Equivalence of reduced, Polyakov, Faddeev-Popov, and Faddeev path-integral quantization of gauge theories

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A geometrical treatment of the path integral for gauge theories with first-class constraints linear in the momenta is performed. The equivalence of reduced, Polyakov, Faddeev-Popov, and Faddeev path-integral quantization of gauge theories is established. In the process of carrying this out we find a modified version of the original Faddeev-Popov formula which is derived under much more general conditions than the usual one. Throughout this paper we emphasize the fact that we only make use of the information contained in the action for the system, and of the natural geometrical structures derived from it.

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As is well known, the quantization of Yang-Mills theories experienced a rapid development thanks mostly to the introduction of functional-integral techniques [1–3]. Very soon after they were first used in this context, a tremendous amount of work, which continues to date, went into generalization and a better understanding of functional-integral methods for gauge theories, not only in particle physics, but also in other areas as well. Nevertheless, despite great progress in such endeavors, which apparently culminated with the work inspired by Vilkovisky and co-workers [4–7], there is still debate over certain questions of principle in this subject. In particular, the relation between the various pre-Becchi-Rouet-Stora-Tyvtin (BRST) methods for path-integral quantization of gauge theories, namely, reduced, Polyakov, Faddeev, and Faddeev-Popov methods is not completely clear. In a previous paper [8] we studied the question of the relation between the "first reduce and then quantize" and Dirac's type quantization, "first quantize and then reduce," within the framework of operator quantization for constrained Hamiltonian systems with only primary first-class constraints, linear in momenta. In this note we will try to elucidate the relationship between the above-mentioned four different versions of the path integral for gauge theories of this type, which include Yang-Mills theories [9]. Our setting here will be based on results obtained in our paper, Ref. [8], to which we will refer the reader for more details. The type of systems of interest to us are characterized by a Lagrangian of the form

$$L=\tfrac{1}{2}G_{AB}\dot{Q}^A\dot{Q}^B-V\ , \tag{1}$$

where G_{AB} and V are functions of configuration-space variables Q^A, $A=1,\dots,N$. G_{AB} is a singular metric tensor of rank $n<N$. The constraints are linear in $P_A=\partial L/\partial\dot{Q}^A=G_{AB}\dot{Q}^B$,

$$\varphi_\alpha=U^A_\alpha P_A,\quad \alpha=1,\dots,k=N-n\ , \tag{2}$$

where the U^A_α's define a basis for the space of null vectors of G_{AB}, $\hat{U}_\alpha=U^A_\alpha(Q)\partial/\partial Q^A$, which in order to guarantee the first-class condition also have to be Killing vectors for G_{AB}:

$$(\mathcal{L}_{\hat{U}_\alpha}G)_{AB}=0\ . \tag{3}$$

The fact that the set of null vectors is closed under the Lie brackets [8] implies that

$$[\hat{U}_\alpha,\hat{U}_\beta]=C^\gamma_{\alpha\beta}(Q)U_\gamma\ , \tag{4}$$

which indeed shows that the constraints are first class; i.e., they have the following Poisson brackets with each other:

$$\{\varphi_\alpha,\varphi_\beta\}=-C^\gamma_{\alpha\beta}\varphi_\gamma\ . \tag{5}$$

Finally, the potential V in Eq. (1) has to satisfy (gauge invariance)

$$\hat{U}_\alpha(V)=U^A_\alpha\frac{\partial V}{\partial Q^A}=0\ . \tag{6}$$

These observations clearly spell the symmetries of the system. The Hamiltonian dynamics are described by

$$H=\tfrac{1}{2}M^{AB}(Q)P_AP_B+V(Q)\ . \tag{7}$$

As shown in [8], $M^{AB}(Q)$ can be chosen to be a nonsingular symmetric matrix function only of Q, satisfying

$$M^{AB}G_{AC}G_{BD}=G_{CD}\ . \tag{8}$$

Hence, the inverse of $M^{AB}(Q)$ naturally provides the

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configuration space with a well-defined (nonsingular) covariant metric tensor. Clearly, the choice of such a metric tensor satisfying Eq. (8) is *highly arbitrary,* and at this point it is not obvious how such arbitrariness could not be a potentially serious problem later on for the quantization of the theory (in fact, as we showed in [8], it is at the root of the inequivalence between Dirac's type and the "reduce first and then quantize" methods). More on this below.

