PHYSICAL REVIEW D VOLUME 33, NUMBER 8 15 APRIL 1986

Pseudoclassical description for a nonrelativistic spinning particle. II. Classical content

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(Received 28 June 1985)

The pure classical content of a pseudoclassical nonrelativistic model of a spinning particle is studied. The only physical meaningful world line is the one without "*Zitterbewegung.*" Interactions with external electromagnetic fields are also studied.

I. INTRODUCTION

The use of Grassmann variables for describing certain attributes of elementary particles such as spin, color, etc., is increasing. These pseudoclassical models[1] are usually described by means of singular Lagrangians which contain Grassmann variables in addition to the usual space-time variables. The most common feature in all these models is that the physical interpretation of Grassmann variables is given after its quantization.

On the other hand, a pure classical interpretation of the spinning particle[2] is an old problem; therefore, one is motivated to relate both formulations. In order to do that it is necessary to apply a pure classical interpretation of Grassmann variables. This can be achieved by using a distribution function on Grassmann variables.[3]

In this work we will use this pure classical interpretation of the Grassmann variables in order to obtain the classical physical content of a nonrelativistic pseudoclassical model.[4] We present the model from a Hamiltonian point of view and we give a graded symplectic structure and two first-class constraints. One of the constraints S is even and corresponds to the Schrödinger equation, the "mass shell" condition; the other one $\chi_{\rm LL}$ is odd and after quantization yields the Levy-Leblond equation.[5] These first-class constraints also are generators of two gauge transformations, i.e., reparametrizations and supergauge.

In order to obtain a classical interpretation of this model it is necessary to define some sort of world line in four-dimensional space-time, starting from the corresponding object in superspace. Taking into account that this model has two gauge invariances, the candidate must be a gauge-invariant object, but any such object is a sheet rather than a line, in superspace. The question is then how to construct a good line in four-dimensional space-time departing from a sheet. As in the relativistic case,[6] one can show that there is no distribution function which gives physical meaning to the gauge-invariant sheet because different lines on the sheet give rise to different world lines in physical space (four-dimensional space-time). Therefore, if we want to have a unique world in this physical space we need to choose it in superspace. We can construct this line by introducing a new constraint $\Phi_{\rm LL}$, which breaks the supergauge invariance and selects a line in each sheet.

The next step will be to find the corresponding world line in physical space starting from a line in superspace. This is realized by a suitable distribution function which averages over Grassmann variables in the submanifold M of the super phase space, defined by $\chi_{\rm LL}$ and $\Phi_{\rm LL}$. Furthermore, this model must verify the world-line condition[7] (WLC); i.e., the canonical and geometrical realizations must coincide up to a reparametrization. We will prove that the WLC is satisfied for any constraint $\Phi_{\rm LL}$, but when we study the equations of motion we will see that there is only a physical interpretation when there is no "*Zitterbewegung;*" this requirement uniquely fixes the constraint $\Phi_{\rm LL}$. We will also study the physics when an external electromagnetic and Yang-Mills field are switched on.

The organization of the work is as follows. In Sec. II we introduce the free-particle model. In Sec. III we discuss the physically invariant object and introduce the constraint $\Phi_{\rm LL}$. In Sec. IV we study the distribution function in the submanifold M. In Sec. V we discuss the WLC and the physical content of the model. In Sec. VI we give a Galilean realization. In Sec. VII we consider the electromagnetic interaction and in Sec. VIII we discuss the Yang-Mills interaction.

II. FREE PARTICLE

A pseudoclassical model for a nonrelativistic spinning particle was presented in paper I, by means of a singular Lagrangian. Here we want to study the model directly in the Hamiltonian formalism; for this reason we will work in the nonrelativistic-phase superspace $[\mathbf{x},t,\mathbf{p},E,\epsilon,\eta,\overline{\eta}]$, where $\mathbf{x},t$ are, respectively, the position and time of the particle, $\mathbf{p}$ and E are the momentum and energy, and $\epsilon,\eta,\overline{\eta}$ are Grassmann variables whose physical meaning is related to the spin of the particle.

This super phase is endowed with a graded symplectic structure

$$\{x^i,p^j\}=\delta^{ij}, \quad \{\epsilon^i,\epsilon^j\}=-i\delta^{ij},$$
$$\{t,E\}=-1, \quad \{\eta,\bar{\eta}\}=i. \tag{2.1}$$

The dynamical content of the model is determined by two first-class constraints

$$S=\mathbf{p}^2-2mE\approx 0\ ,$$
$$\chi_\mu=E\eta-\mathbf{p}\cdot\boldsymbol{\epsilon}+m\bar{\eta}\approx 0\ , \tag{2.2}$$

which after quantization give us the correct wave functions; that is, the first one becomes the Schrödinger equation and the second Levy-Leblond equation.[5]

The Dirac Hamiltonian is

$$H_D=\lambda_0 S+\lambda\chi_{\rm LL}\ , \tag{2.3}$$

where λ_0 and λ are even and odd arbitrary functions of evolution parameter τ. Therefore, the equations of motion are

$$\dot{x}^i=2\lambda_0 p^i-\lambda\epsilon^i\ ,$$
$$\dot{\epsilon}^i=-ip^i\ ,$$
$$\dot{\eta}=-im\ ,$$
$$\dot{\bar{\eta}}=-i\lambda E\ . \tag{2.4}$$

This evolution is confined in a surface defined by the constraints S and $\chi_{\rm LL}$. The first-class constraints S and $\chi_{\rm LL}$ are also generators of gauge transformations that leave the surface unaltered. These constraints give us a foliation of the surface in terms of two-dimensional sheets Σ.

