Pseudoclassical description of a relativistic spinning particle

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(Received 14 February 1985)

A pseudoclassical model for a relativistic spinning particle is studied. The only physically meaningful world line is the one without *Zitterbewegung*. The Poincaré realization for this situation is constructed.

I. INTRODUCTION

The description of a classical relativistic spinning particle is an old problem.[1] Recently, there has appeared a new approach to the problem, making use of the Hamiltonian formalism in terms of constraints[2,3] following the theory developed by Dirac.[4]

In this work we want to study a possible pseudoclassical description of a spinning particle as an example of a constrained system with Grassmann variables. We will consider a model that, apart from the usual space-time variables, has five Grassmann variables: one pseudovector ϵ_μ and one pseudoscalar ϵ_5.[5–7] This model has been extensively studied, since after quantization it gives rise to the Dirac equation.[5]

We present the model from a Hamiltonian point of view. That means we give a graded symplectic structure[8] and two first-class constraints. One of these, χ_0, is even and corresponds to the mass-shell condition; the other one, χ_D, is odd and after quantization yields the Dirac equation.

Since our interest is to deduce the classical description of a spinning particle associated with this pseudoclassical model we will need to define some sort of world line in Minkowski space starting from the corresponding object in superspace (Minkowski space plus Grassmann variables). Taking into account that this model has two gauge invariances, reparametrization and supergauge corresponding to the first-class constraints, the superspace candidate must be gauge invariant, but any such object is a sheet, rather than a line, in superspace. The question is then how to construct a good world line in Minkowski space starting from the sheet. We proceed by introducing a distribution function[6] of Grassmann variables in such a way that we can pass from anticommuting objects in superspace to real numbers in Minkowski space.

In a previous work,[9] we showed that there is no distribution function which gives a physical meaning to the gauge-invariant sheets because different lines on the sheet give rise to different world lines in Minkowski space. So that if we want to have a unique world line in Minkowski space, we need to have a line in superspace. We can construct this line by introducing a new constraint, ϕ_D, that breaks the supergauge invariance and selects a line in each sheet.

The next step will be to find the corresponding world line in Minkowski space starting from the abstract submanifold M on superspace defined by χ_D and ϕ_D. This is realized by a suitable distribution function that averages over Grassmann variables in the submanifold M. Then we must verify that this is a valid world line, which means it has to satisfy the world-line condition[10] (WLC). In other words, the canonical and geometrical realizations must coincide up to a reparametrization. We will see that the WLC is satisfied for any constraint ϕ_D.

Once we have a good geometrical candidate for the world line, we will study the physical properties of this model. In principle it can contain *Zitterbewegung*, an important feature of a relativistic spinning particle,[1] by choosing an appropriate constraint ϕ_D, but a careful analysis reveals that such a constraint produces a spin vector functionally dependent on the position of the particle, which is incorrect for a free particle.

If we want to avoid this contradiction we must accept the parallelism between the four-velocity and the four-momentum of the particle. This requires the constraint ϕ_D to be proportional to ϵ_5.

The organization of the work is as follows. In Sec. II we introduce the model. In Sec. III we discuss the physically invariant object and introduce the constraint ϕ_D. In Sec. IV we study the distribution function in the submanifold M. In Sec. V we discuss the WLC. Section VI is devoted to the physical content of the model, and finally in the last section we give a Poincaré realization for this model.

II. THE MODEL

We start working in phase superspace $(x^\mu, P^\mu, \epsilon^\mu, \epsilon_5)$ with a graded[8] symplectic structure given by the Poisson brackets (PB)

$$\{x^\mu,P^\nu\}=-g^{\mu\nu}\ ,\quad \{\epsilon^\mu,\epsilon^\nu\}=ig^{\mu\nu}\ ,\quad \{\epsilon_5,\epsilon_5\}=-i\ , \tag{2.1}$$

where x^μ represents the instantaneous position of the particle, P^μ is the four-momentum canonically conjugate to x^μ, and ϵ^μ and ϵ_5 are canonical Grassmann variables associated with the spin content of the model, as we shall see later.

We provide the dynamical content of the model by means of[5–7,11] two constraints defined on the superspace,

$$\chi_0=P^2-m^2\ , \tag{2.2}$$

$$\chi_D=P_\mu\epsilon^\mu-m\epsilon_5\ . \tag{2.2a}$$

Note that χ_0 is even and corresponds to the mass-shell condition while χ_D is odd and is such that after quantization it gives rise to the Dirac equation.

If we compute the PB between the constraints we obtain

$$\{\chi_0,\chi_0\}=0\ ,\quad \{\chi_0,\chi_D\}=0\ ,\quad \{\chi_D,\chi_D\}=i\chi_0\ .$$

Therefore χ_0, χ_D are first class. Following the Dirac theory[4] of constrained Hamiltonian systems we know that the evolution is generated by the Hamiltonian

$$H_D=\lambda_0\chi_0+\lambda_D\chi_D\ , \tag{2.3}$$

where λ_0 and λ_D are arbitrary functions of the evolution parameter τ, so that the equations of motion will be

$$\begin{aligned}\dot{x}_\mu&=\{x_\mu,H_D\}=-2\lambda_0P_\mu+\epsilon_\mu\lambda_D\ ,\\ \dot{P}_\mu&=\{P_\mu,H_D\}=0\ ,\\ \dot{\epsilon}_\mu&=\{\epsilon_\mu,H_D\}=-i\lambda_DP_\mu\ ,\\ \dot{\epsilon}_5&=\{\epsilon_5,H_D\}=im\lambda_D\ .\end{aligned} \tag{2.4}$$

This evolution is confined to a sheet S, defined by the constraints χ_0 and χ_D.