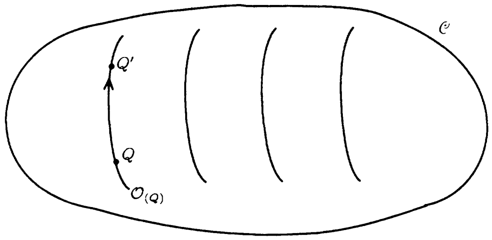
Two comments are now in order. First, these results were all obtained at the *classical level;* no quantum mechanics has been invoked yet. In particular, the Killing vector condition (3), as shown in [8], comes solely from the requirement that the Lagrangian (1) describes a *classical first-class system with no secondary constraints.* As will be elaborated upon below, sometimes it is said that a similar condition has to be imposed on the metric in the space of trajectories. We will see clearly whether or not we will need to make such an assumption here. Second, our previous analysis was done in configuration and phase spaces. We now need to examine the implications of our results in trajectory space, a task which we now turn to.

Consider the space of trajectories, $\mathcal{C}$. In order to study functional integrals on $\mathcal{C}$, we need to know (a) the gauge orbit structure and (b) a gauge-invariant measure on it. At this point we should pause to introduce a very convenient notation. Even though the results of our previous investigations [8] were obtained explicitly in the framework of a quantum-mechanical system, the generalization to include field theories is immediate once one adopts DeWitt's condensed notation: in Q^i, i includes *all* types of indices, both discrete (group, particle type, spin, ...) and continuous (spacetime, ...). Contractions of indices then imply not only discrete sums, but also integrals. In this manner, it is then obvious that $\mathcal{C}$ inherits an orbit structure defined also by Eq. (4) (see Fig. 1). Indeed, infinitesimally, two points Q and $Q+\delta Q$ belong to the same orbit iff [10] (S is the action of the system)

$$\delta S=\frac{\delta S}{\delta Q^i}\delta Q^i=\frac{\delta S}{\delta Q^i}U^i_\alpha(Q)\delta\varepsilon^\alpha\equiv\hat{U}_\alpha(S)\delta\varepsilon^\alpha=0\ . \quad (9)$$

Explicitly [11]

$$\hat{U}_\alpha=U^i_\alpha\frac{\delta}{\delta Q^i}$$
$$\equiv\sum_A\int_{\text{spacetime}}d^mx\ U^A_\alpha(Q(x))\frac{\delta}{\delta Q^A(x)}\ , \quad (10)$$



$S[Q]=S[Q']$, Q, $Q'\in$ Orbit $\mathcal{O}_{(Q)}$.

FIG. 1. Orbit structure of the space of the trajectories, $\mathcal{C}$: $S[Q]=S[Q']$, $Q,Q'\in$ orbit $\mathcal{O}(Q)$.

with the $U^A_\alpha(Q)$ defined as before in configuration space. Clearly, these $\hat{U}_\alpha$'s satisfy the corresponding brackets relationship, analogous to Eq. (4), in trajectory space. Notice that the summation over γ in Eq. (4) will now contain a space-time integration, and that, in general, *the* $C^\gamma_{\alpha\beta}$'s *will be functions of the* Q^i's. The orbit structure is so established. In order to define the measure, we proceed as follows. On $\mathcal{C}$, we can construct a natural metric tensor out of M_{AB}. Indeed, for any two small displacements $\delta Q_1,\delta Q_2$ around the point $Q\in\mathcal{C}$ we have

$$\langle\delta Q_1|\delta Q_2\rangle_Q\equiv\sum_{A,B}\int d^mx\,d^my\,\delta Q_1^A(x)\times\mathcal{G}_{AB}(Q(x),x,y)\delta Q_2^B$$
$$\equiv\delta Q_1^i\mathcal{G}_{ij}(Q)\delta Q_2^j\ , \quad (11)$$

where

$$\mathcal{G}_{ij}(Q)\equiv M_{AB}(Q)\delta^m(x-y)\ , \quad (12)$$

i.e.,

$$\langle\delta Q_1|\delta Q_2\rangle=\sum_{A,B}\int d^mx\,\delta Q_1^A(x)M_{AB}(Q(x))\delta Q_2^B(x)\ . \quad (13)$$