The global symmetries of the model are those associated with the Galilean group. The corresponding transformations are generated, through the Poisson brackets (PB's) (2.1) by

$$G=-\tfrac{1}{2}\omega_{ij}R^{ij}-v_iB^i+a_iP^i-aE\ , \tag{2.5}$$

where (ω_{ij},v_i,a_i,a) are the Galilean infinitesimal transformation parameters and the Galilean generators are given by

$$R^{ij}=x^ip^j-x^jp^i-i\epsilon^i\epsilon^j\ ,$$
$$B^i=mx^i-tp^i+i\eta\epsilon^i\ ,$$
$$P^i=p^i\ ,$$
$$E=E\ . \tag{2.6}$$

Note that this is a realization of Galilean algebra, with the Lie brackets (2.1).

III. PHYSICAL CONTENT

In paper I the physical meaning of Grassmann variables was obtained by means of a quantization procedure; here we want to give a pure classical interpretation to these variables. In order to do that we first introduce a "world line" in superspace as a family of "events" $L(\mathbf{x}(\tau),t(\tau),\epsilon(\tau),\eta(\tau),\bar{\eta}(\tau))$ which are solutions of equations of motion (2.4). However, as in the relativistic case, L is not a gauge-invariant object and so it is not a good candidate for an object with physical meaning. Nevertheless, we can construct such an invariant object by performing all the possible gauge transformations over the line L, obtaining a two-dimensional sheet Σ. At this point it might appear that the physical content of the model is in the sheets, but following a similar analysis as in the relativistic case[6] we can conclude that the sheets are not physical because there does not exist a distribution function on the Grassmann variables which gives the same world line in space-time space from different lines of the sheet related by supergauge transformations.

In order to have a unique world line in the physical space it will be necessary to choose a line L on Σ. We do this by breaking the supergauge symmetry. Explicitly we introduce a new constraint $\Phi_{\rm LL}$ and odd function of $(\mathbf{x},t,\mathbf{p},\epsilon,\eta,\bar{\eta},E)$ in such a manner that $\chi_{\rm LL}$ becomes a second-class constraint.

By requiring the stability of $\Phi_{\rm LL}$ the arbitrary function λ, appearing in Dirac Hamiltonian (2.3), is determined. If $\Phi_{\rm LL}$ is such that $\{\Phi_{\rm LL},\chi_{\rm LL}\}$ has an even non-Grassmann part different from zero, in this case λ becomes

$$\lambda=\left[\frac{\partial\Phi_{\rm LL}}{\partial\tau}+\lambda_0\{\Phi_{\rm LL},S\}\right]\frac{1}{\{\Phi_{\rm LL},\chi_{\rm LL}\}}\ . \tag{3.1}$$

After substitution in Dirac's Hamiltonian (2.3), we can define the new first-class constraint

$$H_D=\lambda_0 S'+\frac{\partial\Phi_{\rm LL}}{\partial\tau}\chi_{\rm LL}\frac{1}{D}\ , \tag{3.2}$$

where

$$S'\equiv S+\frac{\gamma}{D}\chi_{\rm LL} \tag{3.3}$$

and

$$D\equiv\{\Phi_{\rm LL},\chi_{\rm LL}\},\quad \gamma\equiv\{\Phi_{\rm LL},S\}\ , \tag{3.4}$$

where D must have an even non-Grassmann part.

The second-class constraints $\chi_{\rm LL}$ and $\Phi_{\rm LL}$ can be eliminated by means of the introduction of corresponding Dirac brackets (DB's) whose explicit expression is

$$\{A,B\}^*=\{A,B\}+\frac{1}{D^2}\{A,\chi_{\rm LL}\}\{\Phi_{\rm LL},\Phi_{\rm LL}\}\{\chi_{\rm LL},B\}$$
$$-\frac{1}{D}(\{A,\chi_{\rm LL}\}\{\Phi_{\rm LL},B\}+\{A,\Phi_{\rm LL}\}\{\chi_{\rm LL},B\})\ . \tag{3.5}$$

The equations of motion are

$$\dot{x}^i=\lambda_0\left[2p^i-\frac{\gamma}{D}\epsilon^i\right],\quad \dot{\epsilon}^i=-\lambda_0\frac{\gamma}{D}p^i\ ,$$
$$\dot{t}=\lambda_0\left[2m-\frac{\gamma}{D}\eta\right],\quad \dot{\bar{\eta}}=-\lambda_0\frac{\gamma}{D}E,\quad \dot{\eta}=-\lambda_0\frac{\gamma}{D}m\ . \tag{3.6}$$

Now it is possible to define a line on configuration superspace as a one-parameter family of events $L(\mathbf{x}(\tau),t(\tau),\epsilon(\tau),\eta(\tau),\bar{\eta}(\tau))$ which is the solution of differential equations (3.6). This line L is invariant under reparametrization which is the only gauge invariance that remains at this level. At this point we need to pass from this line in superspace to a line in the physical space.