The sheet S, will remain unaltered under gauge transformations generated by the two first-class constraints through the PB, i.e., the sheet is invariant under the variations

$$\delta q=\{q,a\chi_0+\alpha\chi_D\}, \tag{2.5}$$

where q is any variable of the phase superspace, and a and α are the gauge parameters which are even and odd, respectively. Explicitly

$$\begin{aligned}\delta x^\mu&=a\dot{x}^\mu+i\alpha\epsilon^\mu\ ,\\ \delta\epsilon^\mu&=a\dot{\epsilon}^\mu-\alpha P^\mu\ ,\\ \delta\epsilon_5&=a\dot{\epsilon}_5-m\alpha\ .\end{aligned} \tag{2.6}$$

These are gauge transformations, and we call them reparametrization if $\alpha=0$ and supergauge when $a=0$.

The model has global symmetries as well; it is Poincaré invariant. That means it is invariant under transformations generated by

$$G=\tfrac{1}{2}\omega^{\alpha\beta}M_{\alpha\beta}+a^\alpha P_\alpha \tag{2.7}$$

through the PB (2.1), $\omega^{\alpha\beta}$ and a^α being the transformation parameters and

$$\begin{aligned}M_{\alpha\beta}&=x_\alpha P_\beta-x_\beta P_\alpha-i\epsilon_\alpha\epsilon_\beta\ ,\\ P_\alpha&=P_\alpha\end{aligned} \tag{2.8}$$

are the generators which realize the Poincaré algebra.

III. CLASSICAL CONTENT

Now, we want to discuss the physical content of the model at the classical level. We begin by defining what we call a "world line" in superspace as a family of "events" $L(x^\mu(\tau),\epsilon^\mu(\tau),\epsilon_5(\tau))$, which is a solution of differential equations of motion (2.4).

The world line L is mapped onto itself under a reparametrization, but under a supergauge transformation it becomes another line L', that means L is not a gauge-invariant object and so is not a good candidate for an object with physical meaning. Nevertheless we can construct such an invariant object by performing all the possible supergauge transformations over the line L, obtaining the two-dimensional sheet S defined by the constraints χ_0 and χ_D.

After that, we would think that the physical content of the model is in the sheets, but, from the analysis of a previous work[9] we concluded that the sheets S are not physical because there does not exist a distribution function[6] on the Grassmann variables which gives the same world line in Minkowski space from different lines of the sheet related by a supergauge transformation.

In order to have a unique world line in Minkowski space, it will be necessary to choose a line L on S by breaking the supergauge symmetry. We can do that by introducing a new constraint ϕ_D, an odd function of x, P, ϵ, ϵ_5, and τ, such that

$$\{\chi_D,\phi_D\}\neq 0\ . \tag{3.1}$$

In that way, χ_D becomes a second-class constraint. If we require the stability of the constraint ϕ_D by means of (2.3) we obtain

$$\dot{\phi}_D=0=\frac{\partial\phi_D}{\partial\tau}+\{\phi_D,H\}=\frac{\partial\phi_D}{\partial\tau}+\lambda_0\gamma+\lambda_D D\ , \tag{3.2}$$

where

$$\gamma\equiv\{\phi_D,\chi_0\}\ ,\quad D\equiv-\{\phi_D,\chi_D\}\ .$$

Now, if we choose the constraint ϕ_D such that D has an even, non-Grassmann, part different from zero, D^{-1} will exist, and we could use (3.2) to express λ_D as

$$\lambda_D=\left[-\frac{\partial\phi_D}{\partial\tau}-\lambda_0\gamma\right]\frac{1}{D} \tag{3.3}$$

and Dirac's Hamiltonian will be

$$H_D=\lambda_0\left[\chi_0-\frac{\gamma}{D}\chi_D\right]-\frac{\partial\phi_D}{\partial\tau}\frac{1}{D}\chi_D\ , \tag{3.4}$$

from which we can define a new first-class constraint χ_0',

$$\chi_0' \equiv \chi_0 - \frac{\gamma}{D}\chi_D \ . \tag{3.5}$$

In order to eliminate the second-class constraints we introduce the corresponding Dirac bracket whose explicit expression is given by

$$\{A,B\}^* = \{A,B\} + \frac{1}{D^2}\{A,\chi_D\}\{\phi_D,\phi_D\}\{\chi_D,B\} + \frac{1}{D}(\{A,\chi_D\}\{\phi_D,B\} + \{A,\phi_D\}\{\chi_D,B\}) \ . \tag{3.6}$$

The evolution of dynamical variables will now be

$$\begin{aligned} \dot{x}^\mu &= \lambda_0\{x^\mu,\chi_0'\}^* \simeq -\lambda_0\left[2P^\mu - \frac{\gamma}{D}\epsilon^\mu\right] , \\ \dot{P}^\mu &= \lambda_0\{P^\mu,\chi_0'\}^* \simeq 0 , \\ \dot{\epsilon}^\mu &= \lambda_0\{\epsilon^\mu,\chi_0'\}^* \simeq i\frac{\lambda_0}{D}\gamma P^\mu , \\ \dot{\epsilon}_5 &= \lambda_0\{\epsilon_5,\chi_0'\} \simeq i\frac{\lambda_0}{D}\gamma m \ . \end{aligned} \tag{3.7}$$

After that it is possible to define a world line in superspace as a uniparametric family of events $L(x^\mu(\tau),\epsilon^\mu(\tau),\epsilon_5(\tau))$: the solution of the differential equation (3.7). This line will be invariant under gauge transformations (reparametrizations) generated by the only first-class constraint χ_0': they are

$$\begin{aligned} \delta x^\mu &= a\{x^\mu,\chi_0'\}^* = \frac{a}{\lambda_0}\dot{x}^\mu , \\ \delta\epsilon^\mu &= a\{\epsilon^\mu,\chi_0'\}^* = \frac{a}{\lambda_0}\dot{\epsilon}^\mu , \\ \delta\epsilon_5 &= a\{\epsilon_5,\chi_0'\}^* = \frac{a}{\lambda_0}\dot{\epsilon}_5 \ . \end{aligned} \tag{3.8}$$

Therefore, L is a gauge-invariant object which carries the physical content of the model in superspace.