Then, the natural measure on $\mathcal{C}$ is

$$\int_{\mathcal{C}}\Omega=\int_{\mathcal{C}}[dQ][\det\mathcal{G}_{ij}(Q)]^{1/2}$$
$$\equiv\prod_{\substack{A\\ \text{spacetime}}}\int dQ^A(x)[\det M_{AB}(Q(x))]^{1/2}\ , \quad (14)$$

where we used the diagonality of $\mathcal{G}_{ij}$ in the space-time labels to write Eq. (14). This measure is clearly reparametrization invariant, i.e., invariant under invertible mappings $Q^A\to Q'^A(Q)$, and hence in particular it is invariant under mappings $Q^A\to Q'^A$ *which preserve the orbits,* that is, *diffeomorphisms which map points on a given orbit into points on the same orbit.* This subset of the group of diffeomorphisms on $\mathcal{C}$ defines the gauge group $\mathcal{G}$. Therefore Eq. (14) defines a *gauge-invariant measure,* for which we sought. We are now ready to write the full path integral over $\mathcal{C}$:

$$Z=\int_{\mathcal{C}}[dQ](\det M_{AB})^{1/2}e^{iS[Q]}\ . \quad (15)$$

The path integral Eq. (15) suffers from the usual disease: due to the constancy of $S[Q]$ along orbits, there is a redundancy when performing the full integration over $\mathcal{C}$. One way to cure the problem is to work in reduced trajectory space $\mathcal{R}$. In our previous paper we performed the classical reduction of our systems in both the Lagrangian and Hamiltonian versions. A convenient way to do that was to introduce "adapted coordinates" in the entire configuration space Φ, in order to describe the "physical" or "reduced" configuration space $\mathcal{M}$. They are defined as follows: the labels for the orbits, which will hence describe $\mathcal{M}$, are defined by the condition

$$\hat{U}_\alpha(q^a(Q))=0,\quad a=1,\ldots,n;n=\dim\mathcal{M}\ ,$$
$$\alpha=1,\ldots,k=N-n\ . \quad (16)$$

This means that the q^a's are gauge invariant. One adds to the q^a's a set of functions $q^\alpha(Q)$, $\alpha=1,\ldots,k=N-n$, such that $\det|U_\alpha(q^\beta)|\neq 0$. These q^α's parametrize the orbits. We then showed [8] that G_{AB} and M^{AB} defined earlier become [12] in this "adapted coordinate system,"

$$G_{A'B'}=\begin{pmatrix} g_{ab}(q^a) & 0 \\ 0 & 0 \end{pmatrix}, \tag{17}$$

and

$$M^{A'B'}=\begin{pmatrix} g^{ab}(q^a) & m^{a\beta} \\ m^{\alpha b} & m^{\alpha\beta} \end{pmatrix}, \tag{18}$$

where $g^{ab}=(g_{ab})^{-1}$. Here g_{ab} is a nonsingular metric which depends only on the physical variables q^a, and $m^{a\beta}$, $m^{\alpha\beta}$ are completely arbitrary functions of *both* q^a and q^α, with the only requirement that $M^{A'B'}$ be regular. As is shown in [8], when we project our geometrical structures from Φ to the physical configuration space $\mathcal{M}\approx\Phi/\text{orbits}$ we then get a nonsingular covariant metric tensor $g_{ab}(q^a)$. Clearly, our previous analysis on the trajectory space $\mathcal{C}$ can be repeated now on the reduced trajectory space $\mathcal{R}$. The reduced path integral can hence be readily written as

$$Z_R=\int_{\mathcal{R}}[dq^a](\det g_{ab})^{1/2}e^{iS[q^a]} . \tag{19}$$

The action $S[q^a]$ is the one obtained using the Lagrangian (1) with $G_{A'B'}$ as in Eq. (17) (it contains only physical variables). Now, consider the Polyakov path integral for our system:

$$Z_P=\int_{\mathcal{C}}\frac{[dQ^A](\det M_{AB})^{1/2}}{V_{\mathcal{O}(Q)}}e^{iS[Q]}, \tag{20}$$