IV. DISTRIBUTION FUNCTION

Now the model is defined by the first-class constraints S', Eq. (3.3), on the submanifold M on super phase space defined by χ_{LL} and Φ_{LL}. We can pass from super phase space to four-dimensional space-time $(\mathbf{x},t)$ by means of a distribution function ρ' which works in the submanifold M; in this way we can connect an abstract mechanics with observable quantities.

The submanifold M can be parametrized by means of a set of canonical variables $(\tilde{x}^i,\tilde{t},\tilde{p}^i,\tilde{E},\tilde{\epsilon}^i)$ which is a subset of the original set. The procedure to construct this subset is to perform the Shanmugadashan transformation.[8] In this canonical transformation the new canonical coordinates $(\tilde{x}^i,\tilde{t},\tilde{p}^i,\tilde{E},\tilde{\epsilon}^i,\tilde{\eta},\tilde{\bar{\eta}})$ are chosen in such a way that the submanifold M can be characterized by equating some coordinates, i.e., $\tilde{\eta},\tilde{\bar{\eta}}$, to zero. Furthermore, we have the property that the DB's coincides with the PB's in reduced super phase space.

Therefore, the phase-space distribution function on M will be a function $\rho'(\tilde{x}^i,\tilde{t},\tilde{p}^i,\tilde{E},\tilde{\epsilon}^i)$. In order to give the correct meaning to ρ' as a distribution function we must require two conditions.

(i) Normalization conditions

$$\int d\mu'\rho'(\tilde{\epsilon}^i,\tilde{p}^i,\tilde{x}^i,\tilde{t},\tilde{E})=1\ , \qquad (4.1)$$

where $d\mu'$ is the measure in reduced space.

(ii) Liouville equation

$$\frac{\partial\rho'}{\partial\tau}+\{\rho',H_D\}^*=\frac{\partial\rho'}{\partial\tau}+\{\rho',H_D\}_R=0\ , \qquad (4.2)$$

where $\{\ \}^*$ is the DB's defined in (3.5) and $\{\ \}_R$ is the Poisson brackets in the reduced super phase space.

When the ρ' is given, we can calculate the average $\langle A'\rangle$ for any function A' defined on the submanifold by means of the integration

$$\langle A'\rangle=\int d\mu'\rho' A'\ . \qquad (4.3)$$

This procedure would require explicit use of the Shanmugadashan transformation, but we would rather obtain the physical results without explicitly performing that transformation. If we work with DB's we can use redundant variables; therefore, it is possible to give a distribution function of ρ depending on all super-phase-space variables with the condition that ρ must vanish outside of M. In order to ensure that, we write ρ as

$$\rho(\epsilon^i,\eta,\bar{\eta},p^i,E)=\delta(\chi_{LL})\tilde{\rho}(\epsilon^i,\eta,\bar{\eta},p^i,E)\delta(\Phi_{LL})\ . \qquad (4.4)$$

If we want to use this function, we must change the normalization condition (4.1) to

$$\int d\mu\,\rho(\epsilon^i,\eta,\bar{\eta},p^i,E)=1\ , \qquad (4.5)$$

where $d\mu$ is the Grassmann measure depending on all Grassmann variables; that is,

$$d\mu=d\eta\,d\bar{\eta}\,d\epsilon^3 d\epsilon^2 d\epsilon^1\ . \qquad (4.6)$$

Note that the only integration over Grassmann variables different from zero is

$$\int d\mu\,\eta\bar{\eta}\epsilon^i\epsilon^j\epsilon^k=-i\epsilon^{ijk}\ , \qquad (4.7)$$

where ϵ^{ijk} is the Levi-Civita tensor.

Now we can pass from abstract space to the real phase space by means of the average

$$\langle A\rangle=\int d\mu\,\rho A\ , \qquad (4.8)$$

where A is any dynamical variable defined on the submanifold M, but not necessarily expressed in terms of independent variables.