Now we want to pass from this world line in superspace to a "good" world line in Minkowski space. We can do that by a means of a distribution function, ρ, acting in the submanifold M on the phase space defined by χ_D and ϕ_D, such that we can average over the Grassmann variables and in this way pass from the model with Grassman variables to a model with real quantities. In the next section we will explicitly study this distribution function.

IV. DISTRIBUTION FUNCTION ON GRASSMANN VARIABLES

At this point the model is defined by the first-class constraint χ_0' on the submanifold M of the phase superspace defined by χ_D and ϕ_D. As was first demonstrated by Berezin and Marinov,[6] we can pass from the superspace to the Minkowski space by means of a suitable distribution function on the Grassmann variables. In our case the distribution function ρ' works in the submanifold M.

A way to parametrize the coordinates of the submanifold M is to perform a Shanmugadhasan[12] transformation, which is a canonical transformation characterized by the fact that the constraints χ_D and ϕ_D become a subset $(\tilde{\epsilon},\tilde{\epsilon}_5)$ of the new set of canonical variables, $\tilde{P}^\mu,\tilde{x}^\mu,\tilde{\epsilon}_\lambda,\tilde{\epsilon},\tilde{\epsilon}_5$ $(\lambda=1,2,3)$. Evidently we must do the transformation requiring that $\tilde{\epsilon}=0$, $\tilde{\epsilon}_5=0$ determines the same submanifold given by $\chi_D=0$ and $\phi_D=0$. Furthermore, we know the graded symplectic coordinates of M given by $x^\mu,\tilde{P}^\mu,\tilde{\epsilon}_\lambda$. Therefore, the phase-space distribution function on M will be a function $\rho'(\tilde{x}^\mu,\tilde{P}^\mu,\tilde{\epsilon}_\lambda)$.

If we want to give a correct meaning to $\rho'(\tilde{x}^\mu,\tilde{P}^\mu,\tilde{\epsilon}_\lambda)$ as a distribution function we must demand two conditions.

(i) Normalization condition:

$$\int d\mu'\rho'(\tilde{\epsilon}_\lambda P^\mu,\tilde{x}_\mu) = 1 , \tag{4.1}$$

where $d\mu'$ is the measure in the reduced space.

(2) It must also satisfy a Liouville equation with the Hamiltonian (3.4),

$$\frac{\partial\rho'}{\partial\tau} + \{\rho',H_D\}^* = \frac{\partial\rho'}{\partial\tau} + \{\rho,H_D\}_R = 0 \tag{4.2}$$

with $\{\ \}_R$ being the PB in the reduced space.

At this point, if we know ρ' explicitly we can calculate the "average" of any function A' depending on the submanifold variables by writing

$$\langle A'\rangle = \int d\mu'\rho' A' \ . \tag{4.3}$$

Although, in principle, this procedure requires us to explicitly perform the Shanmugadhasan transformation, we can ask if it is possible to obtain the physical results without explicitly performing that transformation. This in fact can be done. In (4.2) we write the Liouville equation in terms of the Dirac brackets, which enables us to work with redundant variables. Therefore we can use a distribution function depending on all superspace phase variables, $\rho(\epsilon,x,P)$, but in such a manner that ρ vanishes outside of M. It is easy to construct this function,

$$\rho(\epsilon,P) = \delta(\chi_D)\tilde{\rho}(\epsilon,P)\delta(\phi_D) , \tag{4.4}$$

with the property

$$\rho(\epsilon,P,x)\,|_M = \rho'(\tilde{\epsilon}_\lambda,\tilde{P},\tilde{x}) \ . \tag{4.5}$$

If we use the distribution function ρ, the normalization condition (4.1) must change since $\rho(\epsilon,P,x)$ is defined in all phase superspace. The new condition becomes

$$\int d\mu\,\rho(P,\epsilon,x) = 1 , \tag{4.6}$$

where the measure $d\mu$ is $d\mu - id^5\epsilon$, and the expression of the Liouville equation (4.2) in terms of (4.4) is given by

$$\frac{\partial\rho}{\partial\tau} + \{\rho,H_D\}^* = 0 \ . \tag{4.7}$$

Now we can pass from the abstract space to the real phase space by means of

$$\langle A\rangle = \int d\mu\,\rho A , \tag{4.8}$$

where A is any dynamical variable of the submanifold M, but not necessarily expressed in terms of independent variables.

In order to construct the function $\tilde{\rho}(\epsilon,P)$ we can generalize the nonrelativistic distribution function[6]

$$\rho_{\rm NR}(\boldsymbol{\epsilon})=\mathbf{c}(t)\cdot\boldsymbol{\epsilon}-\frac{i}{6}\epsilon_{ijk}\epsilon^i\epsilon^j\epsilon^k , \tag{4.9}$$

where $\mathbf{c}$ represents the spin variable of a nonrelativistic particle. We construct the relativistic distribution function by means of a covariantization procedure. We assume (4.9) to apply to the particle at rest. There are two possible reference frames which qualify as rest frames; they are the one corresponding to $\mathbf{U}=0$, where $\mathbf{U}=\langle\dot{\mathbf{x}}\rangle$, and the one which corresponds to $\mathbf{P}=0$. They are not the same in general. However, in this model the first one is not available because after the covariantization, when we are in a frame with arbitrary $\mathbf{U}$, the velocity $\mathbf{U}$ will appear in a function that operates in phase superspace.

