a_1,\ldots,a_N)\ . \quad (4.12)$$

The first terms of these recurrences are given by

$$M_1(s;a)=\sum_{n=1}^{\infty}(an^2)^{-s}=a^{-s}\zeta(2s)\ , \quad (4.13)$$

$$\Delta_1^{(1)}(s;a)=\frac{\Gamma(-s)}{\Gamma(s)}a^s\zeta(-2s)\ , \quad (4.14)$$

$$\begin{aligned}M_2(s;a_1,a_2)=&-\tfrac{1}{2}a_2^{-s}\zeta(2s)+\tfrac{1}{2}\pi^{1/2}a_1^{-1/2}a_2^{1/2-s}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\zeta(2s-1)\\&-\tfrac{1}{2}\pi^{2s-1/2}a_1^{-s}\frac{\Gamma(\frac{1}{2}-s)}{\Gamma(s)}\zeta(1-2s)+\tfrac{1}{2}\pi^{2s-1}a_1^{1/2-s}a_2^{-1/2}\frac{\Gamma(1-s)}{\Gamma(s)}\zeta(2-2s)\\&+\frac{\pi^{2s-1}}{\sqrt{a_1a_2}}\frac{\Gamma(1-s)}{\Gamma(s)}M_2\left[1-s;\frac{1}{a_1},\frac{1}{a_2}\right]. \quad (4.15)\end{aligned}$$

Finally, substituting (4.8) into (4.5) for $s=-1$, we obtain the ζ-function regularized value for the Casimir density corresponding to a massless scalar field between p pairs of parallel plates with Dirichlet boundary conditions. It is given by

$$\epsilon=2^{-q-1}\pi^{q/2+1}\left[\prod_{j=1}^{p}L_j\right]^{-1}\frac{\Gamma(-(q+1)/2)}{\Gamma(-1/2)}\times M_p\left[-\frac{q+1}{2};L_1^{-2},\ldots,L_p^{-2}\right], \quad (4.16)$$

where M_p is to be calculated through the recurrence (4.9). From these results one immediately obtains the ones corresponding to a massless scalar field in $\mathbf{T}^p\times\mathbf{R}^{q+1}$ (Ref. 8).

When $p=1$ what happens is that we are compactifying on a circle. Considering $q=0$, 1, 2, 3, and 4, Eqs. (4.13) and (4.16) yield the results for the Casimir energy density corresponding to a massless scalar field in $\mathbf{S}^1\times\mathbf{R}$, $\mathbf{S}^1\times\mathbf{R}^2$, $\mathbf{S}^1\times\mathbf{R}^3$, $\mathbf{S}^1\times\mathbf{R}^4$, and $\mathbf{S}^1\times\mathbf{R}^5$, respectively,[8] i.e.,

$$\begin{aligned}\epsilon(p=1,q=0)&=-\frac{\pi}{6L^2}\ ,\\ \epsilon(p=1,q=1)&=-\frac{\zeta(3)}{2\pi L^3}\ ,\\ \epsilon(p=1,q=2)&=-\frac{\pi^2}{90L^4}\ ,\\ \epsilon(p=1,q=3)&=-\frac{3\zeta(5)}{4\pi^2L^5}\ ,\\ \epsilon(p=1,q=4)&=-\frac{2\pi^3}{945L^6}\ ,\end{aligned} \quad (4.17)$$

where L is the only compactification distance, which is, in this case, the length of the circumference.

This illustrates the uses one can make of the extended theorem of ζ-function regularization. Of course, things become much easier when the exponent α is strictly smaller than 2. When this is so, the correction terms [e.g., the last one in (4.9)] do not appear and the recursion formulas are certainly simpler. Closed and explicit expressions are given elsewhere[10] for those cases.

ACKNOWLEDGMENTS

This problem was posed to us by Professor A. Actor. We are very thankful to him for his encouragement. Partial economic support has been received from Comisión Asesora de Investigación Científica y Técnica (CAICYT) Contract No. AE87-0016-3. One of us, A.R., has also obtained financial help through an FPPI (Formación de Profesorado y Personal Investigador) grant from Ministério de Educación y Ciencia (Spain).

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