To construct the distribution function, we depart from the nonrelativistic distribution function introduced by Berezin and Marinov[3]

$$\tilde{\rho}(\epsilon)=\mathbf{c}(t)\cdot\boldsymbol{\epsilon}-\frac{i}{6}\epsilon_{ijk}\epsilon^i\epsilon^j\epsilon^k\ , \qquad (4.9)$$

where the vector $\mathbf{c}$ is related to the spin or intrinsic angular moment of the particle. We suppose that this expression is valid in the rest frame characterized by $\mathbf{P}=0$. An expression which is valid for any reference frame and Galilean invariant is given by

$$\tilde{\rho}(\eta,\bar{\eta},\epsilon)=\mathbf{c}\cdot\boldsymbol{\epsilon}-\frac{\mathbf{p}\cdot\mathbf{c}}{m}\eta-\frac{i}{6}\left[\epsilon_{ijk}\epsilon^i\epsilon^j\epsilon^k-\frac{3}{m}\epsilon_{ijk}p^i\epsilon^j\epsilon^k\eta\right]\ . \qquad (4.10)$$

Consider now the most general expression for the constraint Φ_{LL}. If we introduce the notation

$$\epsilon_1=\epsilon_i\ ,\quad \epsilon_4=\bar{\eta}\ ,\quad \epsilon_5=\eta\ , \qquad (4.11)$$

making a Taylor-Grassmann expansion we have

$$\Phi_{LL}=d_a\epsilon^a+f_{abc}\epsilon^a\epsilon^b\epsilon^c+g_{abcde}\epsilon^a\epsilon^b\epsilon^c\epsilon^d\epsilon^e\ ,\quad a,b,c,d,e=1,\ldots,5\ , \qquad (4.12)$$

where d_a, f_{abc}, g_{abcde} are arbitrary functions of the real variables $\mathbf{x}$, t, $\mathbf{p}$, E, and τ. However, the last two terms have no physical relevance because this number of Grassmann variables is too high, and so their real coefficients cannot appear in averaged quantities. Therefore we have

$$p=\delta(\chi_{LL})\tilde{\rho}(\eta,\bar{\eta},\epsilon)\delta(d_a\epsilon^a)\ . \qquad (4.13)$$

The normalization condition

$$\int d\mu\,\rho=1 \qquad (4.14)$$

imposes the relation among the real coefficients in Φ_{LL}.

$$\mathbf{p}\cdot\mathbf{d}-md_5-E\,d_4-\frac{\mathbf{p}^2}{m}d_4=1\ . \qquad (4.15)$$

Note that the functions d_a have dimensions.

The Liouville equation will give us

$$\dot{\mathbf{c}}\cdot\boldsymbol{\epsilon}-\frac{\dot{\mathbf{c}}\cdot\mathbf{p}}{m}\eta-\left[\mathbf{c}\cdot\boldsymbol{\epsilon}-\frac{\mathbf{c}\cdot\mathbf{p}}{m}\eta\right]\delta'(\Phi_{LL})\frac{\partial\Phi_{LL}}{\partial\tau}=0\ . \qquad (4.16)$$

As we wish to describe a physical free spinning particle, c_a must be constant. Therefore Φ_{LL} must be independent of the evolution parameter τ.

V. WORLD-LINE INVARIANCE, AVERAGED QUANTITIES

At this point, we have the mechanism to construct a world line in four-dimensional space-time $\langle x_i \rangle, \langle t \rangle$ which enables us to give a classical physical meaning to the Grassmann variables.

The coordinates of the world line are

$$\langle x_i \rangle = \int d\mu \rho x_i \, , \qquad \langle t \rangle = \int d\mu \rho t \, , \tag{5.1}$$

and the spin content of the model is obtained from the Grassmann part of the generator of the Galilean group corresponding to rotations

$$\langle S^{ij} \rangle \equiv \langle -i\epsilon^i \epsilon^j \rangle = \int d\mu \rho \langle -i\epsilon^i \epsilon^j \rangle \, . \tag{5.2}$$

These objects must have the correct transformation properties under the Galilean group. That means they must satisfy the world-line condition.[7] Before we discuss this condition we must fix the evolution parameter by breaking the gauge symmetry associated with reparametrizations; this means fixing the parameter τ by means of a new constraint Φ such that S' becomes second class; that is,

$$\{S',\Phi\}^* = A0 \, . \tag{5.3}$$

Φ must be chosen such that A has an even non-Grassmann part different from zero. In that case the new DB's is

$$\{f,g\}^{\#} = \{f,g\}^* + \frac{1}{A}(\{f,S'\}^*\{\Phi,g\}^* - \{f,\Phi\}^*\{S',g\}^*) \, . \tag{5.4}$$

The stability of this new constraint Φ enables us to calculate the function λ_0 appearing in (3.2):

$$\lambda_0 = \frac{1}{A}\frac{\partial\Phi}{\partial\tau} \, . \tag{5.5}$$

Now we want to see if the line $L(\langle x_i \rangle, \langle t \rangle)$ has real objectivity or not. We can construct a canonical realization of the Galilean group in terms of DB's (5.4) in super phase space and also a geometrical realization by means of the PB's. The world-line conditions ensure that two inertial observers "see" the same world line; therefore, the only difference between the two kinds of transformations, the canonical and geometrical, must be a reparametrization; i.e.,

$$\langle x^i(\tau+\delta\tau) \rangle + \langle \{x^i,G\} \rangle = \langle x^i(\tau) \rangle + \langle \{x^i,G\}^{\#} \rangle \, ,$$
$$\langle t(\tau+\delta\tau) \rangle + \langle \{t,G\} \rangle = \langle t(\tau) \rangle + \langle \{t,G\}^{\#} \rangle \, ,$$

and the same for the spin variable:

$$\langle S^{ij}(\tau+\delta\tau) \rangle + \langle \{S^{ij},G\} \rangle = \langle S^{ij}(\tau) \rangle + \langle \{S^{ij},G\}^{\#} \rangle \, . \tag{5.6}$$