where $\mathcal{O}(Q)$ is the orbit to which Q belongs, and $V_{\mathcal{O}(Q)}$ its volume. In adapted coordinates,

$$\int_{\mathcal{C}}[dQ^A](\det M_{AB})^{1/2}=\int[dq^a][dq^\alpha](\det M_{A'B'})^{1/2}, \tag{21}$$

where

$$M_{A'B'}=\begin{pmatrix} M_{ab} & M_{b\beta} \\ M_{\alpha b} & M_{\alpha\beta} \end{pmatrix} \tag{22}$$

is the inverse of $M^{A'B'}$ of Eq. (18). Next, [13] we prove the following crucial factorization property:

$$\det M_{A'B'}=\frac{\det M_{\alpha\beta}}{\det g^{ab}} . \tag{23}$$

Proof: Write $M_{A'B'}$ and $M^{A'B'}$ as

$$M_{A'B'}=\begin{pmatrix} A & B \\ B & D \end{pmatrix}, \tag{24}$$

$$M^{A'B'}=\begin{pmatrix} A' & B' \\ B' & D' \end{pmatrix} . \tag{25}$$

Since they are inverse to each other, a number of identities are satisfied by their block components. Important to us are

$$A'A+B'B=I \quad\text{and}\quad A'B+B'D=0 . \tag{26}$$

Next, use the identity

$$\begin{pmatrix} A & B \\ B & D \end{pmatrix}=\begin{pmatrix} I & B \\ 0 & D \end{pmatrix}\begin{pmatrix} A-BD^{-1}B & 0 \\ D^{-1}B & I \end{pmatrix} \tag{27}$$

which implies

$$\det M_{A'B'}=\det D\,\det(A-BD^{-1}B) , \tag{28}$$

with $D=M_{\alpha\beta}$. Finally, using Eq. (28) we get

$$\begin{aligned} A'(A-BD^{-1}B)&=A'A-A^{-1}BD^{-1}B \\ &=A'A+B'DD^{-1}B \\ &=A'A+B'B=I \end{aligned} \tag{29}$$

from which

$$\det M_{A'B'}=\frac{\det D}{\det A'}=\frac{\det M_{\alpha\beta}}{\det g^{ab}} .$$

Q.E.D. Hence the measure over $\mathcal{C}$ can be written in adapted coordinates as

$$\int[dq^a][\det g_{ab}(q^a)]^{1/2}\int[dq^\alpha][\det M_{\alpha\beta}(q^a,q^\alpha)]^{1/2} . \tag{30}$$

In Eq. (30) we can easily identify the integration over q^α as the volume of the orbit labeled by q^a. Indeed, for infinitesimal δQ's along the orbit labeled by q^a (i.e., $\delta q^a=0$) we have, from Eq. (13),

$$\langle\delta Q_1|\delta Q_2\rangle=\sum_{\alpha,\beta}\int d^m x\,\delta Q_1^\alpha(x)M_{\alpha\beta}\delta Q_2^\beta(x) . \tag{31}$$

And hence, the volume for the orbit is

$$V_{\mathcal{O}(qa)}=\int[dq^\alpha](\det M_{\alpha\beta})^{1/2} . \tag{32}$$

We see then, that

$$Z_P=Z_R . \tag{33}$$

Another method of displaying the important factorization property Eq. (30) will be shown in the Appendix, which will have a more geometrical flavor.

Next, we will consider the Faddeev-Popov version. From our previous discussion, it is clear that the reduced trajectory space $\mathcal{R}$ can be considered as the quotient of the full trajectory space $\mathcal{C}$ by the gauge group $\mathcal{G}$ (i.e., the subgroup of the diffeomorphism group of $\mathcal{C}$ which preserves the orbits): $\mathcal{R}\approx\mathcal{C}/\mathcal{G}$. This means that we can always implement $\mathcal{R}$, at least locally, [14] by means of a surface embedded in $\mathcal{C}$ which intersects the orbits only once (see Fig. 2). More precisely, for any $Q\in\mathcal{C}$ there is a unique $\omega\in\mathcal{G}$ such that $F^\alpha(Q^{\omega^{-1}})=C^\alpha$, with C^α a set of numbers. Our task is to try to rewrite the path integral Eq. (19) over $\mathcal{R}$ as a path integral over $\mathcal{C}$. Clearly, this will imply the use of some notion of gauge fixing which contains the information about the embedding of $\mathcal{R}$ in $\mathcal{C}$. To achieve this we follow the usual insertion of unity in the reduced path integral in terms of a path integral along the orbit $\mathcal{O}(q^a)$. In adapted coordinates

$$1=\int_{\mathcal{O}(q^a)}[dq^\alpha](\det M_{\alpha\beta})^{1/2}\delta[F^\alpha(Q)-C^\alpha]\mu[Q] . \tag{34}$$

