Classical motions from pseudoclassical spin-$\frac{1}{2}$ particle models

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The classical trajectory and spin precessions of Bargmann, Michel, and Telegdi are deduced from a pseudoclassical model of a relativistic spin-$\frac{1}{2}$ particle. The corresponding deduction from a nonrelativistic model is also given.

I. INTRODUCTION

In a recent paper we studied the classical content of a pseudoclassical model of a free relativistic spin-$\frac{1}{2}$ particle.[1] The model, an example of Dirac's constraint Hamiltonian dynamics,[2] contains, in addition to the usual space-time variables, five Grassmann variables which carry the spin content of the model.[3-5] The model is introduced by two constraints which are first class by virtue of the Poisson brackets (PB's) expressing the graded symplectic structure in the phase superspace.[6]

It was earlier determined that quantization of the model would express the two constraints as the Dirac and Klein-Gordon (KG) wave equations.[3-5] For this reason the model carries the spin-$\frac{1}{2}$ label.

The two first-class constraints imply two gauge invariances: reparametrization and supergauge. To extract the classical content of the model, the supergauge symmetry must be broken to confine the evolution in phase superspace to a unique, reparametrization-invariant line.[1] This is done by introducing an appropriate gauge-fixing constraint. The phase superspace equations of motion are then reevaluated with respect to the Dirac brackets.[2] By means of a distribution function,[4] which allows Grassmann variables to be averaged over, the classical equations of motion are obtained. These are *Zitterbewegung* free and the world-line conditions (WLC's) in Minkowski space are shown to be satisfied.[1]

An identical analysis was done for a pseudoclassical model of a free nonrelativistic spin-$\frac{1}{2}$ particle.[7] Here the global invariance group is Galilean rather than Poincaré and the spin-$\frac{1}{2}$ label is applicable because quantization was shown to yield the Levy-Leblond and Schrödinger wave equations.[8]

In this paper we extend the analysis of both the relativistic and nonrelativistic models to include the coupling with external electromagnetic fields, including the effect of an anomalous moment. In Sec. II we treat the relativistic case. The two constraints are systematically generated from the free-particle model. The results of quantization are quoted.[9] The classical equations of motion are then obtained as in the free case.[1] The supergauge symmetry-breaking constraint is introduced and the phase-superspace equations of motion are obtained. A distribution function is then used to pass to Minkowski space, where the classical equations of motion are just those of Bargmann, Michel, and Telegdi[10] (BMT).

In Sec. III a similar analysis is carried out for the nonrelativistic case.

II. RELATIVISTIC MOTION

As mentioned in the Introduction, we want to be systematic about the introduction of electromagnetic coupling at the pseudoclassical level. This should be done in accord with the procedure used at the quantum level, or we have no assurance that the spin-$\frac{1}{2}$ label is correctly worn.[3]

The phase-superspace variables are, as in the free case,[1] x_μ, the instantaneous position four-vector; P_μ, the four-momentum canonically conjugate to x_μ; plus the five Grassmann variables, ϵ_μ and ϵ_5. The graded symplectic structure is given by the nonzero PB's:

$$(x_\mu, P_\nu) = -g_{\mu\nu}, \quad (\epsilon_\mu, \epsilon_\nu) = i g_{\mu\nu}, \quad (\epsilon_5, \epsilon_5) = -i \ . \qquad (1)$$

The first departure from the free case is minimal,[3] that is,

$$P_\mu \to \pi_\mu = P_\mu - eA_\mu \ , \qquad (2)$$

and then

$$(\pi_\mu, \pi_\nu) = eF_{\mu\nu} \ . \qquad (3)$$

The odd Grassmann constraint is then

$$\chi_D = \pi_\mu \epsilon^\mu - m\epsilon_5 \approx 0 \ . \qquad (4)$$

This constraint is then used to generate the even Grassmann constraint, i.e., $(\chi_D, \chi_D) = i\chi_0$, where

$$\chi_0 = \pi_\mu \pi^\mu - m^2 - ieF_{\mu\nu}\epsilon^\mu \epsilon^\nu \approx 0 \ ; \qquad (5)$$

therefore the two constraints χ_D and χ_0 are first class.

Following the usual quantization rules,[6] Eq. (4) yields the minimally coupled Dirac equation and (5) yields the corresponding second-order equation.

