Pseudoclassical description for a nonrelativistic spinning particle. I. The Levy-Leblond equation

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A pseudoclassical model for a spinning nonrelativistic particle is presented. The model contains two first-class constraints which after quantization give rise to the Levy-Leblond equation for a spin-$\frac{1}{2}$ particle.

I. INTRODUCTION

Recent interest in Grassmann variables in particle physics comes from dual models.[1] Later their use in supergravity theories[2] has proven to be very useful. In systems with a finite number of degrees of freedom, Grassmann variables have been useful in order to describe particle attributes such as spin, isospin, color, etc. The pseudomechanics[3] that arise from these models can be considered as a quasiclassical limit of a general quantum system.[4] The classical content of these pseudoclassical models was given by Berezin and Marinov,[5] via the introduction of a distribution function over these Grassmann variables.

As an example, consider supergravity in one dimension;[6] this model after quantization[5-8] yields the Dirac equation. A pure classical interpretation of this model has also been given.[9]

In this work we want to study the Galilean counterpart of the relativistic model mentioned above. We seek a pseudoclassical model which after quantization yields the Levy-Leblond equation.[10] To this end we introduce a Lagrangian for a free nonrelativistic particle invariant under reparametrization; this Lagrangian has one constraint which after quantization yields the Schrödinger equation. We then generalize this Lagrangian by introducing Grassmann variables in such a way that the new Lagrangian has two first-class constraints, which after quantization yields the Levy-Leblond equation. We also consider interactions with an external electromagnetic field. If, furthermore, we introduce additional internal degrees of freedom we can consider the interaction with an external Yang-Mills field.

In this paper the physical meaning of the Grassmann variables is understood via quantization. The classical content will be discussed in a separate paper.[11]

The paper is organized as follows: In the second section, we introduce the free-particle model. In Sec. III we study the global and gauge symmetries of the model. Section IV is devoted to the study of the interaction with an external electromagnetic field. In Sec. V we consider the interaction with an external Yang-Mills field. Section VI is devoted to quantization.

II. THE FREE PARTICLE

The Lagrangian for a free nonrelativistic particle invariant under reparametrization is

$$L=\tfrac{1}{2}m\frac{\dot{\mathbf{x}}^2}{\dot{t}}\ , \tag{2.1}$$

where $(\mathbf{x},t)$ are coordinates of an event and $\dot{\mathbf{x}},\dot{t}$ are the "velocities" with respect to an arbitrary parameter τ.

The conjugate momenta are

$$\mathbf{p}=\frac{\partial L}{\partial\dot{\mathbf{x}}}=m\frac{\dot{\mathbf{x}}}{\dot{t}}\ , \tag{2.2a}$$

$$E=-\frac{\partial L}{\partial\dot{t}}=\frac{1}{2}m\frac{\dot{\mathbf{x}}^2}{\dot{t}^2}\ , \tag{2.2b}$$

from which we can construct the constraint

$$S=\mathbf{p}^2-2mE\approx 0\ . \tag{2.3}$$

If we introduce the symplectic structure

$$\{x_i,p_j\}=\delta_{ij},\quad \{t,E\}=-1 \tag{2.4}$$

we can immediately see the first-class character of the constraint S.

Now if we want to quantize this model we must impose the constraint (2.3) over the physical states:

$$(\hat{\mathbf{p}}^2-2m\hat{E})\,|\,\psi\rangle=0\ , \tag{2.5}$$

where $\hat{\mathbf{p}}$ and $\hat{E}$ are the quantum operators for momentum and energy. Note that this condition is nothing but the Schrödinger equation.

As we are interested in the description of a free spinning, nonrelativistic particle, it is necessary to generalize the above model. We will take as a guide the formulation of a relativistic spinning particle,[5-8] where a singular Lagrangian was introduced such that quantization of the model yields the Dirac equation. Therefore, we seek a singular Lagrangian model which after quantization yields the Levy-Leblond equation.[10]

The Lagrangian is

$$L=\frac{1}{2}m\frac{(\dot{\mathbf{x}}+\chi\boldsymbol{\epsilon})^2}{\dot{t}+\chi\eta}-m\chi\eta-\frac{i}{2}(\eta\dot{\bar{\eta}}+\bar{\eta}\dot{\eta})+\frac{i}{2}\boldsymbol{\epsilon}\cdot\dot{\boldsymbol{\epsilon}} . \tag{2.6}$$

This Lagrangian contains, apart from the space-time variables $(\mathbf{x},t)$, Grassmann variables whose dimensions are chosen to ensure correct dimensionality of the action. If we denote by $[\hbar]$ and $[u]$ the dimensions of the action and velocity, respectively, we make the following assumptions:

$$[\eta]=[\hbar]^{1/2}[u]^{-1},\quad [\epsilon^i]=[\hbar]^{1/2} , \tag{2.7a}$$

$$[\bar{\eta}]=[\hbar]^{1/2}[u],\quad [\chi]=[\hbar]^{1/2}[m]^{-1}[u]^{-1} . \tag{2.7b}$$