We say that the line has a real objectivity if it is possible to find a real $\delta\tau$. Using evolution equations (3.6) with the corresponding λ_0, (5.5), and also using the constraints S and χ_{LL} it is possible to isolate $\delta\tau$ obtaining

$$\delta\tau = \frac{\left\langle -\frac{1}{D}2\bar{\eta}\{\Phi_{LL},G\} \right\rangle + \left\langle \frac{1}{A}\left[2E - \frac{\gamma}{D}\bar{\eta}\right]\{\Phi,G\}^* \right\rangle}{\left\langle \frac{1}{A}\frac{\partial\Phi}{\partial\tau}\left[2E - \frac{1}{2}\frac{\gamma}{D}\bar{\eta}\right] \right\rangle} \, . \tag{5.7}$$

The denominator never vanishes because A and $\partial\Phi/\partial\tau$ must have an even non-Grassmann part different from zero; therefore, $\langle (1/A)(\partial\Phi/\partial\tau)2E \rangle$ will have a part which is scalar; on the other hand, if $\langle (1/2A)(\gamma/D)\bar{\eta}(\partial\Phi/\delta\tau) \rangle$ is different from zero, it will be explicitly dependent on the Levi-Civita tensor and therefore will be a pseudoscalar.

Therefore, the WLC does not add any new restriction on the constraints Φ_{LL} and Φ. We can conclude that the line $\langle \mathbf{x}(\tau) \rangle$, $\langle t(\tau) \rangle$, $\langle S^{ij}(\tau) \rangle$ is an appropriate geometrical object to characterize the world line of a nonrelativistic spinning particle.

As in the relativistic case,[6] this line would only have physical meaning for a restricted class of constraints Φ_{LL}. For example, consider the averaged value of the Grassmann spin tensor:

$$S^{ij} = -\epsilon^{ijk}\left[E_4\left[Ec_k + \frac{\mathbf{p}\cdot\mathbf{c}}{m}p_k\right] - [c_k m d_5 - (\mathbf{p}\cdot\mathbf{c})d_k]\right] \, . \tag{5.8}$$

Because of the tensor character of S^{ij} and ϵ^{ijk} it is necessary that d_k be a three-vector and that $d_4=0$. Furthermore, the free-particle spin tensor cannot depend on the position of the particle; therefore, the coefficients $\mathbf{d}$ are only functions of the momenta. We have

$$\Phi_{LL} = f\left[\frac{1}{\mathbf{p}^2}\right]\mathbf{p}\cdot\boldsymbol{\epsilon} + d_5\eta \, , \tag{5.9}$$

where f is a general function of their arguments. Furthermore, the coefficients of ϵ and η must verify the normalization condition (4.15). With these constraints one can see from the equations of motion (3.9) that there are not *Zitterbewegung*. If we choose Φ_{LL} as

$$\Phi_{LL} = \frac{\eta}{m} \tag{5.10}$$

the term associated with the spin has a clear physical meaning:

$$S^k = \tfrac{1}{2}\epsilon^{kij}\langle S^{ij} \rangle = c^k \, . \tag{5.11}$$

The equations of motion for the physical quantities are

$$u^i = \langle \dot{x}^i \rangle = 2\lambda_0 p^i \, , \qquad \dot{t} = 2m\lambda_0 \, , \qquad \dot{c}^i = 0 \, . \tag{5.12}$$

When we choose the time coordinate as the evolution parameter, by means of the constraint

$$\Phi = t - \tau \approx 0 \ , \qquad (5.13)$$

the arbitrary function λ_0 becomes

$$\lambda_0 = \frac{1}{2m} \ , \qquad (5.14)$$

and we recover the equations of motion for a nonrelativistic spinning particle without *Zitterbewegung*.

VI. GALILEAN REALIZATION

We construct the Galilean realization for this model for the case $\Phi_{\rm LL} = \eta/m$. To do that we follow the Shanmugadashan method.[8] We define the conjugate variables

$$\tilde{\eta} = \frac{\eta}{m} \ , \qquad (6.1a)$$

$$\tilde{\tilde{\eta}} = \chi_{\rm LL} + \frac{\eta}{2m} S \ , \qquad (6.1b)$$

where $\chi_{\rm LL}$ and S are given by Eqs. (2.2). The new variables satisfy the PB's:

$$\{\tilde{\tilde{\eta}}, \tilde{\eta}\} = i, \ \{\tilde{\tilde{\eta}}, \tilde{\tilde{\eta}}\} = 0, \ \{\tilde{\eta}, \tilde{\eta}\} = 0 \ . \qquad (6.2)$$