To include the effects of an anomalous moment at the pseudoclassical level, we modify (4) to read[9]

$$\chi_D=\pi_\mu\epsilon^\mu-m\epsilon_5-\frac{iea}{2m}F_{\mu\nu}\epsilon^\mu\epsilon^\nu\epsilon_5\approx 0\ , \tag{6}$$

and then use χ_D to generate the new χ_0 via $(\chi_D,\chi_D)=i\chi_0$. In this way we find

$$\chi_0=\pi_\mu\pi^\mu-m^2-ie(1+a)F_{\mu\nu}\epsilon^\mu\epsilon^\nu-\frac{2iea}{m}\pi^\mu F_{\mu\nu}\epsilon^\nu\epsilon_5 + \frac{e^2a^2}{4m^2}F_{\mu\nu}F_{\rho\sigma}\epsilon^\mu\epsilon^\nu\epsilon^\rho\epsilon^\sigma\approx 0\ . \tag{7}$$

Now following the usual quantization rules, we have the Dirac and second-order Dirac equations describing the interaction of a spin-$\frac{1}{2}$ particle, with gyromagnetic ratio $(e/m)(1+a)$, with prescribed external electromagnetic fields.

To extract the classical content of this model, we first note that at this point the Dirac Hamiltonian is

$$H=\lambda_0\chi_0+\lambda_D\chi_D\ , \tag{8}$$

with χ_0 given by (7), χ_D given by (6), and with λ_0 and λ_D arbitrary. Introducing the gauge-fixing constraint[1]

$$\phi_D=\epsilon_5\approx 0 \tag{9}$$

(which selects the physical motions in the free case) requires $(\epsilon_5,H)\approx 0$ for stability. This fixes the relationship

$$\lambda_D=\lambda_0 f\ , \tag{10}$$

with

$$f=i\left[\frac{2ea}{m}\pi^\mu F_{\mu\nu}\epsilon^\nu\right]\Big/\left[m+\frac{iea}{2m}F_{\mu\nu}\epsilon^\mu\epsilon^\nu\right] \tag{11}$$

between the previously arbitrary coefficients so that

$$H^*=\lambda_0(\chi_0+f\chi_D) \tag{12}$$

is the one first-class constraint in the theory.

The phase-superspace equation of motion for any variable A is now $\dot{A}=(A,H)^*=(A,H^*)$. Thus we have

$$\dot{x}_\mu=\lambda_0(2\pi_\mu+\epsilon_\mu)\ , \tag{13}$$

$$\dot{\pi}_\mu=\lambda_0(2eF_{\mu\nu}\pi^\nu+efF_{\mu\nu}\epsilon^\nu)\ , \tag{14}$$

and

$$\dot{\epsilon}_\mu=\lambda_0\left[2e(1+a)F^\nu_\mu\epsilon_\nu+if\pi_\mu+\frac{ie^2a^2}{m^2}F_{\mu\nu}F_{\rho\sigma}\epsilon^\nu\epsilon^\rho\epsilon^\sigma\right]\ . \tag{15}$$

Notice that $\dot{\pi}_\mu\pi^\mu=0$ so that $\pi^2=\text{const}$, and that

$$\dot{x}_\mu\dot{x}^\mu=4\lambda_0^2(\pi_\mu\pi^\mu+f\pi_\mu\epsilon^\mu)\ . \tag{16}$$

Therefore, to fix λ_0 so that $\dot{x}_\mu\dot{x}^\mu=1$, i.e., to fix the temporal parameter to be the particle proper time, requires

$$\lambda_0=\frac{1}{2m}\left[1+\frac{ie(1+a)}{m^2}F_{\mu\nu}\epsilon^\mu\epsilon^\nu + \frac{e^2a^2}{4m^4}F_{\mu\nu}F_{\rho\sigma}\epsilon^\mu\epsilon^\nu\epsilon^\rho\epsilon^\sigma\right]^{-1/2}\ , \tag{17}$$

where we have used (6), (7), and (9).

At this point the model is defined by the Hamiltonian (12) on the submanifold of the phase superspace defined by χ_D and ϕ_D, Eqs. (6) and (9), respectively. We can now pass from the superspace to the Minkowski space by means of a suitable distribution function on the Grassmann variables.[4]

For the distribution function we take, as in the free case,[1]

$$\rho=\delta(\chi_D)\tilde{\rho}\delta(\phi_D) \tag{18}$$

where now

$$\tilde{\rho}=\frac{V_\mu\epsilon^\mu}{\pi^2}-\frac{i}{6\pi^2}\epsilon^{\mu\nu\rho\sigma}\pi_\mu\epsilon_\nu\epsilon_\rho\epsilon_\sigma\ ; \tag{19}$$

thus, normalization is maintained, i.e.,

$$\int d\mu\,\rho=1 \tag{20}$$

and

$$\langle A\rangle=\int d\mu\,\rho A\ , \tag{21}$$

where if A is any variable in the phase superspace the $\langle A\rangle$ is the corresponding classical variable.

