PHYSICAL REVIEW D VOLUME 38, NUMBER 12 15 DECEMBER 1988

Spin-$\frac{3}{2}$ gravitational trace anomaly

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(Received 19 April 1988)

It is argued that previous computations of the spin-$\frac{3}{2}$ anomaly have spurious contributions, as there is Weyl-invariance breaking already at the classical level. The genuine, gauge-invariant, spin-$\frac{3}{2}$ gravitational trace anomaly is computed here.

I. INTRODUCTION AND REVIEW OF THE SPIN-0, -$\frac{1}{2}$, AND -1 TRACE ANOMALIES

The aim of this work is the computation of the spin-$\frac{3}{2}$ gravitational trace anomaly in four dimensions. We will consider a vector-spinor spin-$\frac{3}{2}$ quantum field, which is only coupled to an arbitrary classical gravitational field and is otherwise free. Furthermore, its Lagrangian should be invariant under local dilatations or Weyl transformations. The integration over the quantum field leads to an effective action which is a functional of the gravitational field and its derivatives. The exact result is given by the one-loop contribution. From the effective action one obtains the vacuum expectation value of the stress-energy-momentum tensor. It is not zero, but a combination of local terms which depend exclusively on the Riemann tensor. This is the gravitational trace anomaly. The symmetry which is broken by the quantization is the Weyl symmetry. Thus, the dilatation current is anomalous. Its divergence is precisely the trace of the stress(-energy-momentum) tensor.

For spins lower than $\frac{3}{2}$ the trace anomaly is well known. We will review it shortly in this introduction. The next sections will present what is known for spin $\frac{3}{2}$ and our way of approaching the computation. Some relevant steps of the computation will be presented next. We wind up with the result and some concluding comments.

Let us consider spin-0, -$\frac{1}{2}$, and -1 fields in a gravitational background. Most references concerning these fields can be found in Birrell and Davies[1] and we will refer only to those of more immediate relevance to our work. We will also follow their notation.

The Lagrangian density of the spin-0 field in a gravitational background is

$$\mathcal{L}_0(x)=\tfrac{1}{2}[-g(x)]^{1/2}\{g^{\mu\nu}(x)\phi(x)_{,\mu}\phi(x)_{,\nu} -[m^2+\xi R(x)]\phi^2(x)\}\ , \tag{1.1}$$

where $g(x)\equiv\det g_{\mu\nu}(x)$ and $R(x)$ is the scalar curvature. For the Majorana spin-$\frac{1}{2}$ field the Lagrangian density reads

$$\mathcal{L}_{1/2}(x)=\tfrac{1}{2}V(x)\left[\frac{i}{2}\{\bar{\psi}(x)\gamma^\mu\nabla_\mu\psi(x)-[\nabla_\mu\bar{\psi}(x)]\gamma^\mu\psi(x)\} -m\bar{\psi}(x)\psi(x)\right]\ , \tag{1.2}$$

where $V(x)\equiv\det V^\alpha{}_\mu(x)\equiv[-g(x)]^{1/2}$, $V^\alpha{}_\mu(x)$ being a vierbein, and ∇_μ is the covariant derivative acting on a spinor via a spinorial connection. Finally, for a spin-1 gauge field the Lagrangian density is

$$\mathcal{L}_1(x)=-\tfrac{1}{4}[-g(x)]^{1/2}F_{\mu\nu}(x)F^{\mu\nu}(x) \tag{1.3}$$

with the field strength given by

$$F_{\mu\nu}(x)\equiv A_\mu(x)_{;\nu}-A_\nu(x)_{;\mu} = A_\mu(x)_{,\nu}-A_\nu(x)_{,\mu}\ , \tag{1.4}$$

and to which a (covariant) gauge-fixing term

$$\mathcal{L}_G=-\frac{1}{2a}[-g(x)]^{1/2}[A^\mu(x)_{;\mu}]^2 \tag{1.5}$$

as well as the ghost term

$$\mathcal{L}_g=[-g(x)]^{1/2}g^{\mu\nu}(x)\partial_\mu C(x)\partial_\nu C^*(x) \tag{1.6}$$

should be added.