To complete the canonical transformation, we must solve the equations

$$\{f, \tilde{\tilde{\eta}}\} = 0, \ \{f, \tilde{\eta}\} = 0 \ . \qquad (6.3)$$

We seek the solutions of (6.3) from which the Grassmann variables have no position dependence, i.e.,

$$\tilde{\epsilon}^i = \tilde{\epsilon}^i(E, \mathbf{P}, \epsilon, \eta, \tilde{\eta}) \ . \qquad (6.4)$$

Equations (6.3) and the canonical condition

$$\{\tilde{\epsilon}^i, \tilde{\epsilon}^j\} = -\delta^{ij} i \qquad (6.5)$$

give us

$$\tilde{\epsilon}^i = \epsilon^i - \frac{p^i}{m}\eta \ . \qquad (6.6)$$

For the remaining variables, $\tilde{x}, \tilde{t}$ and $\tilde{p}, \tilde{\tilde{t}}$, we choose

$$\tilde{p}^i = p^i \ , \qquad \tilde{E} = E \qquad (6.7)$$

and

$$\tilde{x}^i = x^i + f^i(p, E, t, \epsilon_i, \eta, \tilde{\eta}) \ , \qquad \tilde{t} = t + g(p, E, t, \epsilon_i, \eta, \tilde{\eta}) \ . \qquad (6.8)$$

By imposing the corresponding canonical condition

$$\{\tilde{x}^i, \tilde{x}^j\} = \{\tilde{t}, \tilde{x}^i\} = \{\tilde{x}^i, \tilde{\epsilon}^j\} = \{\tilde{x}, \tilde{\eta}\} = \{x^i, \tilde{\tilde{\eta}}\} = 0 \ , \qquad \{t, \tilde{\epsilon}^i\} = \{\tilde{t}, \tilde{\tilde{\eta}}\} = \{\tilde{t}, \tilde{\eta}\} = 0 \ , \qquad (6.9)$$

and

$$\{x^i, p^j\} = \delta^{ij} \ , \qquad \{\tilde{t}, E\} = -1 \ . \qquad (6.10)$$

A solution is

$$\tilde{x}^i = x^i - i\frac{\epsilon^i \eta}{m} \ , \qquad \tilde{t} = t \ . \qquad (6.11)$$

The Galilean generators (2.7) in terms of new variables become

$$R^{ij} = \tilde{x}^i p^i - \tilde{x}^j p^i - i\tilde{\epsilon}^i \epsilon^j \ , \qquad B^i = -tp^i + mx^i \ , \qquad P^i = p^i \ , \qquad E = E \ . \qquad (6.12)$$

This is an 11-dimensional realization of Galilean algebra with respect to the brackets (3.5) which is explicitly

$$\{A, B\}^* = \{A, B\} - i(\{A, \tilde{\tilde{\eta}}\}\{\eta, B\}\{A, \eta\}\{\tilde{\tilde{\eta}}, B\}) \qquad (6.13)$$

which coincides with the PB's in the reduced space $\{ \ \}_{R'}$ defined by

$$\tilde{\eta} = 0, \ \tilde{\tilde{\eta}} = 0 \ . \qquad (6.14)$$

Furthermore, we can fix the evolution parameter by choosing the constraint

$$\Phi = t - \tau \approx 0 \ . \qquad (6.15)$$

Then S' becomes a second-class constraint. That enables us to construct a nine-dimensional realization. In fact, we can eliminate two superflows degrees of freedom with

$$t = \tau \ , \qquad E = \frac{\mathbf{p}^2}{2m} \ . \qquad (6.16)$$

Then the corresponding DB's which realize the algebra are

$$\{A, B\}^{\#} = \{A, B\}^* + \{A, S'\}\{\Phi, B\}^* - \{A, \Phi\}^*\{S', B\}^* \ . \qquad (6.17)$$

Because of the Galilean scalar character of $\chi'_{\rm LL}$ and S' we have

$$\{G, G'\}^{\#} = \{G, G'\}^* = \{G, G'\} \ , \qquad (6.18)$$

where G or G' denote any of the Galilean generators:

$$\tilde{S}^i = -\tfrac{1}{2} i \epsilon^{ijk} \tilde{\epsilon}^j \tilde{\epsilon}^k \ . \qquad (6.19)$$

The distribution function is

$$\rho(p^i, \tilde{\eta}, \epsilon, \eta) = \delta(\mathbf{p}\cdot\boldsymbol{\epsilon}) \left[\mathbf{c}\cdot\mathbf{e} - \frac{\mathbf{p}\cdot\mathbf{c}}{m}\eta - \frac{i}{6}\left[\epsilon_{ijk}\epsilon^i\epsilon^j\epsilon^k - \frac{3}{m}\epsilon_{ijk}p^i\epsilon^j\epsilon^k\eta \right] \right] \delta\left[\frac{\eta}{m}\right] \qquad (6.20)$$

which enables us to calculate the average position and spin variables:

$$\langle \tilde{\mathbf{x}} \rangle = \langle \mathbf{x} \rangle = \mathbf{x} \ , \quad (6.21)$$

$$\langle \mathbf{S} \rangle = \langle \mathbf{S} \rangle = \mathbf{c} \ , \quad (6.22)$$

where

$$\tilde{S}^i = -\tfrac{1}{2} i \epsilon^{ijk} \tilde{\epsilon}^i \tilde{\epsilon}^k \ . \quad (6.23)$$