The only Grassmann object to survive this "classicalization" is the spin tensor:

$$S_{\mu\nu}=\langle -i\epsilon_\mu\epsilon_\nu\rangle=\frac{1}{\pi^2}\epsilon_{\mu\nu\rho\sigma}\pi^\rho V^\sigma\ . \tag{22}$$

Equation (22) can be inverted to give the spin pseudovector in terms of the spin tensor

$$V^\mu=\epsilon^{\mu\nu\rho\sigma}\pi_\nu S_{\rho\sigma}\ , \tag{23}$$

as long as $V^\mu\pi_\mu=0$ is maintained by the classical equations of motion.

These equations of motion must be obtained iteratively to each order in e.[11] For the verification we seek, let us retain terms to order e. Thus the classical equations of motion are

$$u_\mu=\langle\dot{x}_\mu\rangle=2\lambda_0\left[\pi_\mu-\frac{ae}{m^2}\pi^\sigma F^\alpha_\sigma S_{\alpha\mu}\right]\ , \tag{24}$$

$$\dot{\pi}_\mu=\langle\dot{\pi}_\mu\rangle=2\lambda_0(eF_{\mu\nu}\pi^\nu)\ , \tag{25}$$

and, with $\dot{S}_{\mu\nu}=\langle -i\epsilon_\mu\dot{\epsilon}_\nu-i\dot{\epsilon}_\mu\epsilon_\nu\rangle$,

$$\dot{S}_{\mu\nu}=2\lambda_0\left[e(1+a)(F^\alpha_\mu S_{\alpha\nu}-F^\alpha_\nu S_{\alpha\mu}) - \frac{ae}{m^2}\pi_\sigma F^{\sigma\alpha}(\pi_\nu S_{\mu\alpha}-\pi_\mu S_{\nu\alpha})\right]\ , \tag{26}$$

or, equivalently for the spin,

$$\dot{V}^\mu=2\lambda_0\left[e(1+a)F^{\mu\nu}V_\nu-\frac{ae}{m^2}(V^\alpha F_{\alpha\beta}\pi^\beta)\pi^\mu\right]\ . \tag{27}$$

To this order in e we can replace π_μ by $u_\mu/2\lambda_0$ for an arbitrary temporal parameter and use $\lambda_0=\frac{1}{2}m$ to specify the proper time. Then Eqs. (25) and (26) or (27) are the usual BMT equations.[10] To this order $V^\mu\pi_\mu=0$ is maintained by the equations of motion.

Finally, we note that, to this order in e, the Liouville equation

$$\frac{\partial\rho}{\partial\tau}+(\rho,H^*)=0 \tag{28}$$

is satisfied by the equations of motion. In the literature attempts are made to deduce the equation of motion for V_μ by demanding that (28) be satisfied and putting it in the form $\dot V^\mu\epsilon_\mu+A^\mu\epsilon_\mu=0$ to assert that $\dot V^\mu=A^\mu$ is the sought-after equation of motion.[4] This is wrong as evidenced by (27), i.e., the second term is weakly orthogonal to ϵ_μ, so the BMT equation would not be recovered this way.

III. NONRELATIVISTIC MOTIONS

The phase-superspace variables are, as in the free case,[7,8] x_i, the instantaneous position three-vector; P_i, the three-momentum conjugate to x_i; t, the universal time; E, the energy conjugate to t; plus the five Grassmann variables, ϵ_i, η, and $\overline{\eta}$. The graded symplectic structure is given by the nonzero PB's:

$$\begin{gathered}(x_i,P_j)=\delta_{ij},\quad (t,E)=-1\ ,\\ (\epsilon_i,\epsilon_j)=-i\delta_{ij},\quad (\eta,\overline{\eta})=i\ .\end{gathered} \tag{29}$$

The first departure from the free case is minimal; that is, Eq. (2); then Eq. (3) is separated into

$$(\pi_i,\pi_j)=eF_{ij},\quad (\pi_0,\pi_i)=eF_{0i}\ . \tag{30}$$

The odd Grassmann constraint is then[7,8]

$$\chi_{LL}=\pi_0\eta-\pi_i\epsilon_i+m\overline{\eta}\approx 0\ . \tag{31}$$

Since $(\chi_{LL},\chi_{LL})=iS$, where

$$S=2m\pi_0-\pi^2-ie(F_{0i}\eta\epsilon_i-F_{ij}\epsilon_i\epsilon_j)\approx 0\ , \tag{32}$$

the constraints χ_{LL} and S are first class.

To include the effects of an anomalous moment we modify (31) to read

$$\chi_{LL}=\pi_0\eta-\pi_i\epsilon_i+m\overline{\eta}+\frac{iea}{2m}F_{ij}\epsilon_i\epsilon_j\eta\approx 0\ . \tag{33}$$

Since $(\chi_{LL},\chi_{LL})=iS$, where

$$\begin{aligned}S=2m\pi_0-\pi^2-ieF_{0i}\eta\epsilon_i+ie(1+a)F_{ij}\epsilon_i\epsilon_j\\ +\frac{2iea}{m}\pi_iF_{ij}\epsilon_j\eta\approx 0\ ,\end{aligned} \tag{34}$$

the constraints χ_{LL} and S are first class.