Therefore, we will write the covariant generalization of Eq. (4.9) by supposing that this is the expression of the phase-superspace distribution function in the frame where $\mathbf{P}=0$; the result is

$$\tilde{\rho}(\epsilon,P)=\frac{V\cdot\epsilon}{m}-\frac{i}{6m}\epsilon^{\mu\nu\alpha\beta}P_\mu\epsilon_\nu\epsilon_\alpha\epsilon_\beta , \tag{4.10}$$

where V^μ is the Pauli-Lubansky four-vector which satisfies

$$V\cdot P=0 , \tag{4.11}$$

and in the rest frame, $\mathbf{P}=0$, V^μ is

$$V^\mu=(0,\mathbf{c}) . \tag{4.12}$$

Now we can analyze the consequences of conditions (4.1) and (4.2) or equivalently (4.6) and (4.7). First of all we take the most general expression for ϕ_D:

$$\begin{aligned}&\phi_D=a\epsilon+b\epsilon_5+(n\epsilon\epsilon\epsilon)+m(\epsilon\epsilon\epsilon\epsilon)\epsilon_5 ,\\&(n\epsilon\epsilon\epsilon)\equiv\epsilon^{\alpha\beta\gamma\delta}n_\alpha\epsilon_\beta\epsilon_\gamma\epsilon_\delta ,\\&(\epsilon\epsilon\epsilon\epsilon)\equiv\epsilon^{\alpha\beta\gamma\delta}\epsilon_\alpha\epsilon_\beta\epsilon_\gamma\epsilon_\delta ,\end{aligned} \tag{4.13}$$

where a_α, b, n_α, and n' are arbitrary functions of x, P, or τ. The terms of third and fifth order in ϵ contribute in the normalization of the distribution function (4.6), but to calculate the average for a function of Grassmann variables, the last two terms of ϕ_D will be multiplied by at least three Grassmann variables, so the number of them will be more than five and these terms will not contribute to the averages. As a consequence, we can choose $n_\alpha=n'=0$, without loss of generality.

From the condition (4.6), the relation between a^α and b is

$$a\cdot P-mb=1 . \tag{4.14}$$

After that $\rho(\epsilon,P)$ is given by

$$\rho(\epsilon,P)=\delta(P\epsilon-m\epsilon_5)\frac{1}{m}\left[V\epsilon-\frac{i}{6}(P\epsilon\epsilon\epsilon)\right]\delta(a\epsilon+b\epsilon_5) . \tag{4.15}$$

The Liouville equation (4.7) also adds information about ρ: We have

$$\left.\frac{\partial\rho}{\partial\tau}\right|_{\delta(\chi_D)=\delta(\phi_D)=1}=\frac{\dot{V}^\alpha\epsilon_\alpha}{m}+\frac{V^\alpha\epsilon^\alpha}{m}\delta'(\phi_D)\frac{\partial\phi_D}{\partial\tau}$$

and

$$\{\rho,H_D\}^*=\frac{V^\alpha\dot{\epsilon}_\alpha}{m}-\frac{i}{2m}(P\dot{\epsilon}\epsilon\epsilon) ,$$

but from Eqs. (3.7) and (4.11) we conclude

$$\left.\frac{\partial\rho}{\partial\tau}\right|_\Sigma+\{\rho,H_D\}^*=\frac{\dot{V}^\alpha\epsilon_\alpha}{m}+\frac{V_\alpha\epsilon^\alpha}{m}\delta'(\phi_D)\frac{\partial\phi_D}{\partial\tau}=0 .$$

As we wish to describe a physical free spinning particle, we must require that

$$\dot{V}_\alpha=0 , \tag{4.16}$$

and therefore this gives us a restriction on ϕ_D. In fact ϕ_D must not depend explicitly on τ.

V. WORLD-LINE INVARIANCE

In Sec. III we introduced the physically relevant object in superspace, the line $L(x^\mu(\tau),\epsilon^\mu(\tau),\epsilon_5(\tau))$. In the preceding section we discussed the way to transform this line L into a world line in Minkowski space, explicitly by means of a "distribution function" $\rho(x,P,\epsilon)$ acting on phase superspace. This function $\rho(x,P,\epsilon)$ must be a suitable distribution function in the sense that it works in the subspace defined by $\phi_D=0$ and $\chi_D=0$, and that it satisfies a Liouville equation.

At the moment the important fact is that we have a mechanism to construct a world line in Minkowski space

$$\langle x^\mu\rangle=\int d\mu\,\rho x^\mu , \tag{5.1}$$

where $d\mu$ is the measure in the phase superspace. With the same mechanism we can determine the classical spin content of the model. From (2.8) the classical spin tensor should be

$$\langle S^{\mu\nu}\rangle=\langle -i\epsilon^\mu\epsilon^\nu\rangle=\int d\mu\,\rho(-i\epsilon^\mu\epsilon^\nu) . \tag{5.2}$$

Now is the moment to determine whether the objects $\langle x^\mu\rangle$ and $\langle S^{\mu\nu}\rangle$ have the correct geometrical transformation properties, i.e., whether the world-line condition is satisfied. In order to discuss this condition we need to fix the evolution parameter by breaking the gauge symmetry associated with reparametrizations, which means to fix the parameter τ by introducing a new constraint ϕ_0 which depends on τ such that χ_0' becomes second class:

$$\{\chi_0',\phi_0\}^*=A\neq0 . \tag{5.3}$$

The corresponding Dirac brackets will be

$$\{f,g\}^{\#}=\{f,g\}^*+\frac{1}{A}(\{f,\chi_0'\}^*\{\phi_0,g\}^*-\{f,\phi_0\}^*\{\chi_0',g\}^*) , \tag{5.4}$$

where we must choose a constraint ϕ_0 such that A has a non-Grassmann part different from zero in order to ensure that A^{-1} exists. By requiring the stability of this new constraint, ϕ_0,

$$\{\phi_0,\lambda_0\chi_0'\}^* + \frac{\partial\phi_0}{\partial\tau} = 0 \,, \tag{5.5}$$

the arbitrary function λ_0 is determined to be

$$\lambda_0 = -\frac{1}{A}\frac{\partial\phi_0}{\partial\tau} \,. \tag{5.6}$$

It is then meaningful to call $\langle x^\mu(\tau)\rangle$ the world line in Minkowski space.