The generators of the Galilean group in terms of averaged quantities are

$$\begin{aligned} J^i &= \tfrac{1}{2}\epsilon^{ijk} R_{jk} = (\tilde{\mathbf{x}} \times \mathbf{p})^i + c^i \ , \\ B_i &= -\tau p^i + m x_i \ , \\ P^i &= p^i \ , \\ E &= \frac{\mathbf{p}^2}{2m} \ , \end{aligned} \quad (6.24)$$

which is a realization under the operator

$$O[\langle A \rangle, \langle B \rangle] = \frac{\partial \langle A \rangle}{\partial x^i}\frac{\partial \langle B \rangle}{\partial p^i} - \frac{\partial \langle A \rangle}{\partial p^i}\frac{\partial \langle B \rangle}{\partial x^i} + \epsilon^{ijk} C^k \frac{\partial \langle A \rangle}{\partial C^i}\frac{\partial \langle B \rangle}{\partial C^i} \ . \quad (6.25)$$

VII. ELECTROMAGNETIC INTERACTIONS

We introduce the interaction of the nonrelativistic spinning particle with an external electromagnetic field by means of the two first-class constraints:[4]

$$\begin{aligned} S^{\mathrm{EM}} &= (\mathbf{p} - e\mathbf{A})^2 - 2m(E - eA^0) - ie(2F^{0i}\eta\epsilon^i + F^{ij}\epsilon^i\epsilon^j) \ , \\ \chi_{\mathrm{LL}}^{\mathrm{EM}} &= (E - eA^0)\eta - (\mathbf{p} - e\mathbf{A})\cdot\boldsymbol{\epsilon} + m\bar{\eta} \ , \end{aligned} \quad (7.1)$$

where (A^0, A) are the electromagnetic potentials and

$$F^{ij} \equiv \frac{\partial A^i}{\partial x_j} - \frac{\partial A^i}{\partial x_j}, \quad F^{0i} \equiv \frac{\partial A^0}{\partial x_i} - \frac{\partial A^i}{\partial t} \quad (7.2)$$

are the electromagnetic fields. The equations of motion are generated by the Dirac Hamiltonian

$$H_D = \lambda \chi_{\mathrm{LL}}^{\mathrm{EM}} + \lambda_0 S^{\mathrm{EM}} \quad (7.3)$$

with the PB's (2.1); their explicit expression is given in paper I, Eq. (4.12).

To study the classical content we must once again construct a world line in physical space time. In order to do that it is necessary to define a line in the surface of super phase space defined by the constraints (7.1); that is, we need to introduce an extra constraint Φ_{LL}. Furthermore as we want to recover the free-particle model of the previous section when the potentials are turned off the only physical meaningful constraint is $\Phi_{\mathrm{LL}} = \eta/m$. In that case the Dirac Hamiltonian reduces to

$$H_D = \lambda_0 S^{\mathrm{EM}} \ . \quad (7.4)$$

Now we can construct the distribution function by introducing minimal electromagnetic coupling into the free-particle distribution function, i.e.,

$$\rho^{\mathrm{EM}} = \delta(\chi_{\mathrm{LL}}^{\mathrm{EM}}) \left[\mathbf{c}\cdot\boldsymbol{\epsilon} - \frac{\mathbf{p}_{\mathrm{EM}}\cdot\mathbf{c}}{m}\eta - \frac{i}{6}\left[\epsilon_{ijk}\epsilon^i\epsilon^j\epsilon^k - \frac{3}{m}\epsilon_{ijk} p^i_{\mathrm{EM}} \epsilon^j \epsilon^k \right] \right] \delta(\phi_{\mathrm{LL}}) \ , \quad (7.5)$$

where

$$\mathbf{p}_{\mathrm{EM}} = \mathbf{p} - e\mathbf{A} \ . \quad (7.6)$$

This distribution function has to satisfy the normalization condition and the Liouville equation. This last equation gives the evolution of the spin:

$$\dot{\mathbf{c}} = 2\lambda_0 \mathbf{B} \times \mathbf{C} \ . \quad (7.7)$$

We can also calculate the evolution of other quantities. In particular, for the acceleration we have

$$\dot{u}^i = 2\lambda_0 \left[u^j F^{ij} e + 2\lambda_0 e \left[mF^{0i} - \frac{i}{2} F^{kj}{}_{,i} \epsilon^{kjl} C^l \right] \right] \ . \quad (7.8)$$

When the evolution parameter is the time coordinate of the particle, Eq. (7.8) is just the usual Lorentz-Dirac force law.