Recall that the Lagrangian densities $\mathcal{L}_0$, $\mathcal{L}_{1/2}$, and $\mathcal{L}_1$ are, in $n=4$ dimensions with $m=0$ and $\xi=\frac{1}{6}$, invariant under local dilatations or Weyl transformations of the metric

$$g_{\mu\nu}(x)\to\Omega^2(x)g_{\mu\nu}(x) \tag{1.7}$$

if the fields transform as

$$\begin{aligned}&\phi(x)\to\Omega^{-1}(x)\phi(x),\quad \psi(x)\to\Omega^{-3/2}(x)\psi(x)\ ,\\ &A^\mu(x)\to\Omega^{-2}(x)A^\mu(x),\quad A_\mu(x)\to A_\mu(x)\ .\end{aligned} \tag{1.8}$$

It should be mentioned here that neither $\mathcal{L}_G$ nor $\mathcal{L}_g$ are Weyl invariant. The breaking of Weyl invariance by the gauge-fixing term has been a subject of much controversy. As we will have to face a similar problem for the spin-$\frac{3}{2}$ particle it will be very useful to review the present understanding for the spin-1 particle. This will be done along this section.

The stress tensor corresponding to a classical action S is given by

$$T_{\mu\nu}(x)=\frac{2}{[-g(x)]^{1/2}}\frac{\delta S}{\delta g^{\mu\nu}(x)}=\frac{V^{\alpha}{}_{\mu}(x)}{V(x)}\frac{\delta S}{\delta V^{\alpha\nu}(x)} . \quad (1.9)$$

For the gauge-fixing term (1.5) one immediately finds, for the trace of the corresponding stress tensor,

$$T^{\mu}{}_{\mu G}(x)=-\frac{2}{a}(A^{\lambda}{}_{;\lambda}A^{\mu})_{;\mu} , \quad (1.10)$$

which is a divergence. Be redefining the dilatation current we could get rid of this term.

In the quantum theory the vacuum expectation value of the renormalized stress tensor, $\langle T^{\mu}_{\text{ren}\ \mu}\rangle$ is given again by (1.9) where S is substituted by W_{ren}, the renormalized effective action. Of course the unrenormalized effective action is ultraviolet divergent, and requires regularization. The work of Brown and Cassidy[2] and Duff[3] shows that dimensional regularization is the most convenient one. The regularized effective action carries then a pole at $n=4$, n being the dimension. This infinity is removed by a counterterm $\Delta W(n)$ the residue of which is Weyl invariant. It is thus necessarily a combination of

$$F\equiv R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}-2R_{\mu\nu}R^{\mu\nu}+\tfrac{1}{3}R^2 ,$$
$$G\equiv R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}-4R_{\mu\nu}R^{\mu\nu}+R^2 , \quad (1.11)$$

where F is, in four dimensions, the Weyl tensor squared and G gives, by the Gauss-Bonnet theorem, the Euler-Poincaré characteristic, so that both lead to Weyl-invariant functionals. A total divergence in n dimensions, $\Box R$, has of course been disposed of. Thus

$$\langle T^{\mu}_{\text{ren}\ \mu}\rangle=\frac{2}{(-g)^{1/2}}g^{\mu\nu}\left[\frac{\delta W(n)}{\delta g^{\mu\nu}}+\frac{\delta\Delta W(n)}{\delta g^{\mu\nu}}\right]_{n=4} . \quad (1.12)$$

The first term of the right-hand side (RHS) is zero. The trace anomaly comes from the second term, the subtraction.[3] Recalling that

$$\Delta W(n)=\frac{1}{(4\pi)^2}\frac{1}{n-4}\int d^nx(-g)^{1/2}[\alpha F(x)+\beta G(x)] \quad (1.13)$$

and using the formulas

$$\frac{2}{(-g)^{1/2}}g^{\mu\nu}\frac{\delta}{\delta g^{\mu\nu}}\int d^nx(-g)^{1/2}F=-(n-4)(F-\tfrac{2}{3}\Box R) ,$$
$$\frac{2}{(-g)^{1/2}}g^{\mu\nu}\frac{\delta}{\delta g^{\mu\nu}}\int d^nx(-g)^{1/2}G=-(n-4)G , \quad (1.14)$$

one finally finds the trace anomaly

$$\langle T^{\mu}_{\text{ren}\mu}\rangle=-\frac{1}{16\pi^2}[\alpha(F-\tfrac{2}{3}\Box R)+\beta G] . \quad (1.15)$$

Notice that there are strong restrictions on the *a priori* general form

$$\langle T^{\mu}_{\text{ren}\ \mu}\rangle=-\frac{1}{16\pi^2}[\alpha(F-\tfrac{2}{3}\Box R)+\beta G+\gamma\Box R+\delta R^2] \quad (1.16)$$

as (1.15) implies $\gamma=\delta=0$.