The usual quantization rules yield a Levy-Leblond and Schrödinger wave equation characteristic of a spin-$\frac{1}{2}$ particle with gyromagnetic ratio $(e/m)(1+a)$.

To extract the classical content of this model, we first note that at this point the Dirac Hamiltonian is

$$H=\lambda_sS+\lambda_{LL}\chi_{LL}\ , \tag{35}$$

with S given by (34), χ_{LL} given by (33), and with λ_s and λ_{LL} arbitrary.

Introducing the gauge-fixing constraint[7]

$$\phi_{LL}=\eta\approx 0 \tag{36}$$

requires $(\eta,H)\approx 0$ for stability. This forces $\lambda_{LL}=0$, so that

$$H^*=\lambda_sS \tag{37}$$

is the one first-class constraint in the theory.

The phase-superspace equations of motion are

$$\dot x_i=-2\lambda_s\pi_i\ , \tag{38}$$

$$\dot t=-2m\lambda_s\ , \tag{39}$$

$$\dot\pi_i=2\lambda_s(-emF_{i0}+eF_{ij}\pi_j)\ , \tag{40}$$

$$\dot\pi_0=2\lambda_seF_{0i}\pi_i\ , \tag{41}$$

$$\dot\epsilon_i=2\lambda_se(1+a)F_{ij}\epsilon_j\ , \tag{42}$$

and

$$\dot{\overline{\eta}}=-\frac{2ea}{m}\pi_iF_{ij}\epsilon_j\ . \tag{43}$$

Notice that

$$\lambda_s=-\tfrac{1}{2}m \tag{44}$$

ensures

$$\dot t=1\ .$$

For the distribution function, we take, as in the free case,[7]

$$\rho=\delta(\chi_{LL})\widetilde{\rho}\delta(\phi_{LL})\ , \tag{45}$$

where now

$$\widetilde{\rho}=C_i\epsilon_i-\frac{\pi_i\epsilon_i}{m}\eta-\frac{i}{6m}\left[\epsilon_{ijk}\epsilon_i\epsilon_j\epsilon_k-\frac{3}{m}\epsilon_{ijk}\pi_i\epsilon_j\epsilon_k\eta\right]\ . \tag{46}$$

Normalization is maintained and the only Grassmann object to survive "classicalization" is the spin tensor:

$$S_{ij}=\langle -i\epsilon_i\epsilon_j\rangle=m\epsilon_{ijk}C_k\ . \tag{47}$$

Equation (47) is equivalent to

$$C_i=\tfrac{1}{2}\epsilon_{ijk}S_{jk}\ . \tag{48}$$

In this case the classical equations of motion are exact, and are given by

$$u_i=\langle\dot x_i\rangle=-2\lambda_s\pi_i\ , \tag{49}$$

$$\dot t=\langle\dot t\rangle=-2m\lambda_s\ , \tag{50}$$

$$\dot\pi_i=\langle\dot\pi_i\rangle=2\lambda_s(-emF_{i0}+eF_{ij}\pi_j)\ , \tag{51}$$

$$\dot\pi_0=\langle\dot\pi_0\rangle=2\lambda_seF_{0i}\pi_i\ , \tag{52}$$

and

$$\dot S_{ij}=2\lambda_s[e(1+a)(F_{ik}S_{kj}-F_{jk}S_{ki})]\ , \tag{53}$$

or

$$\dot C_i=2\lambda_se(1+a)F_{ij}C_j\ . \tag{54}$$

With $\lambda_s=-\frac{1}{2}m$, these are the nonrelativistic BMT equations, or the usual Lorentz-Dirac force law, and the usual magnetic dipolar presessional law for a particle with gyromagnetic ratio $(e/m)(1+a)$. Further, the Liouville equation (28) is satisfied by the equations of motion.

IV. CONCLUSIONS

We have extracted the classical content, i.e., trajectory and spin precessional equation of motion, from two pseudoclassical models of a spin-$\frac{1}{2}$ particle interacting with external electromagnetic fields. The first, a relativistic model whose quantal content is the Dirac-Klein-Gordon system, and the second, a nonrelativistic model whose quantal content is the Levy-Leblond—Schrödinger system.

In each case the classical equations of motion are obtained after introducing a supergauge symmetry-breaking (gauge-fixing) constraint to confine the evolution to a line in phase superspace. A distribution function which averages over Grassmann variables is then used to pass from the phase-superspace evolution to the classical equations of motion.

The classical relativistic motions are those of BMT, while the classical nonrelativistic motions are the expected trajectory and spin precessions.

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