The canonical momenta are

$$\mathbf{p}=\frac{\partial L}{\partial \dot{\mathbf{x}}}=m\frac{\dot{\mathbf{x}}+\chi\boldsymbol{\epsilon}}{\dot{t}+\chi\eta} , \tag{2.8a}$$

$$E=-\frac{\partial L}{\partial \dot{t}}=\frac{1}{2}m\frac{(\dot{\mathbf{x}}+\chi\boldsymbol{\epsilon})^2}{(\dot{t}+\chi\eta)^2} , \tag{2.8b}$$

$$\Pi_i=\frac{\partial L}{\partial \dot{\epsilon}_i}=-\frac{i}{2}\epsilon_i , \tag{2.8c}$$

$$\Pi_\eta=\frac{\partial L}{\partial \dot{\eta}}=\frac{i}{2}\bar{\eta} , \tag{2.8d}$$

$$\Pi_{\bar{\eta}}=\frac{\partial L}{\partial \dot{\bar{\eta}}}=\frac{i}{2}\eta , \tag{2.8e}$$

$$\Pi=\frac{\partial L}{\partial \dot{\chi}}=0 . \tag{2.8f}$$

The primary constraints are

$$S=\mathbf{p}^2-2mE\approx 0 , \tag{2.9a}$$

$$\Psi_i=\Pi_i+\frac{i}{2}\epsilon_i\approx 0 , \tag{2.9b}$$

$$\Psi_\eta=\Pi_\eta-\frac{i}{2}\bar{\eta}\approx 0 , \tag{2.9c}$$

$$\Psi_{\bar{\eta}}=\Pi_{\bar{\eta}}-\frac{i}{2}\eta\approx 0 , \tag{2.9d}$$

$$\Pi=0 . \tag{2.9e}$$

If we introduce the graded symplectic structure[3]

$$\begin{aligned}&\{x_i,p_j\}=\delta_{ij},\quad \{\epsilon_i,\Pi_j\}=-\delta_{ij} ,\\ &\{t,E\}=-1,\quad \{\eta,\Pi_\eta\}=1 ,\\ &\{\chi,\Pi\}=-1,\quad \{\eta,\Pi_{\bar{\eta}}\}=1 ,\end{aligned} \tag{2.10}$$

we can see that the constraints S and Π defined in (2.9a) and (2.9e) are first class while the others are second class.

The canonical Hamiltonian is

$$Hc=\chi(E\eta-\mathbf{p}\cdot\boldsymbol{\epsilon}+m\bar{\eta})+\dot{\chi}\Pi . \tag{2.11}$$

The stability of the primary constraints yields one secondary constraint,

$$\chi_{\mathrm{LL}}=E\eta-\mathbf{p}\cdot\boldsymbol{\epsilon}+m\bar{\eta}\approx 0 , \tag{2.12}$$

which is also first class.

The Dirac formalism enables us to omit the second-class constraints, by introducing the Dirac brackets[3] (DB's) explicitly:

$$\begin{aligned}\{A,B\}^*=\{A,B\}+i(&\{A,\Psi_\eta\}\{\Psi_{\bar{\eta}},B\}+\{A,\Psi_{\bar{\eta}}\}\{\Psi_\eta,B\}\\ &-\delta_{ij}\{A,\Psi_i\}\{\Psi_j,B\}) .\end{aligned} \tag{2.13}$$

From now on we will work in the reduced phase space defined by $\mathbf{x},\mathbf{p},E,t,\boldsymbol{\epsilon},\eta,\bar{\eta},\chi,\Pi$, with the DB's among these variables:

$$\begin{aligned}&\{x_i,p_j\}^*=\delta_{ij},\quad \{\epsilon_i,\epsilon_j\}^*=-i\delta_{ij} ,\\ &\{t,E\}^*=-1,\quad \{\eta,\bar{\eta}\}^*=i,\quad \{\chi,\Pi\}^*=1 .\end{aligned} \tag{2.14}$$

χ plays the role of a supplementary variable, which can be easily eliminated by means of a gauge constraint Φ_π, which reduces the constraint $\Pi\approx 0$ to second class. We choose

$$\Phi_\pi=\chi-\lambda(\mathbf{p},\mathbf{x},\boldsymbol{\epsilon},\eta,\bar{\eta},t,E)\approx 0 , \tag{2.15}$$

where λ is an arbitrary function. Now we have

$$\{\Pi,\Phi_\pi\}^*=-1 \tag{2.16}$$

and we construct the new DB's

$$\begin{aligned}\{A,B\}'=\{A,B\}^*&+\{A,\Pi\}^*\{\Phi_\pi,B\}^*\\ &+\{A,\Phi_\pi\}^*\{\Pi,B\}^* .\end{aligned} \tag{2.17}$$

These brackets are the fundamental object in the super phase space spanned by $\mathbf{x},\mathbf{p},t,E,\boldsymbol{\epsilon},\eta,\bar{\eta}$. In this space the evolution is generated by Dirac's Hamiltonian H_D, which is a linear combination of the first-class constraints.

The equations of motion are

$$\dot{x}_i=\{x_i,H_D\}'=2\lambda_0 p_i-\lambda\epsilon_i , \tag{2.18a}$$

$$\dot{\epsilon}_i=\{\epsilon_i,H_D\}'=-i\lambda p_i , \tag{2.18b}$$

$$\dot{\eta}=\{\eta,H_D\}'=-im\lambda , \tag{2.18c}$$

$$\dot{\bar{\eta}}=\{\bar{\eta},H_D\}'=-i\lambda E , \tag{2.18d}$$

in the surface defined by S and χ_{LL}.

III. SYMMETRIES

In this section we study the transformation properties of the Lagrangian (2.6) under Galilean transformations and the gauge symmetries of the Lagrangian.

The Galilean transformations for the position $\mathbf{x}$, momentum $\mathbf{p}$, time t, and energy E are known but the transformation properties of the Grassmann variables are not known *a priori*. In order to deduce these Galilean transformation properties we require the first-class constraints to be Galilean scalars. This implies the following transformation properties for the Grassman variables:

$$\begin{aligned}&\epsilon'^i=R^i{}_j\epsilon^j+v^i\eta ,\\ &\bar{\eta}'=v_iR^i{}_j\epsilon^j+\bar{\eta}+\eta\mathbf{v}^2 ,\\ &\eta'=\eta .\end{aligned} \tag{3.1}$$

We also require that the variable χ be a Galilean scalar.