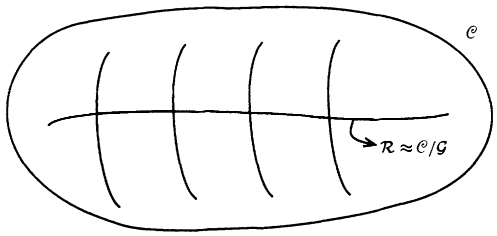


FIG. 2. Embedding of the reduced trajectory space $\mathcal{R}\approx\mathcal{C}/\mathcal{G}$ into $\mathcal{C}$.

The factor μ is inserted in order to assure the proper normalization. Change variables from q^α to F^α (the definition of F^α guarantees that this is always possible, at least locally),

$$1=\int[dF^\alpha](\det\tilde{M}_{\alpha\beta})^{1/2}\delta[F^\alpha-C^\alpha]\mu[Q(F)] \\ =(\det\tilde{M}_{\alpha\beta})^{1/2}\mu[Q(F)]|_{F=C}\ , \tag{35}$$

and hence

$$\mu[Q(F)]|_{F=C}=(\det\tilde{M}_{\alpha\beta})^{-1/2}_{F=C}\ . \tag{36}$$

$\tilde{M}_{\alpha\beta}$ is the metric $M_{\alpha\beta}$ in the coordinates F^α (recall we are using DeWitt's notation):

$$\tilde{M}_{\alpha\beta}=\frac{\partial q^\sigma}{\partial F^\alpha}\frac{\partial q^\rho}{\partial F^\beta}M_{\sigma\rho} \tag{37}$$

from which

$$\mu[Q(F)]|_{F=C}=\left|\det\frac{\partial F^\beta}{\partial q^\alpha}\right|(\det M_{\alpha\beta})^{-1/2}\ . \tag{38}$$

Inserting unity as in Eq. (34) into the reduced path integral, with μ given by Eq. (38) we obtain

$$Z_R=\int[dq^a][dq^\alpha](\det g_{ab})^{1/2}(\det M_{\alpha\beta})^{1/2}(\det M_{\alpha\beta})^{-1/2}\left|\det\frac{\partial F^\beta}{\partial q^\alpha}\right|_{F=C}\delta[F^\alpha(Q)-C^\alpha]e^{iS[q^a]} \\ =\int[dQ](\det M_{AB})^{1/2}[\det M_{\alpha\beta}(Q)]^{-1/2}\left|\det\frac{\partial F^\beta}{\partial q^\alpha}\right|_{F=C}\delta[F^\alpha-C^\alpha]e^{iS[Q]}\ . \tag{39}$$

In Eq. (39) we have used the gauge invariance of the action, $S[q^a]=S[Q]$. This expression can be related to the usual one as follows: consider the orbit-preserving diffeomorphism $(\delta q^a=0)\delta q^\alpha$ as an infinitesimal gauge transformation [see Eq. (9)]:

$$\delta q^\alpha=U^\alpha_\lambda(Q)\delta\varepsilon^\lambda\ . \tag{40}$$

Then

$$\det\frac{\partial F^\beta}{\partial q^\alpha}\bigg|_{F=C}=\left\{\det\frac{\partial F^\beta}{\partial\varepsilon^\lambda}\right\}\bigg|_{F=C}(\det U^\alpha_\lambda)^{-1}\bigg|_{F=C}\ , \tag{41}$$

so finally we get our version of the Faddeev-Popov result:

$$Z_R=\tilde{Z}_{\rm FP}=\int[dQ][\det M_{AB}(Q)]^{1/2}[\det M_{\alpha\beta}(Q)]^{-1/2}(\det U^\alpha_\lambda)^{-1}\left|\det\frac{\partial F^\beta}{\partial\varepsilon^\lambda}\right|\delta[F^\alpha(Q)-C^\alpha]e^{iS[Q]} \tag{42}$$