Now we can check whether or not this line has an objective reality. In other words, we can construct a canonical realization of the Poincaré group in terms of the Dirac brackets (5.4) in phase superspace and also a geometrical realization in terms of the Poisson brackets. If the world line has an objective reality,[10] the only difference between these two kinds of realizations must be a reparametrization, that is,

$$\langle x^\mu(\tau+\delta\tau)\rangle + \langle\{x^\mu,G\}\rangle = \langle x^\mu(\tau)\rangle + \langle\{x^\mu,G\}^\#\rangle \,, \tag{5.7a}$$

$$\langle S^{\mu\nu}(\tau+\delta\tau)\rangle + \langle\{S^{\mu\nu},G\}\rangle = \langle S^{\mu\nu}(\tau)\rangle + \langle\{S^{\mu\nu},G\}^\#\rangle \,, \tag{5.7b}$$

$$\left\langle -\frac{1}{A}\frac{\partial\phi_0}{\partial\tau}\left[2P^\mu - \frac{\gamma}{D}\epsilon^\mu\right]\right\rangle\delta\tau = \left\langle -\frac{1}{D}\epsilon^\mu\{\phi_D,G\}\right\rangle + \left\langle -\frac{1}{A}\left[2P^\mu - \frac{\gamma}{D}\epsilon^\mu\right]\{\phi_0,G\}^*\right\rangle \,, \tag{5.8a}$$

$$\left\langle -\frac{1}{A}\frac{\partial\phi_0}{\partial\tau}\frac{\gamma}{D}(P^\mu\epsilon^\nu - P^\nu\epsilon^\mu)\right\rangle\delta\tau = \left\langle -\frac{1}{D}(\epsilon^\mu P^\nu - \epsilon^\nu P^\mu)\{\phi_D,G\}\right\rangle + \left\langle \frac{1}{A}\frac{\partial\phi_0}{\partial\tau}\frac{\gamma}{D}(\epsilon^\mu P^\nu - \epsilon^\nu P^\mu)\{\phi_0,G\}^*\right\rangle \,. \tag{5.8b}$$

It is easy to see that (5.8b) can be obtained from (5.8a), so it need not be studied separately. In (5.8a) we multiply by P^μ, which commutes with the averaging procedure. We obtain

$$\left\langle -\frac{1}{A}\frac{\partial\phi_0}{\partial\tau}\left[2P^2 - \frac{\gamma}{D}\epsilon\cdot P\right]\right\rangle\delta\tau = -\left\langle \frac{1}{D}\epsilon\cdot P\{\phi_D,G\}\right\rangle - \left\langle \frac{1}{A}\left[2P^2 - \frac{\gamma}{D}P\cdot\epsilon\right]\{\phi_0,G\}^*\right\rangle \,,$$

and, by making use of the constraints

$$\left\langle -\frac{1}{A}\frac{\partial\phi_0}{\partial\tau}\left[2m - \frac{\gamma}{D}\epsilon_5\right]\right\rangle\delta\tau = -\left\langle \frac{1}{D}\epsilon_5\{\phi_D,G\}\right\rangle - \left\langle \frac{1}{A}\left[2m - \frac{\gamma}{D}\epsilon_5\right]\{\phi_0,G\}^*\right\rangle \,,$$

so that we can isolate

$$\delta\tau = \frac{-\left\langle \frac{1}{D}\epsilon_5\{\phi_D,G\}\right\rangle - \left\langle \frac{1}{A}\left[2m - \frac{\gamma}{D}\epsilon_5\right]\{\phi_0,G\}^*\right\rangle}{\left\langle -\frac{1}{A}\frac{\partial\phi_0}{\partial\tau}\left[2m - \frac{\gamma}{D}\epsilon_5\right]\right\rangle} \,. \tag{5.9}$$

The denominator never vanishes because

$$\left\langle 2m\frac{1}{A}\frac{\partial\phi_0}{\partial\tau}\right\rangle$$

has a pure scalar term due to the fact that A and ϕ_0 have non-Grassmann parts different from zero. On the other hand,

$$\left\langle \frac{1}{A}\frac{\gamma}{D}\epsilon_5\frac{\partial\phi_0}{\partial\tau}\right\rangle$$

has a pseudoscalar character derived from the presence of the Levi-Civita tensor from integration over Grassmann variables. Therefore, imposing the world-line condition in Minkowski space does not lead to new restrictions on the constraints ϕ_0 or ϕ_D. We can conclude that geometrical lines $\langle X^\mu(\tau)\rangle$ and $\langle S^{\mu\nu}(\tau)\rangle$ are appropriate objects to characterize the world line of a relativistic spinning particle. Now it is necessary to study in detail the physics of the model given by $\langle X^\mu(\tau)\rangle$ and $\langle S^{\mu\nu}(\tau)\rangle$, which will be done in the next section.