At this point we are in a position to verify the Galilean invariance of the Lagrangian (2.8). Consider the infinitesimal Galilean transformation:

$$\begin{aligned} &\delta x_i = R_{ij}x_j + v_i t + a_i \ , \\ &\delta t = a \ , \\ &\delta \epsilon_i = \omega_{ij}\epsilon_j + v_i \eta \ , \\ &\delta \overline{\eta} = \mathbf{v}\cdot\boldsymbol{\epsilon} \ , \\ &\delta \eta = 0 \ , \\ &\delta \chi = 0 \ , \end{aligned} \tag{3.2}$$

where ω_{ij}, v_i, a_i, and a are infinitesimal parameters under these variations; the Lagrangian transforms as

$$\delta L = \frac{d}{d\tau}(m v_i x_i) \ . \tag{3.3}$$

Therefore, the Galilean group is a global symmetry of the Lagrangian. The generator in super phase space is

$$G = -\tfrac{1}{2}\omega_{ij}R_{ij} - v_i B_i + a_i P_i - aE \ , \tag{3.4}$$

which reproduces the transformation laws (3.10) by means of the DB's in (2.14). The explicit form of the generators is

$$\begin{aligned} &\text{Rotations: } R_{ij} = x_i p_j - x_j p_i - i\epsilon_i \epsilon_j \ , \\ &\text{Pure Galilean transformation: } B_i = m x_i - t p_i + i\eta\epsilon_i \ , \\ &\text{Spatial translations: } P_i = p_i \ , \\ &\text{Temporal translations: } E = E \ . \end{aligned} \tag{3.5}$$

These form a realization of the extended Galilean group with neutral element m under the DB's in (2.14).

We now analyze the local gauge symmetries of this model. The existence of such symmetries is suggested by the appearance of first-class constraints.[12] There are two types of gauge transformation: reparametrizations and supergauge. They are parametrized, respectively, with a and α, which are infinitesimal arbitrary functions of τ, even and odd, respectively. These transformations are

$$\begin{aligned} &\delta x_i = a\dot{x}_i - i\alpha\epsilon_i \ , \\ &\delta \epsilon_i = a\dot{\epsilon}_i + \alpha m \frac{\dot{\mathbf{x}} + \chi\boldsymbol{\epsilon}}{\dot{t} + \chi\eta} \ , \\ &\delta t = a\dot{t} - i\alpha\eta \ , \end{aligned} \tag{3.6}$$

$$\begin{aligned} &\delta \chi = a\dot{\chi} + \dot{a}\chi + \dot{\alpha} i \ ; \\ &\delta \eta = a\dot{\eta} + \alpha m \ , \\ &\delta \overline{\eta} = a\dot{\overline{\eta}} + \alpha m \frac{(\dot{x} + \chi\epsilon)^2}{(\dot{t} + \chi\eta)^2} \ . \end{aligned} \tag{3.7}$$

The variation of the Lagrangian (2.6) is a total derivative with respect to the evolution parameter in both cases, i.e., when $\alpha = 0$

$$\delta L = \frac{d}{d\tau}(aL) \ . \tag{3.8}$$

Note that the reparametrization invariance is not obvious because the Lagrangian (2.6) is not homogeneous of degree one in the velocities. When $a = 0$

$$\delta L = \frac{d}{d\tau}\left[\frac{i\alpha}{2}(\mathbf{p}\cdot\boldsymbol{\epsilon} - m\overline{\eta} + E\eta)\right] , \tag{3.9}$$

where $\mathbf{p}$ and E are given by (2.8a) and (2.8b).

The generators of these transformations are, respectively,[12]

$$\begin{aligned} G_{\text{rep}} &= a\left[\frac{i}{2m}(\mathbf{p}^2 - 2mE) + \chi(m\overline{\eta} - \mathbf{p}\cdot\boldsymbol{\epsilon} + E\eta)\right] \\ &\quad + \frac{d}{d\tau}(a\chi)\Pi \ , \\ G_{\text{sup}} &= i\alpha(\eta E - \mathbf{p}\cdot\boldsymbol{\epsilon} + m\overline{\eta}) + \dot{\alpha} i \Pi \ . \end{aligned} \tag{3.10}$$

IV. ELECTROMAGNETIC INTERACTIONS

In this section we construct the action of a nonrelativistic spinning particle interacting with an external electromagnetic field. In order to construct this action, we first introduce the interaction through the constraints in such a way that the new constraints remain first class. We impose this requirement in order to recover the free-particle model of the previous section when we turn off the interaction.

The standard way to introduce the electromagnetic interaction is through minimal substitution

$$\begin{aligned} &\mathbf{p} \to \mathbf{p} - e\mathbf{A} \ , \\ &E \to E - eA^0 \ , \end{aligned} \tag{4.1}$$

where $(A^0, \mathbf{A})$ are, respectively, the scalar and vector electromagnetic potentials. If we make this substitution in first-class constraints (2.8a) and (2.12), they become

$$\widetilde{S} = (\mathbf{p} - e\mathbf{A})^2 - 2m(E - eA^0) \ , \tag{4.2}$$

$$\chi_{\text{LL}}^{\text{EM}} = (E - eA^0)\eta - (\mathbf{p} - e\mathbf{A})\cdot\boldsymbol{\epsilon} + m\overline{\eta} \ . \tag{4.3}$$