VIII. YANG-MILLS INTERACTION

If we now consider a nonrelativistic spinning particle with internal degrees of freedom, described by a set of Grassmann variables θ_α and θ_α^*, we can consider the interaction with an external Yang-Mills field.[9] At the pseudoclassical level the model is defined in paper I by the two first-class constraints

$$\begin{aligned} S^{\mathrm{YM}} &= (\mathbf{p} - gI^0 \mathbf{A}_{a'})^2 - 2m(E - gI^0 A^0_{a'}) \\ &\quad - ig(2F^{0i}{}_{a'} I^{a'} \eta\epsilon^i + F^{ij}{}_{a'} I^{a'} \epsilon^i \epsilon^j) \simeq 0 \ , \end{aligned} \quad (8.1a)$$

$$\chi_{\mathrm{LL}}^{\mathrm{YM}} = (E - gI^a \mathbf{A}^0_{a'})\eta - (\mathbf{p} - gI^{a'} A^{a'})\epsilon + m\bar{\eta} \approx 0 \ , \quad a' = 1, \ldots, n \ , \quad (8.1b)$$

where $I^{a'} = \theta\tau^{a'}\theta$ are a realization of the Lie algebra of the internal-symmetry group and $(A^0_{a'}\, \mathbf{A}_{a'})$ are the Yang-Mills potentials. The equations of motion at the pseudoclassical level are given in paper I. In order to extract the classical content we consider, as in the preceding sections, a line in the surface of super phase space defined $\chi^{\mathrm{YM}}\ S^{\mathrm{YM}}$. Explicitly, we introduce the constraint $\Phi_{\mathrm{LL}} = \eta/m$.

The distribution function must contain all Grassmann variables. By construction $\epsilon, \eta, \bar{\eta}$ and $\theta_\alpha, \theta_\alpha^*$ commute; this suggests the introduction of two distribution functions $\rho_i(\epsilon, \eta, \bar{\eta}, \rho)$ and $\rho_2(\theta_\alpha, \theta_\alpha^*)$ with the properties

$$\begin{aligned} &\int \rho_1(\epsilon, \eta, \bar{\eta}, P) d^5\epsilon = 1 \ , \\ &\int \rho_2(\theta_\alpha, \theta_\alpha^*) d\mu(\theta_\alpha, \theta_\alpha^*) = 1 \ , \end{aligned} \quad (8.2)$$

where $d\mu$ is the measure associated with $\theta_\alpha, \theta_\alpha^*$ whose explicit form will depend on the internal group considered. Furthermore, we will require that ρ_1 and ρ_2 satisfy separately the Liouville equation. In that way we are sure that the total distribution function

$$\rho=\rho_1(\epsilon,\eta,\bar{\eta},b)\rho_2(\theta_\alpha,\theta^*_\alpha) \tag{8.3}$$

has the correct properties.

The distribution function ρ_1 associated with the spin variables is

$$\rho^{\rm YM}=\delta(\chi_{\rm LL}^{\rm YM})\left[\mathbf{c}\cdot\boldsymbol{\epsilon}-\frac{\mathbf{p}_{\rm YM}\cdot\mathbf{c}}{m}\eta -\frac{i}{6}\left[\epsilon_{ijk}\epsilon^i\epsilon^j\epsilon^k -\frac{3}{m}\epsilon_{ijk}p^i_{\rm YM}\epsilon^j\epsilon^k\right]\right]\delta(\phi_{\rm LL})\,, \tag{8.4}$$

where

$$\mathbf{p}_{\rm YM}=\mathbf{p}-g\,\mathbf{A}^{a'}I_{a'},\ \ a'=1,\ldots,N\,. \tag{8.5}$$

The explicit form of ρ_2 will depend on the internal-symmetry group. The Liouville equation for ρ_1 gives the evolution of spin, namely,

$$\dot{c}^{\,i}=-2g\lambda_0 I^{a'}F^j_{a'}c^i\,. \tag{8.6}$$

Note that $I^{a'}$ contains Grassmann variables. Therefore to obtain a classical equation we need to consider the average with respect to ρ_2. Explicit examples are found in Ref. 10.

IX. CONCLUSIONS

We have studied the classical content of a pseudoclassical model for a nonrelativistic spinning particle. We have shown that in order to define a world line in a four-dimensional space-time we need to consider a line in superspace and a suitable distribution function on the submanifold defined by $\chi_{\rm LL}$ and $\Phi_{\rm LL}$. The world line in physical space defined in this way has an objective reality, i.e., verifies the WLC for an arbitrary choice of $\Phi_{\rm LL}$. However, the physical content of this model is not independent of the constraint $\Phi_{\rm LL}$. In fact, we conclude that the only possible meaningful choice of $\Phi_{\rm LL}$ is the one given by $\Phi_{\rm LL}=\eta/m$ that yields an evolution without *Zitterbewegung*. The Galilean realization for this situation is constructed by means of the Shanmugadashan transformation. Finally we have studied the interactions with external electromagnetic and Yang-Mills fields. In both cases the expected classical equations of motion are obtained.

ACKNOWLEDGMENTS

We are grateful to Professor Kenneth Rafanelli for a careful reading of the manuscript. One of us (J.G.) would like to acknowledge Professor B. Sakita for the hospitality at the High Energy Group of City College. Partial financial support by Comite Conjunto Hispano Norteamericano para la Cooperaction Cientifica y Tecnologica is acknowledged.

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