VI. EQUATIONS FOR THE AVERAGED QUANTITIES

Now we want to study the physics of the model in Minkowski space; the relevant quantities are

$$\langle\epsilon^\mu\rangle = \langle\epsilon_5\rangle = 0 \,,\quad \langle x^\mu\rangle = x^\mu \,,\quad \langle P^\mu\rangle = P^\mu \,,$$

$$\langle\epsilon^\mu\epsilon^\nu\rangle = i\epsilon^{\mu\nu\alpha\beta}\left[\frac{V_\alpha}{m}(bP_\beta + ma_\beta)\right] \,, \tag{6.1}$$

$$\langle\epsilon^\mu\epsilon_5\rangle = \frac{i}{m}\epsilon^{\mu\nu\alpha\beta}a_\nu V_\alpha P_\beta \,.$$

The classical trajectory is obtained from

$$\frac{d}{d\tau}\langle x^\mu\rangle = \left\langle\frac{dx^\mu}{d\tau}\right\rangle = \langle\{x^\mu,\lambda_0\chi_0'\}^*\rangle \equiv U^\mu \,. \tag{6.2}$$

Explicitly, we have then

$$U^\mu = RP^\mu + B\epsilon^{\mu\nu\alpha\beta}P_\nu x_\alpha V_\beta \,, \tag{6.3}$$

where R and B are functions of P^α and x^α. The exact functional dependence will be determined fixing the gauge in both supergauge and reparametrizations.

The classical spin motion is given by

$$\langle\dot S^{\mu\nu}\rangle = \frac{d}{d\tau}(-\langle\epsilon^\mu\epsilon^\nu\rangle) = \tilde A\epsilon^{\mu\nu\alpha\beta}V_\alpha P_\beta + \tilde B\epsilon^{\mu\nu\alpha\beta}V_\alpha x_\beta \,, \tag{6.4}$$

where $\widetilde{A}$ and $\widetilde{B}$ are functions of x^α and P^α.

From these last two equations it seems possible to find some constraint ϕ_D which allows *Zitterbewegung*,[1] the noncolinearity of the four-velocity and the four-momentum. However, such a constraint gives us a strange behavior for the spin. In fact, by assuming the most general form

$$\phi_D=[a_1(x,P)x_\alpha+a_2(x,p)P^\alpha]\epsilon^\alpha+b(x,P)\epsilon_5 , \qquad (6.5)$$

we obtain

$$\langle \epsilon^\mu\epsilon^\nu\rangle = i\epsilon^{\mu\nu\alpha\beta}\left[\frac{V_\alpha}{m}(b+a_2m)P_\beta+ma_1x_\beta\right] . \qquad (6.6)$$

The spin depends on the position of the particle, which is not a characteristic of a free relativistic spinning particle. In order to avoid this we can impose

$$a_1=0 ,$$

$$a_2=\text{independent of } x ,$$

$$b=\text{independent of } x,$$

but in this case we will have for the four-velocity

$$U^\mu=-2\lambda_0P^\mu \qquad (6.7)$$

and, for the evolution of the spin part,

$$\langle \dot{S}^{\mu\nu}\rangle=0 . \qquad (6.8)$$

Therefore, the problem of a position-dependent spin tensor can only be avoided at the expense of *Zitterbewegung*. Actually this should not be surprising since by writing (4.10) as the covariant generalization of (4.9) we have defined the nonrelativistic limit as $\mathbf{P}\rightarrow 0$. Therefore, unless (4.9) can be covariantly generalized so that it holds in the velocity rest frame rather than the momentum rest frame, any *Zitterbewegung* intrinsic to the abstract model will not be manifested at the classical level. So we are left with, at most,

$$\phi_D=a_2P\cdot\epsilon+b\epsilon_5 , \qquad (6.9)$$

but taking into account the normalization condition and the constraint χ_D it is easy to see that this general constraint reduces to

$$\phi_D=\frac{\epsilon_5}{m} .$$

At this point we should note the following. When we studied the world-line condition (WLC) in superspace we proved that the only possible constraint was $\phi_D=a\epsilon_5$. In the case that we have studied here the situation is not the same, and the world-line condition in Minkowski space does not give us any restriction on the constraint ϕ_D, but when we want to give a physical meaning to the averaged quantities we recover the same result: ϕ_D must be given by (6.10). Therefore, we can infer that two studies, the WLC in superspace and the WLC in Minkowski space, are not in contradiction, but are in fact in agreement.

VII. POINCARÉ REALIZATION

In this section we construct a canonical realization of the Poincaré group, with the degrees of freedom corresponding to a spinning particle. We begin by following the Shanmugadhasan method.[12] In order to find the subset of canonical variables characterizing the reduced space

$$\chi_D=P\cdot\epsilon-m\epsilon_5\simeq 0 , \quad \phi_D=\frac{\epsilon_5}{m}\simeq 0 ,$$

we choose as new canonical variables

$$\tilde{\epsilon}_5=\epsilon_5 , \qquad \tilde{\epsilon}=\frac{P\cdot\epsilon}{m} , \qquad (7.1)$$

and the reduced space will be

$$\tilde{\epsilon}_5\simeq 0 , \qquad \tilde{\epsilon}\simeq 0 , \qquad (7.2)$$

which coincides with $\chi_D\simeq 0$ and $\phi_D\simeq 0$.