Making use of the DB's in (2.17) we can verify whether those constraints are first class. We begin with $\chi_{\text{LL}}^{\text{EM}}$:

$$\begin{aligned} \{\chi_{\text{LL}}^{\text{EM}}, \chi_{\text{LL}}^{\text{EM}}\}' = -i[&(\mathbf{p} - e\mathbf{A})^2 - 2m(E - eA^0) \\ &- ie(2F^{0i}\eta\epsilon^i - F^{ij}\epsilon^i\epsilon^j)] \ , \end{aligned} \tag{4.4}$$

where

$$F^{ij} \equiv \frac{\partial A^i}{\partial x^j} - \frac{\partial A^j}{\partial x^i} \ ,$$

$$F^{0i} \equiv -\frac{\partial A^0}{\partial x^i} - \frac{\partial A^i}{\partial t}$$

are the electromagnetic fields. The right-hand side of (4.4) is not exactly the constraint $\widetilde{S}$, because of the additional term involving the interaction between the Grassmann variables and the electromagnetic field. This suggests a Pauli-type modification of the Schrödinger constraint. Consider the new constraint

$$S^{\rm EM}\equiv(\mathbf{p}-e\mathbf{A})^2-2m(E-eA^0)$$
$$-ie(2F^{0i}\eta\epsilon^i-F^{ij}\epsilon^i\epsilon^j)\approx 0 \qquad (4.5)$$

and therefore (4.4) is written as

$$\{\chi_{\rm LL}^{\rm EM},\chi_{\rm LL}^{\rm EM}\}'=-iS^{\rm EM} \qquad (4.6)$$

and we have

$$\{\chi_{\rm LL}^{\rm EM},S^{\rm EM}\}'=0\ . \qquad (4.7)$$

This means that $\chi_{\rm LL}^{\rm EM}$ and $S_{\rm LL}^{\rm EM}$ are first class.

We can now construct the Lagrangian which gives us first-class constraints $S^{\rm EM}$ and $\chi_{\rm LL}^{\rm EM}$. Making use of the inverse Legendre transformation we obtain

$$L_{\rm EM}=\frac{1}{2}m\left[\frac{(\dot{\mathbf{x}}+\chi\boldsymbol{\epsilon})^2}{\dot t+\chi\eta}-\frac{ie\dot t}{m^2}(F^{ij}\epsilon^i\epsilon^j+2F^{0i}\eta\epsilon^i)-\chi\overline{\eta}\right]$$
$$-\frac{i}{2}(\eta\dot{\overline{\eta}}+\overline{\eta}\dot\eta)+\frac{i}{2}\boldsymbol{\epsilon}\cdot\dot{\boldsymbol{\epsilon}}+e(\dot{\mathbf{x}}\mathbf{A}-\dot tA^0)\ , \qquad (4.8)$$

the Lagrangian consists of the two parts

$$L_{\rm EM}=L_0+L_{\rm int}^{\rm EM}\ , \qquad (4.9)$$

where

$$L_{\rm int}^{\rm EM}=\frac{-ie\dot t}{m^2}(F^{ij}\epsilon^i\epsilon^j+F^{0i}\eta\epsilon^i)+e(\dot x^iA^i+\dot tA^0)\ , \qquad (4.10)$$

which differs from minimal coupling by the appearance of the Pauli-type term.

The Dirac Hamiltonian is

$$H_D=\lambda\chi_{\rm LL}^{\rm EM}+\lambda_0S^{\rm EM} \qquad (4.11)$$

and with DB's in (2.17) Hamilton's equation of motion is

$$\dot x^i=-\lambda\epsilon^i+\lambda_02(p^i-eA^i)\ , \qquad (4.12a)$$

$$\dot t=-\lambda\eta+\lambda_02m\ , \qquad (4.12b)$$

$$\dot p^i=\lambda e(\epsilon^jF^{ij}-F^{0i}\eta)+eA^i$$
$$+2e\lambda_0[mF^{0i}-(p^j-eA^j)F^{ij}$$
$$-i(F^{0j}{}_{,i}\eta\epsilon^j+\tfrac{1}{2}F^{kj}{}_{,i}\epsilon^k\epsilon^j)]\ , \qquad (4.12c)$$

$$\dot E=-\lambda eF^{0i}\epsilon^i+2\lambda_0(p^i-eA^i)F^{0i}e+e\dot A^0\ , \qquad (4.12d)$$

$$\dot\epsilon^i=i\lambda(p^i-eA^i)-2\lambda_0e(F^{ij}\epsilon^j-F^{0i}\eta)\ , \qquad (4.12e)$$

$$\dot\eta=im\lambda\ , \qquad (4.12f)$$

$$\dot{\overline{\eta}}=\lambda(E-eA^0)i+2\lambda_0eF^{0i}\epsilon^i\ , \qquad (4.12g)$$

in the surface defined by $S^{\rm EM}$ and $\chi_{\rm LL}^{\rm EM}$ where

$$F^{0j}{}_{,i}\equiv\partial F^{0j}/\partial x^i,\ \ F^{kj}{}_{,i}\equiv\partial F^{kj}/\partial x^i\ .$$

From the Galilean realization (3.13), we can see that at the superspace level there is a nonorbital part of the angular momentum that we can identify with the spin, i.e.,

$$S_{ij}=-i\epsilon_i\epsilon_j \qquad (4.13)$$

from which we can define the three-vector

$$S_k=\tfrac{1}{2}\epsilon_{ijk}S_{ij}\ . \qquad (4.14)$$

The evolution of S_k is given by

$$\dot S^k=2\lambda_0e\epsilon^{ikl}S^iB^l+\epsilon^{ijk}[\lambda\epsilon^i(p^j-eA^j)$$
$$+2i\lambda_0eF^{0i}\eta\epsilon^j]\ . \qquad (4.15)$$

The first term of Eq. (4.15) is the nonrelativistic limit of the Bargmann-Michel-Telegdi equation.[13] The other terms disappear at a purely classical level.[11]

The acceleration is

$$\ddot x^k=\lambda_0\epsilon^{kil}\dot x^iB^l+4e\lambda_0\left[-\frac{\lambda}{2}F^{kj}\epsilon^j-i\lambda_0(F^{0j}{}_{,k}\eta\epsilon^j+\tfrac{1}{2}F^{ij}{}_{,k}\epsilon^i\epsilon^j)\right] \qquad (4.16)$$

which shows the usual precessional velocity about the magnetic field and other terms depending on arbitrary functions and spatial derivatives of magnetic and electric field.