$$=\int[dQ][\det M_{AB}(Q)]^{1/2}(\det\Theta_{\alpha\beta})^{-1/2}\left|\det\frac{\partial F^\beta}{\partial\varepsilon^\lambda}\right|\delta[F^\alpha(Q)-C^\alpha]e^{iS[Q]}\ , \tag{43}$$

where

$$\Theta_{\alpha\beta}=\hat{U}_\alpha\cdot\hat{U}_\beta=U^A_\alpha U^B_\beta M_{AB} \\ =U^\rho_\alpha U^\sigma_\beta M_{\rho\sigma} \tag{44}$$

is the matrix of scalar products of the $\hat{U}_\alpha$'s, which has been expressed in adapted coordinates (in which $U^a_\alpha=0$). Equation (44) shows that $\Theta_{\alpha\beta}$, by virtue of the linear independence of the $\hat{U}_\alpha$'s, and the nondegeneracy of M_{AB}, is regular. This fact, combined with $|U^\beta_\alpha|\neq0$ (by definition of the q^α's) shows that $M_{\rho\sigma}$ is regular, which is a necessary consistency condition within our framework, as can be seen in Eq. (23).

In Eq. (43) we have omitted the instruction $F^\alpha=C^\alpha$ in the corresponding determinants since the presence of the δ function explicitly enforces it. Notice that we differ from the usual treatment by a factor $(\det\Theta_{\alpha\beta})^{-1/2}$ which in general will be nontrivial. Now we turn to the Hamiltonian, or Faddeev, version of the gauge fixing. As shown in paper [8], the dynamics of the reduced system is encoded in the Lagrangian

$$L=\tfrac{1}{2}g_{ab}\dot{q}^a\dot{q}^b-V[q^a]\ , \tag{45}$$

which is to be used in Eq. (19). This equation can then be rewritten in the Hamiltonian form (using the standard result for Gaussian path integrals)

$$Z_R=\int[dq^a][dp_a]e^{iI[q^a,p_a]}\ , \tag{46}$$

where

$$I[q^a,p_a]=\int d^mx[\dot{q}^ap_a-h(q^a,p_a)] \tag{47}$$

and

$$h=\tfrac{1}{2}g^{ab}p_ap_b+V(q^a)\ . \tag{48}$$

We have reverted temporarily to the explicit notation in Eqs. (47) and (48) for the sake of clarity. Now, in order to go to integrations over *all* phase-space variables Q^A,P_A, in Eq. (46), let us introduce the corresponding representations for unity:

$$1=\int[dq^\alpha]\delta[F^\alpha-C^\alpha]\Delta_F \tag{49}$$

and

$$1=\int[dp_\alpha]\delta[\varphi_\alpha]\Delta_\varphi\ , \tag{50}$$

where, clearly

$$\Delta_F=\left|\det\frac{\partial F^\alpha}{\partial q^\beta}\right|_{F=C} \tag{51}$$

and

$$\Delta_\varphi=|\det U_\alpha^\beta|\ . \tag{52}$$

Then, since

$$\begin{aligned}\left|\det\{F^\alpha,\varphi_\beta\}\right|&=\left|\det\left[\frac{\partial F^\alpha}{\partial q^\gamma}\frac{\partial\varphi_\beta}{\partial p_\gamma}\right]\right|\\&=\left|\det\frac{\partial F^\alpha}{\partial q^\gamma}\right|\left|\det U_\beta^\gamma\right|\end{aligned} \tag{53}$$

we obtain

$$\begin{aligned}Z_R&=Z_F\\&=\int[dQ^A][dP_A]|\det\{F^\alpha,\varphi_\beta\}|\\&\quad\times\delta[\varphi_\alpha]\delta[F^\alpha-C^\alpha]e^{iI[Q,P]}\ ,\end{aligned} \tag{54}$$