The corresponding Poisson brackets are

$$\{\tilde{\epsilon}_5,\tilde{\epsilon}_5\}=-i , \quad \{\tilde{\epsilon},\tilde{\epsilon}\}=i , \quad \{\tilde{\epsilon}_5,\tilde{\epsilon}\}=0 . \qquad (7.3)$$

In order to complete the canonical transformation, we can elect as new Grassmann variables

$$\tilde{\epsilon}_\lambda=\epsilon^\mu_\lambda\epsilon_\mu , \qquad (7.4)$$

where ϵ^μ_λ are the polarization vectors

$$\epsilon^0_\lambda=\frac{P^\lambda}{m}, \quad \epsilon^k_\lambda=\delta^k_\lambda+\frac{P^kP_\lambda}{m(P^0+m)} . \qquad (7.5)$$

We then have

$$\{\tilde{\epsilon}_\lambda,\tilde{\epsilon}_{\lambda'}\}=i\delta_{\lambda\lambda'}, \quad \{\tilde{\epsilon}_\lambda,\tilde{\epsilon}_5\}=\{\tilde{\epsilon}_\lambda,\tilde{\epsilon}\}=0 . \qquad (7.6)$$

With regard to the remaining variables x_μ,P_μ, we suppose

$$\tilde{P}_\mu=P_\mu$$

and

$$\tilde{x}_\mu=x_\mu+f_\mu(P,\epsilon_\mu,\epsilon_5) .$$

By imposing the corresponding canonical conditions

$$\{\tilde{x}_\mu,\tilde{\epsilon}_5\}=\{\tilde{x}_\mu,\tilde{\epsilon}\}=\{\tilde{x}_\mu,\tilde{\epsilon}_\lambda\}=\{\tilde{x}_\mu,\tilde{x}_\nu\}=0$$

and

$$\{x^\mu,P^\nu\}=-g^{\mu\nu} ,$$

we obtain

$$\tilde{x}^\mu=x^\mu+A(P^2)P^\mu+\frac{1}{m+P^0}\left[S^{0\mu}-\frac{P^\mu P^\nu}{P^2}S^{0\nu}-\frac{P^\nu S^{\mu\nu}}{m}\right]. \qquad (7.7)$$

The Lorentz generators are

$$M^{\mu\nu}=x^\mu P^\nu-x^\nu P^\mu-i\epsilon^\mu\epsilon^\nu , \qquad (7.8)$$

which, with $\widetilde{P}^{\mu}=P^{\mu}$, close the Poincaré algebra with the Poisson bracket. These generators in terms of new variables will be

$$\widetilde{M}^{\mu\nu}=\widetilde{x}^{\mu}P^{\nu}-\widetilde{x}^{\nu}P^{\mu}+\widetilde{S}^{\mu\nu} ,$$

and are explicitly

$$\widetilde{S}^{0k}=\widetilde{S}^{k0}=-i\frac{\sum_{\lambda}P^{\alpha}\widetilde{\epsilon}_{\lambda}}{m+P^{0}}\widetilde{\epsilon}^{k} , \tag{7.9}$$

$$\widetilde{S}^{jk}=-i\widetilde{\epsilon}^{j}\widetilde{\epsilon}^{k} , \tag{7.10}$$

and if

$$\widetilde{S}^{i}=\tfrac{1}{2}\epsilon^{0i}{}_{jk}\widetilde{S}^{jk} ,\quad M^{i}=\tfrac{1}{2}\epsilon^{0i}{}_{jk}\widetilde{M}^{jk} ,\quad K^{i}=\widetilde{M}^{0i} , \tag{7.11}$$

the Poincaré generators on the reduced space (7.2) are

$$P^{\mu}=P^{\mu} ,$$
$$\mathbf{M}=\widetilde{\mathbf{X}}\times\mathbf{P}+\widetilde{\mathbf{S}} , \tag{7.12}$$
$$\mathbf{K}=\widetilde{X}_{0}\mathbf{P}-\widetilde{\mathbf{X}}P_{0}+\frac{\mathbf{P}\times\widetilde{\mathbf{S}}}{m+P_{0}} .$$

This is an 11-dimensional realization of the Poincaré algebra in terms of canonical variables, with the corresponding bracket being

$$\{f,g\}^{*}=\{f,g\}+\frac{i}{m^{2}}\{f,\chi_{D}\}\{\chi_{D},g\}$$
$$+i(\{f,\chi_{D}\}\{\phi_{D},g\}+\{f,\phi_{D}\}\{\chi_{D},g\}) , \tag{7.13}$$

which coincides with the Poisson bracket in the reduced space $\{\ \}'_{R}$.

By introducing the constraint

$$\phi_{0}\equiv\widetilde{x}^{0}-\tau , \tag{7.14}$$

we fix the evolution parameter at each point of the trajectory. Doing that, the constraint χ'_{0} which in our case $(\gamma=0)$ coincides with χ_{0} becomes second class. That enables us to write a nine-dimensional Poincaré realization which realizes the Poincaré algebra through the corresponding Dirac bracket. In fact we can eliminate two superfluous degrees of freedom making use of the constraints Φ_{0},χ_{0}:

$$x^{0}=\tau ,\quad P^{0}=(\mathbf{P}^{2}+m^{2})^{1/2} , \tag{7.15}$$

and the Dirac bracket is

$$\{f,g\}^{\#}=\{f,g\}^{*}+\frac{1}{2P^{0}}(\{f,\chi_{0}\}^{*}\{\phi_{0},g\}^{*}$$
$$-\{f,\phi_{0}\}^{*}\{\chi_{0},g\}^{*}) . \tag{7.16}$$

Due to the Poincaré scalar character of χ_{D} and χ'_{0}, we can ensure

$$\{M^{\alpha\beta},M^{\gamma\delta}\}=\{M^{\alpha\beta},M^{\gamma\delta}\}^{*}=\{M^{\alpha\beta},M^{\gamma\delta}\}^{\#} ,$$
$$\{M^{\alpha\beta},P^{\gamma}\}=\{M^{\alpha\beta},P^{\gamma}\}^{*}=\{M^{\alpha\beta},P^{\gamma}\}^{\#} ,$$
$$\{P^{\alpha},P^{\beta}\}=\{P^{\alpha},P^{\beta}\}^{*}=\{P^{\alpha},P^{\beta}\}^{\#} .$$