It is worth mentioning the global symmetries. If A^μ is an object which transforms as a four-vector under the Poincaré group, it is clear that Galilean symmetry will be broken. Nevertheless, it is possible to maintain this symmetry by requiring that A^0 and $\mathbf{A}$ transform as the potentials of Galilean electromagnetism.[14] There exist two different kinds of Galilean electromagnetism: "magnetic" and "electric." In each, the respective potentials behave differently under Galilean transformations. If we study the variation of Lagrangian (4.8) under these two possible kinds of Galilean transformations we will see that the Galilean group is a symmetry only when the electromagnetic fields are of the magnetic type, which is not a surprising result. It is due to the fact that we have introduced the interaction through minimal coupling.

V. YANG-MILLS INTERACTIONS

In this section we study the interaction of a nonrelativistic spinning particle with an external Yang-Mills field.[15] We assume that in addition to the degrees of freedom corresponding to spin the particle also has internal degrees of freedom which are described by Grassmann variables.[17,18] In this way we will automatically have a finite-dimensional representation of the internal-symmetry group.

The free Lagrangian is assumed to depend on the same space-time, and Grassmann spin variables, as before, and a new set of Grassmann variables θ_α $(\alpha=1,\dots,n)$ associated with the internal degrees of freedom. These new variables belong to a representation R of a given symmetry group. We consider $R=R_1+R_1^*$. In this case, the Lagrangian has a term L_0 depending on the old variables and another depending on the new ones

$$L_F=L_0+\frac{i}{2}\sum_{\alpha=1}^{m}(\theta_\alpha^*\dot\theta_\alpha-\dot\theta_\alpha^*\theta_\alpha)\ . \qquad (5.1)$$

The canonical momenta of this Lagrangian are the old momenta (2.8) and the new momenta conjugate to the new variables

$$\Pi_\alpha^* = \frac{\partial L_F}{\partial \dot\theta_\alpha^*} = -\frac{i}{2}\theta_\alpha ,$$
$$\Pi_\alpha = \frac{\partial L}{\partial \dot\theta_\alpha} = -\frac{i}{2}\theta_\alpha^* . \tag{5.2}$$

As a consequence, we have as primary constraints the old constraints (2.9) plus $2n$ new ones:

$$\Psi_\alpha^* = \Pi_\alpha^* + \frac{i}{2}\theta_\alpha \approx 0 ,$$
$$\Psi_\alpha = \Pi_\alpha + \frac{i}{2}\theta_\alpha^* \approx 0 . \tag{5.3}$$

If we enlarge the graded symplectic structure[16] introduced previously with a new set of Poisson brackets (PB's)

$$\{\theta_\alpha, \Pi_\beta\} = -\delta_{\alpha\beta},\ \ \{\theta_\alpha^*, \Pi_\beta^*\} = -\delta_{\alpha\beta} , \tag{5.4}$$

$$\{\theta_\alpha, \Pi_\beta^*\} = \{\theta_\alpha^*, \Pi_\beta\} = 0 , \tag{5.5}$$

and these new variables have vanishing PB's with all other variables. We can see the character of all the constraints. In particular (2.9b), (2.9c), (2.9d), and (5.3) are second class. If we want to eliminate them, we should introduce new Dirac brackets:

$$\{A,B\}^\dagger = \{A,B\}^* - \sum_{\alpha=1}^{n} i(\{A,\Psi_\alpha\}^*\{\Psi_\alpha^*,B\}^* + \{A,\Psi_\alpha^*\}^*\{\Psi_\alpha,B\}^*) , \tag{5.6}$$

where the starred DB's are given by (2.12). In the super phase space $(x^i,p^i,t,E,\epsilon^i,\eta,\bar\eta,\theta_\alpha,\theta_\alpha^*)$ the DB's structure is the old one (2.14) with $\{\ \}^+$ in the place of $\{\ \}^*$ plus the new DB's:

$$\{\theta_\alpha,\theta_\beta\}^\dagger = \{\theta_\alpha^*,\theta_\beta^*\}^\dagger = 0 ,$$
$$\{\theta_\alpha,\theta_\beta^*\}^\dagger = -i\delta_{\alpha\beta} . \tag{5.7}$$

Let us now study the symmetry properties of the Lagrangian (5.1) under global transformations of the internal-symmetry group given by

$$\delta\theta = -i\alpha_a\tau^a,\ \ \delta\theta^* = i\theta^*\alpha_a\tau^a , \tag{5.8}$$

where τ^a, $a=1,\ldots,n$ are the generators of the internal-symmetry group in the representation R and α are the infinitesimal parameters. This transformation can be obtained through the DB's in (5.6) by means of the generator

$$I^a = \theta^*\tau^a\theta \tag{5.9}$$

which verifies

$$\{I^a,I^b\}^\dagger = f_c^{ab} I_c , \tag{5.10}$$

where the f_c^{ab} are the structure of constants of G. If we calculate the corresponding variation of the Lagrangian, δL_F, we have

$$\delta L_F = 0 . \tag{5.11}$$

Therefore, as expected, the transformation (5.8) is a symmetry transformation of our Lagrangian, (5.1).