with

$$I[Q,P]=\int d^mx[\dot{Q}^AP_A-H(Q,P)]\ , \tag{55}$$

where

$$H(Q,P)=\tfrac{1}{2}M^{AB}P_AP_B+V(Q)\ . \tag{56}$$

In order to arrive at Eq. (54) we have used the following facts: (i) the form of the constraints φ_α in adapted coordinates, $\varphi_\alpha=U_\alpha^\beta p_\beta$ (i.e., $U_\alpha^a=0$); (ii) $H(Q,P)|_{\varphi=0}=[\tfrac{1}{2}M^{A'B'}P_{A'}P_{B'}+V(Q)]|_{\varphi=0}=h(q^a,p^a)$. Notice that our expression Eq. (54) coincides, as expected, with Faddeev's formula [15]. Finally, as a check of consistency, we should be able to integrate out the momenta in formula (54) and reobtain our Faddeev-Popov formula Eq. (43). First, our previous comment obviously shows that in Eq. (54) we can use the expression for $H(Q,P)$ in terms of the reduced variables (q^a,p_a), i.e., $H\to h=\dot{q}^ap_a-h(q,p)$. Next, since the U_α^β's only depend on the Q's, the p_α integration is readily performed:

$$\int[dp_\alpha]\delta[\varphi_\alpha]=|\det U_\alpha^\beta|^{-1}\ , \tag{57}$$

which cancels a similar factor in Eq. (53), leaving only $|\det(\partial F^\alpha/\partial q^\beta)|$ which is a function of the Q's only. We then integrate out the p_a's:

$$\int[dp_a]e^{iI[q^a,p_a]}=(\det g_{ab})^{1/2}e^{iS[q^a]}\ ,$$

with $S[q^a]$ defined as in Eq. (19). We then obtain

$$\int[dq^a][dq^\alpha](\det g_{ab})^{1/2}\left|\frac{\partial F^\alpha}{\partial q^\beta}\right|\delta[F^\alpha-C^\alpha]e^{iS[q^a]}\ . \tag{58}$$

Finally, using the factorization Eq. (23) and the gauge invariance of the action, $S[q^a]=S[Q]$, we get

$$\int[dQ^A](\det M_{AB})^{1/2}(\det M_{\alpha\beta})^{-1/2}\left|\det\frac{\partial F^\alpha}{\partial q^\beta}\right|\times\delta[F^\alpha-C^\alpha]e^{iS[Q]}\ ,$$

which is nothing but Eq. (39), from which our Faddeev-Popov formula (43) follows immediately. This concludes the body of results of this paper, which can be summarized as follows

(a) The reduced path integral Eq. (19) coincides with the Polyakov path integral, Eq. (20).

(b) We have obtained a new Faddeev-Popov formula, Eq. (43), which is *guaranteed* to coincide with the reduced path integral.

(c) The usual Faddeev formula [15], Eq. (54) is obtained. The consistency of our approach is then demonstrated, when we integrate out the momenta from Eq. (54) and get back our Faddeev-Popov formula, Eq. (43).

We would like to emphasize that our approach is based entirely on the natural geometrical properties of the trajectory space $\mathcal{C}$ for our type of systems. In particular, we have a prescription to define the invariant measure on $\mathcal{C}$, Eq. (12), which in turn is defined by Eq. (8). This last equation is derived directly from the Lagrangian (1), and *nothing else,* which is very satisfying. Another nice feature in our treatment is that, by virtue of the full equivalence of the different methods with the reduced path integral, *the original ambiguity in the selection of* M^{AB} *plays no role.*

At this point it is also clear that there is no need for extra *a priori* conditions on the metric $\mathcal{G}_{ij}$ on $\mathcal{C}$. This is in contrast with the usual approaches, in which it is demanded that [16] (a) the gauge group on $\mathcal{C}$ is cut down to a Lie subgroup, which is characterized by *structure constants,* and (b) $\mathcal{G}_{ij}$ be a Killing metric for the generators of this Lie subgroup. We have seen that our generators $\hat{U}_\alpha$, Eq. (10), *close with structure functions,* which is enough to define the gauge orbits on $\mathcal{C}$. Moreover, this degree of generality in our treatment is reflected in Eq. (43): the factor $(\det\Theta_{\alpha\beta})^{-1/2}$ makes the path integral invariant under arbitrary rescaling *(including Q-dependent ones)* of the parameter $\delta\varepsilon^\lambda$ which appear in the Faddeev-Popov determinant.