Now, we are ready to perform the Poincaré realization in the phase space; the appropriate distribution function is, from (4.15),

$$\rho(\epsilon,P)=\delta(P\epsilon)\frac{1}{m}\left[VE-\frac{i}{6}(P\epsilon\epsilon\epsilon)\right]\delta\left[\frac{\epsilon_{5}}{m}\right] ,$$

which enables us to calculate the average position variable

$$\langle\widetilde{\mathbf{x}}\rangle=\langle\mathbf{x}\rangle+\frac{1}{P^{0}(m+P^{0})}\mathbf{P}\times\langle\mathbf{S}\rangle$$

and also the spin variable

$$\langle\widetilde{\mathbf{S}}\rangle=\frac{1}{m}\left[\mathbf{V}-\frac{V^{0}\mathbf{P}}{m+P^{0}}\right] .$$

We see that $\langle\widetilde{\mathbf{x}}\rangle$ is the position variable introduced by Pryce,[13] and $\langle\widetilde{\mathbf{S}}\rangle$ has the same meaning as the Thomas spin variable, which appears in the usual representation of a spinning particle.

Lastly, one can obtain the Poincaré realization, which is the same as (7.12) but with the position, the four-momentum, and the spin variables replaced by their averaged values.

It still remains to specify what is the simplectic structure which acts in the phase space. We must have an operator which, when acting on the averaged quantities like $\langle\widetilde{S}_{i}\rangle,\langle\widetilde{x}_{\mu}\rangle$, verifies

$$O[\langle\widetilde{S}_{i}\rangle,\langle\widetilde{S}_{j}\rangle]=\epsilon_{ijk}\langle\widetilde{S}_{k}\rangle ,$$
$$O[\langle\widetilde{x}_{\mu}\rangle,\langle\widetilde{P}_{\nu}\rangle]=-g_{\mu\nu} .$$

This operator can be defined by

$$O[\langle A\rangle,\langle B\rangle]=\frac{\partial\langle A\rangle}{\partial\langle\widetilde{x}^{\mu}\rangle}\frac{\partial\langle B\rangle}{\partial\langle\widetilde{P}^{\mu}\rangle}-\frac{\partial\langle A\rangle}{\partial\langle\widetilde{P}^{\mu}\rangle}\frac{\partial\langle B\rangle}{\partial\langle\widetilde{x}^{\mu}\rangle}$$
$$+\epsilon_{ijk}\langle\widetilde{S}_{k}\rangle\frac{\partial\langle A\rangle}{\partial\langle\widetilde{S}_{i}\rangle}\frac{\partial\langle B\rangle}{\partial\langle\widetilde{S}_{j}\rangle} .$$

With this operator it is straightforward to prove that the realization (7.12) in terms of averaged quantities satisfies the Poincaré algebra.

VIII. CONCLUSIONS

We have studied a pseudoclassical model for a relativistic spinning particle. This model contains, apart from the standard space-time variables, five Grassmann variables:[5–7] one pseudoscalar ϵ_{5} and one pseudovector ϵ_{μ}. It has two first-class constraints χ_{0} and χ_{D}, that, at the quantum level, give rise to the Dirac wave equation.

For a pure classical point of view we have seen that in order to define a world line in Minkowski space we need to introduce another constraint ϕ_{D} and a suitable distribution function ρ that averages over Grassmann variables on the submanifold defined by χ_{D} and ϕ_{D}. The world line in Minkowski space defined this way has an objective reality, i.e., it verifies the world-line condition[10] for an arbitrary choice of the constraint ϕ_D. However, the physical content of this model is not independent of the constraint ϕ_D; in fact, there are choices of ϕ_D that give rise to the *Zitterbewegung*.[14] On closer inspection we conclude that the only possible meaningful choice of ϕ_D is the one given by $\phi_D \sim \epsilon_5$ that yields a world line without *Zitterbewegung*. We note also that this severe restriction imposed by the physics coincides with the one given by the world-line condition at the superspace level.[9] Finally, the Poincaré realization for this situation is constructed by means of the Shanmugadhasan transformation.[12]

1H. C. Corben, *Classical and Quantum Theories of Spinning Particles* (Holden-Day, San Francisco, 1968); Harvey Mudd College report, 1984 (unpublished).

2K. Rafanelli, Phys. Rev. D **30**, 1707 (1984).

3R. R. Aldinger, A. Bohm, P. Kielanowski, M. Loewe, P. Magnollay, N. Mukunda, W. Drechsler, and S. R. Komy, Phys. Rev. D **28**, 3020 (1983).

4P. A. M. Dirac, *Lectures on Quantum Mechanics* (Belfer Graduate School of Science, Yeshiva University, New York, 1964).

5A. Barducci, R. Casalbuoni, and L. Lusanna, Nuovo Cimento **35A**, 377 (1976).

6F. A. Berezin and M. S. Marinov, Ann. Phys. (N.Y.) **104**, 1863 (1977).

7C. A. P. Galvao and C. Teitelboim, J. Math. Phys. **21**, 1863 (1980).

8R. Casalbuoni, Nuovo Cimento **33A**, 389 (1976).

9J. Gomis, P. Mato, and M. Novell, Nuovo Cimento B (to be published).

10E. C. G. Sudarshan, N. Mukunda, and J. N. Goldberg, Phys. Rev. D **23**, 2218 (1984).

11L. Brink, S. Deser, B. Zumino, P. di Vecchia, and P. Howe, Phys. Lett. **64B**, 485 (1976).

12S. Shanmugadhasan, J. Math. Phys. **14**, 677 (1973).

13M. H. L. Pryce, Proc. R. Soc. London **A195**, 62 (1968).

14K. Rafanelli, J. Math. Phys. **8**, 1440 (1967).