Let us now consider the local transformations

$$\delta\theta = -i\alpha_a(\mathbf{x},t)\tau^a\theta ,$$
$$\delta\theta^* = i\theta\alpha_a(\mathbf{x},t)\tau^a , \tag{5.12}$$

where $\alpha^a(x,t)$ are arbitrary functions. Under these transformations the variation of the Lagrangian is given by

$$\delta L_F = \theta^*(\alpha_{a,i}\dot x^i + \alpha_{a,t}\dot t)\tau^a\theta \tag{5.13}$$

and from (5.9) we can write

$$\delta L_F = (\alpha_{a,i}\dot x^i + \alpha_{a,t}\dot t) I^a . \tag{5.14}$$

In order to compensate for this term, we could add another term to the Lagrangian, which depends on the Yang-Mills fields A_a^i and A_a^0 with the correct transformation law under the action of group G:

$$\delta A_a^i = -\frac{1}{2}\frac{\partial\alpha_a}{\partial x^i} + f_a^{bc}\alpha_b A_c^i ,$$
$$A_a^0 = \frac{1}{2}\frac{\partial\alpha_a}{\partial t} + f_a^{bc}\alpha_b A_c^0 . \tag{5.15}$$

This term appears in L as a proper interaction term:

$$L_{\rm int} = g(A_a^i\dot x^i - A_a^0\dot t) I^a . \tag{5.16}$$

With this term the total Lagrangian becomes

$$L = L_F + L_{\rm int} \tag{5.17}$$

and

$$\delta L = \delta L_F + \delta L_{\rm int} = 0 .$$

We can rewrite the Lagrangian (5.17) as

$$L = L_0 - \frac{i}{2}\theta^*\left[\dot\theta - \frac{i}{2}g(A_a^i\dot x^i - A_a^0\dot t)\tau^a\theta\right] - \frac{i}{2}\left[\dot\theta^* + \frac{i}{2}g\theta^*(A_a^i\dot x - A_a^0\dot t)\tau^a\right]\theta . \tag{5.18}$$

This Lagrangian is the same as L_F but with

$$\dot\theta \to \dot\theta - \frac{i}{2}g(A_a^i\dot x^i - A_a^0\dot t)\tau_a\theta . \tag{5.19}$$

The new canonical moments conjugate to the space-time variables are

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{x}}} = \frac{\partial L_0}{\partial \dot{\mathbf{x}}} + gI^a\mathbf{A}_a = \mathbf{p}_0 + gI^a\mathbf{A}_a , \tag{5.20}$$

$$E = -\frac{\partial L}{\partial \dot t} = -\frac{\partial L_0}{\partial \dot t} + gI^a A_a^0 = E_0 + gI^a A_a^0 , \tag{5.21}$$

and the momenta conjugate to the Grassmann variables are not affected by the Yang-Mills interaction, so that the primary constraints are the same as in the free case, except the constraint (2.9a) which is now given by

$$\tilde S = (\mathbf{p} - gI^a\mathbf{A}_a)^2 - 2m(E - gI^a A_a^0) \approx 0 , \tag{5.22}$$

which is nothing but (2.9a) with the usual "minimal coupling."

Following the usual procedure, we can calculate the canonical Hamiltonian:

$$H_c = \chi[(E - gI_a A_a^0)\eta - (\mathbf{p} - gI^a \mathbf{A}_a)\cdot\epsilon + m\bar{\eta}] + \dot{\chi}\Pi \ . \tag{5.23}$$

If we require the stability of the primary constraints, we obtain a secondary constraint

$$\chi_{\rm LL}^{\rm YM} = (E - gI^a A_a^0)\eta - (\mathbf{p} - gI^a \mathbf{A}_a)\cdot\epsilon + m\bar{\eta} \approx 0 \ . \tag{5.24}$$

Following the same reasoning as in the electromagnetic case, these constraints S and $\chi_{\rm LL}^{\rm YM}$ must be a first class. The DB's in (5.6) of $\chi_{\rm LL}^{\rm YM}$ with itself is

$$\{\chi_{\rm LL}^{\rm YM}, \chi_{\rm LL}^{\rm YM}\}' = i[\tilde{S} - ig(2F^{0i}{}_a I^a \eta\epsilon^i - F^{ij}{}_a I^a \epsilon^i\epsilon^j)] \ , \tag{5.25}$$

where

$$F^{ij}{}_a \equiv \frac{\partial A_a^i}{\partial x^j} - \frac{\partial A_a^j}{\partial x^i} - gA_c^i A_b^j f^{cb}{}_a$$

and

$$F^{0i}{}_a \equiv -\frac{\partial A_a^0}{\partial x^i} - \frac{\partial A_a^i}{\partial t} - gA_c^0 A_b^i f^{cb}{}_a \ . \tag{5.26}$$

Equation (5.25) shows us that as in the electromagnetic case, it is necessary to add Lagrangian (5.18), another term expressing the interaction between the spin and Yang-Mills fields. Therefore, if we want to describe a particle with spin and internal degrees of freedom interacting with an external Yang-Mills field minimal coupling is not efficient. A possible modification would be to add an interaction term such as

$$L_{\rm intII} = -\frac{\dot{t}}{2m} igI^a(2F^{0i}{}_a \eta\epsilon^i - F^{ij}{}_a \epsilon^i\epsilon^j) \tag{5.27}$$

which is gauge invariant under the gauge group G. The final Lagrangian is

$$L = L_F + L_{\rm intI} + L_{\rm intII} \tag{5.28}$$

which gives us two first-class constraints $S^{\rm YM}$:

$$S^{\rm YM} = (\mathbf{p} - gI^a \mathbf{A}^a)^2 - 2m(E - gI^a A_a^0) - ig(2F^{0i}{}_a I^a \eta\epsilon^i + F^{ij}{}_a I^a \epsilon^i\epsilon^j) \approx 0 \tag{5.29}$$

and $\chi_{\rm LL}^{\rm YM}$ given by (5.24).