This can be seen as follows: a change in $\delta\varepsilon^\lambda \to \delta\tilde{\varepsilon}^\alpha = [\Lambda^{-1}(Q)]^\alpha_\beta \delta\varepsilon^\beta$ is equivalent to a change in U^α_β by means of

$$\delta q^\alpha = U^\alpha_\beta(Q)\delta\varepsilon^\beta = \tilde{U}^\alpha_\beta(Q)\delta\tilde{\varepsilon}^\beta , \tag{59}$$

i.e.,

$$\tilde{U}^\alpha_\beta(Q) = \Lambda^\gamma_\beta(Q) U^\alpha_\gamma(Q) . \tag{60}$$

Then $\Theta_{\alpha\beta}$ changes as

$$\Theta_{\alpha\beta} \to \tilde{\Theta}_{\alpha\beta} = \hat{\tilde{U}}_\alpha \cdot \hat{\tilde{U}}_\beta = \Lambda^\gamma_\alpha \Lambda^\delta_\beta \Theta_{\gamma\delta} \tag{61}$$

and therefore

$$\left| \frac{\delta F^\alpha}{\delta\varepsilon^\beta} \right| \left| \det\Theta_{\alpha\beta} \right|^{-1/2} = \left| \frac{\delta F^\alpha}{\delta\tilde{\varepsilon}^\beta} \right| \left| \det\tilde{\Theta}_{\alpha\beta} \right|^{-1/2} \tag{62}$$

which shows the announced invariance. The change of the $\hat{U}_\alpha$'s under rescaling, Eq. (60), can be recast in terms of the constraints φ_α: the gauge transformation

$$\delta Q^A = \{Q^A, \varphi_\alpha\}\delta\varepsilon^\alpha , \tag{63}$$

can also be written, since Λ depends only on Q, as

$$\delta Q^A = \{Q^A, \tilde{\varphi}_\beta\}\delta\tilde{\varepsilon}^\beta , \tag{64}$$

where

$$\tilde{\varphi}_\alpha = \Lambda^\beta_\alpha(Q)\varphi_\beta . \tag{65}$$

Hence an equivalent way of characterizing this rescaling invariance is to say that our path integrals are invariant under arbitrary rescaling (including Q-dependent ones) of the constraints φ_α. This type of invariance plays a very important role in Kuchař's program [17] which he developed in order to make reduced and his version of Dirac's quantizations coincide. Another important ingredient in this approach was invariance under point transformations in phase space, which in our case is clearly guaranteed given the way we obtained Faddeev's formula. These observations lead us to conjecture that our path integral formulation for the gauge theories considered here realizes his program.

Concurrently with and previously to our work other authors have done work similar to ours. Of particular relevance to us is the work of Ellicott, Kunstatter, and Toms [18] and Kunstatter [19]. In the former paper, a derivation of the Faddeev-Popov formula is performed, following geometrical methods very close to ours. However, after inspection of their Eqs. (2.14), (3.3), and (3.4), it is clear that their final formula (3.31) *will not reproduce the reduced path integral, which ours does.* One way to see the difference is to pay attention to our treatment of the Polyakov method: we divide by the volume of the gauge orbit, whereas they leave a determinant factor as can be seen in Eq. (2.14) of their paper. The latter paper is partly based on the results of the former, and hence we would still seem to be in disagreement with some of its conclusions. Clearly a further study is needed to elucidate the relationship between the results in these papers and ours. Finally, it would be interesting to continue our work along the following lines: (a) the consequences of our modified Faddeev-Popov formula (if any), (b) the relationship with BRST methods, and (c) the study of reparametrization-invariant systems (gravity, strings, etc.).

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[9] Rigorously speaking Yang-Mills theory is a theory with secondary constraints also. Nevertheless, as in the usual treatments, one can get rid of the pair (A^a_0, P^a_0) altogether from the outset which will render the system as one of those considered here.
[10] This of course comes from the well-known fact that primary first-class constraint generate gauge transformations when there are no secondary class constraints.
[11] Here $\int d^m x$ denotes the invariant measure in the space-time for the theory.
[12] From now on primed indices indicate adapted coordinates.
[13] As will be shown below, $M_{\alpha\beta}$ is regular.
[14] We are ignoring Gribov ambiguities in this paper.

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