Now we can calculate the equations of motion for the relevant variables with the Dirac Hamiltonian:

$$H_D = \lambda\chi_{\rm LL}^{\rm YM} + \lambda_0 S^{\rm YM} \ . \tag{5.30}$$

We obtain

$$\begin{aligned}
\dot{x}^i &= -\lambda\epsilon^i + 2\lambda_0(p^i - eI^a A_a^i) \ , \\
\dot{t} &= -\lambda\eta + 2m\lambda_0 \ , \\
\dot{p}^i &= eI^a A_a^i - \lambda g(F^{0i}{}_a \eta - F^{ij}{}_a \epsilon^j)I^a \\
&\quad + 2\lambda_0 gI^a(F^{ij}{}_a p^j + mF^{0i}{}_a + \eta F^{0j}{}_{a,i}\epsilon^j - \tfrac{1}{2}\epsilon^k\epsilon^j F^{kj}{}_{a,i}) \ , \\
\dot{\epsilon} &= \lambda_0 gI^a 2(F^{0i}{}_a \eta - F^{ij}{}_a \epsilon^j) + i\lambda(p^i - eI^a A_a^i) \ , \\
\dot{\bar{\eta}} &= i\lambda E + 2\lambda_0 gI^a F^{0i}{}_a \epsilon^i \ , \\
\dot{\eta} &= im\lambda \ ,
\end{aligned} \tag{5.31}$$

in the surface defined by $S^{\rm YM}$ and $\chi_{\rm LL}^{\rm YM}$ where we used the notation of a semicolon for the covariant derivative:

$$F^{0j}{}_{a;i} \equiv \frac{\partial F^{0j}{}_a}{\partial x^i} + gA_c^i f^{cb}{}_a F^{0j}{}_b \ ,$$

$$F^{ij}{}_{a;k} \equiv \frac{\partial F^{ij}}{\partial x^k} + gA_c^k f^{cb}{}_a F^{ij}{}_b \ .$$

As in the electromagnetic case, these equations will acquire a clear physical meaning only when we introduce a suitable distribution function. It is possible to maintain the global Galilean symmetry by using the nonrelativistic Yang-Mills fields introduced by Palumbo.[18]

VI. QUANTIZATION

In this section we study the quantization of the pseudoclassical models introduced in the preceding sections. We will follow the usual procedure which associates self-adjoint operators with dynamical variables, acting on a suitable Hilbert space, i.e., if A,B are dynamical variables, which after quantization become observables. Furthermore,

$$i\hbar\{A,B\}^\dagger = [\hat{A},\hat{B}]_\pm \ , \tag{6.1}$$

where the DB's become commutators or anticommutators depending on the character of the variables[3] and the first class are restrictions upon the states of the Hilbert space.

For the space-time and spin variables we must find a set of self-adjoint operators such that

$$[\hat{x}^i, \hat{p}^j] = i\hbar\delta_{ij}, \quad [\hat{t}, \hat{E}] = -i\hbar \ , \tag{6.2}$$

$$\begin{aligned}
&[\epsilon^i, \epsilon^j]_+ = \delta_{ij\hbar}, \quad [\hat{\eta}, \hat{\bar{\eta}}]_+ = -\hbar \ , \\
&[\epsilon^i, \hat{\eta}]_+ = [\hat{\epsilon}^i, \bar{\eta}]_+ 0 \ .
\end{aligned} \tag{6.3}$$

For the operators appearing in (6.2) there is no problem; they are the usual energy, momentum, position, and time operators with respect to others, (6.3), a possible realization is

$$\begin{aligned}
\hat{\epsilon}_i &= \left[\frac{\hbar}{2}\right]^{1/2} \gamma_5\gamma^i \ , \\
\hat{\eta} &= \frac{1}{u}\left[\frac{\hbar}{2}\right]^{1/2} \gamma_5(\gamma_0 - 1) \ ,
\end{aligned}$$

and

$$\hat{\bar{\eta}} = u\frac{1}{2}\left[\frac{\hbar}{2}\right]^{1/2} \gamma_5(\gamma_0 + 1) \ , \tag{6.4}$$

where γ^μ, γ_5 are the Dirac matrices, satisfying

$$[\gamma_\mu, \gamma_\nu]_+ = 2g_{\mu\nu} \ ,$$

$$[\gamma_5, \gamma_5]_+ = 2I, [\gamma_5, \gamma_\mu]_+ = 0 \ ,$$

and u is a constant having the dimensions of velocity.

A. Free case

We impose the constraints S in (2.9a), and $X_{\rm LL}$ in (2.12), over the states

$$(2m\hat{E}-\hat{\mathbf{p}}^2)\,|\,\Psi\rangle=0\ , \tag{6.5}$$

$$\left[\frac{\hbar}{2}\right]^{1/2}\left[\gamma_5(\gamma_0-1)\hat{E}\frac{1}{u}-\mathbf{p}\cdot\gamma_5\boldsymbol{\gamma}+u\frac{\gamma_5}{2}(\gamma_0+1)m\right]|\,\Psi\rangle=0\ . \tag{6.6}$$

Equation (6.5) is the Schrödinger equation. Equation (6.6) is equivalent to the Levy-Leblond equation.

Equation (6.6) has the following advantage over the original equation of Levy-Leblond[10] because of the first character of $X_{\rm LL}$ and S. We have, after quantization,

$$[\hat{X}_{\rm LL},\hat{X}_{\rm LL}]_+=\hbar S\ , \tag{6.7}$$

which guarantees that the square of $\hat{X}_{\rm LL}$ is the Schrödinger operator whereas the Levy-Leblond equation does not share this property. Ψ is a four-component object which can be written as two bispinors ϕ,χ:

$$\Psi=\begin{bmatrix}\phi\\ \chi\end{bmatrix}\ . \tag{6.8}$$

Making the substitution (6.9) in (6.6) we obtain

$$\begin{aligned}&-\frac{2\hat{E}}{u}\chi+\boldsymbol{\sigma}\cdot\mathbf{p}\phi=0\ ,\\ &-\boldsymbol{\sigma}\cdot\hat{\mathbf{p}}\chi+um\phi=0\ ,\end{aligned} \tag{6.9}$$

where the $\boldsymbol{\sigma}$ are the Pauli matrices. It is possible to eliminate the dependence on the dimensional parameter u by redefining the wave-function components if we define $\tilde{\phi}=u\phi$; we obtain

$$\begin{aligned}&-2\hat{E}\chi+\boldsymbol{\sigma}\cdot\mathbf{p}\tilde{\phi}=0\ ,\\ &-\boldsymbol{\sigma}\cdot\hat{\mathbf{p}}\chi+m\tilde{\phi}=0\ .\end{aligned} \tag{6.10}$$

Under a Galilean transformation the wave function transforms as

$$\begin{bmatrix}\tilde{\phi}'(\mathbf{x}',t')\\ \chi'(\mathbf{x}',t')\end{bmatrix}=e^{if(\mathbf{x},t)}\begin{bmatrix}D^{1/2}(R) & \boldsymbol{\sigma}\cdot\mathbf{v}D^{1/2}(R)\\ 0 & D^{1/2}(R)\end{bmatrix}\begin{bmatrix}\tilde{\phi}(\mathbf{x},t)\\ \chi(\mathbf{x},t)\end{bmatrix}\ , \tag{6.11}$$

where $D^{1/2}(R)$ is the two-dimensional representation of the rotation group and $F(x,t)$ is the phase factor:

$$f(\mathbf{x},t)=\tfrac{1}{2}m\mathbf{v}^2t+m\mathbf{v}\cdot R\mathbf{x}\ . \tag{6.12}$$

B. Electromagnetic interaction

In this case, when we quantize the constraints $X_{\rm LL}^{\rm EM}$ and $S^{\rm EM}$ in (4.5) we obtain[10]

$$\begin{aligned}&-2(\hat{E}-eA^0)\chi+\boldsymbol{\sigma}\cdot(\mathbf{p}-e\,\mathbf{A})\tilde{\phi}=0\ ,\\ &-\boldsymbol{\sigma}\cdot(\mathbf{p}-e\,\mathbf{A})\chi+m\tilde{\phi}=0\ ,\end{aligned} \tag{6.13}$$

by making use of the same redefinition of wave function as in the free case. If we express $\tilde{\phi}$ in terms of Φ we obtain

$$\left[\frac{(\hat{\mathbf{p}}-eA)^2}{2m}-(E-eA^0)+\frac{eh}{2m}B^k\sigma^k\right]\chi=0\ , \tag{6.14}$$

which is the Pauli equation, with the gyromagnetic ratio equal to 2. Further, if we use (6.15) to obtain the equation for Φ we obtain

$$\left[\frac{(\hat{\mathbf{p}}-e\,\mathbf{A})^2}{2m}-(\hat{E}-eA^0)+\frac{eh}{2m}\mathbf{B}\cdot\boldsymbol{\sigma}\right]\tilde{\phi}-ie2h\sigma^iF^{0i}\chi=0\ . \tag{6.15}$$

Equations (6.14) and (6.15) are not but the direct quantization of the constraint $S^{\rm EM}$ in (4.5).

C. Yang-Mills interaction

Quantizing the variables describing the internal degrees of freedom, we obtain, for the quantization $X_{\rm LL}^{\rm YM}$,

$$\begin{aligned}&-2(\hat{E}-gI^aA_a^0)\chi+\boldsymbol{\sigma}\cdot(\hat{\mathbf{p}}-gI^a\mathbf{A}_a)\tilde{\phi}=0\ ,\\ &-\boldsymbol{\sigma}\cdot(\mathbf{p}-I^a\mathbf{A}_a)\chi+m\tilde{\phi}=0\ ,\end{aligned} \tag{6.16}$$

which reduces for the variable to χ.

$$\left[\frac{\hat{\mathbf{p}}-gI^a\mathbf{A}^a)^2}{2m}-(\hat{E}-gI^aA_a^0)+\frac{1}{2m}\epsilon^{ijk}\sigma^kgI^a(F^{ij}{}_a+gA_c^iA_b^jf^{cb}{}_a)\right]\chi=0\ , \tag{6.17}$$

which is the Pauli equation for the Yang-Mills case.

VI. CONCLUSIONS

In this paper we have studied a pseudoclassical model for a nonrelativistic particle. The model is presented through a singular Lagrangian. Apart from second-class constraints, related to kinetic terms in Grassmann variables, we have two first-class constraints.

By supposing definite transformation properties for Grassmann variables, the action becomes Galilean invariant. Furthermore, we have two local gauge symmetries associated with the first-class constraints: reparametrization and supergauge invariance.

From the Dirac Hamiltonian, given as a linear combination of two first-class constraints, we obtain the equations of motion at the pseudoclassical level. The graded symplectic structure is given by (2.17).

At this level, the Grassmann variables do not have a definite physical meaning. A possible interpretation of these variables is obtained after quantization, which we have done in Sec. V; there we have seen that a quantized Grassmann variable corresponds to an element of a Clifford algebra.

The Hilbert space of physical states is obtained by requiring the following conditions:

$$X_{\rm LL}\,|\,\Psi\rangle=0,\quad S\,|\,\Psi\rangle=0\ .$$

These conditions are nothing but the Levy-Leblond wave equation for a nonrelativistic particle with spin one-half.

Therefore, our dynamical system defined by the Lagrangian (2.7) is nothing but the pseudoclassical model of the Levy-Leblond wave equation.

We have also studied the interactions of this particle with external electromagnetic and Yang-Mills fields. The results are in good agreement with what is expected at the quantum level. As a final comment, we want to point out that a pure classical interpretation of this model is available[11] by using a suitable distribution function.

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