For scalar fields

$$\alpha_0=\tfrac{1}{120},\ \ \beta_0=-\tfrac{1}{360} \quad (1.17)$$

and for Majorana fermion fields

$$\alpha_{1/2}=\tfrac{1}{40},\ \ \beta_{1/2}=-\tfrac{11}{720} . \quad (1.18)$$

For photons, after subtracting twice the spin-0 ghost contribution (which is not the spin-0 anomaly as the ghosts do not couple to the curvature), the result is

$$\alpha_1=\tfrac{1}{10},\ \ \beta_1=-\tfrac{31}{180} . \quad (1.19)$$

This result is gauge independent. Indeed Brown and Cassidy[2] have proven that for covariant gauges and within dimensional regularization the effective action is gauge independent. Thus α and β do not depend on the gauge parameter a.

When other regularizations are used the anomaly is no longer of the type (1.15) but of the more general form (1.16) (with $\delta=0$), and Endo has shown using path integrals and ζ regularization that γ now has a $\ln a$ contribution.[4] Indeed he found

$$\gamma=\frac{-2+\ln a}{12} . \quad (1.20)$$

The gauge-independent term in (1.20) comes from a subtlety which happens in dimensional regularization which is clearly explained in Ref. 2. It is due to a synergetic effect in which the following facts play a role: (i) $g^{\mu}{}_{\mu}=n$ in n dimensions; (ii) scalar fields, as they come from index summations of vector fields are not conformal, but have $\xi=0$; (iii) this implies a R^2 term in the effective action which, from

$$\frac{2}{(-g)^{1/2}}g^{\mu\nu}\frac{\delta}{\delta g^{\mu\nu}}\int d^4x(-g)^{1/2}R^2=-12\Box R , \quad (1.21)$$

implies a new, gauge-independent, $\Box R$ contribution. This explains the first term on the RHS of (1.20). The second term is the gauge-dependent contribution which appears when one does not use dimensional regularization.

We are not interested in regularization-dependent, gauge-dependent terms which furthermore [see (1.21)] can be taken away by adding a local counterterm to the Lagrangian. The genuine anomaly is of the form (1.15) and the aim of this work is the computation of $\alpha_{3/2}$ and $\beta_{3/2}$.

Before finishing this introduction let us recall here that most studies of spin-$\frac{3}{2}$ fields have been performed in connection with supergravity, as only then the well-known inconsistency problems of spin-$\frac{3}{2}$ fields coupled to gravity are overcome. Our point of view here is different and more modest. We do not assume anything about the gravity Lagrangian, in particular we do not assume it to be of the Einstein-Hilbert type. Our gravitational field is a background field without specified dynamics, and there is no inconsistency problem. In fact our primary symmetry will be Weyl invariant which, if assumed for the gravitational field too, would exclude the Einstein-Hilbert action. In short, our approach is to consider the trace anomaly issue in a setting unconstrained by supergravity, as we think that the knowledge of the trace anomaly is of interest independently of the status of supergravity. It is true that if supergravity eventually turns out to be the right avenue to quantum gravity, it has been shown by Grisaru, Nielsen, Siegel, and Zanon and by Duff[5] that the total (anomalous plus nonanomalous) trace is of more relevance, since it is independent of the field representation and it is part of the corresponding supertrace. But even then our result will still be the anomalous contribution of the spin-$\frac{3}{2}$ vector-spinor field to the gravitational energy-momentum trace, as we will explain in more detail at the end of the next section.

II. REVIEW OF THE SPIN-$\frac{3}{2}$ GRAVITATIONAL TRACE-ANOMALY COMPUTATIONS

The first computation was done simultaneously by several authors. Christensen and Duff obtained[6,1]

$$(\alpha+\beta)_{\rm CD}=-\tfrac{233}{720},\quad \beta_{\rm CD}=0\ ,\qquad \gamma_{\rm CD}=-\tfrac{233}{1080},\quad \delta_{\rm CD}=-\tfrac{61}{1440}\ . \tag{2.1}$$

This result is not of type (1.15). This is not surprising, since their Lagrangian is not Weyl invariant. Indeed, their starting Lagrangian density is the one obtained from supergravity:

$$\mathcal{L}^{\rm SS}_{3/2}(x)=\tfrac{1}{2}V(x)i\bar\psi^\rho(x)(\nabla\!\!\!/ g_{\rho\tau}+\gamma_\rho\nabla\!\!\!/\gamma_\tau-\gamma_\tau\nabla_\rho-\gamma_\rho\nabla_\tau)\psi^\tau(x)\ , \tag{2.2}$$

where $\nabla\!\!\!/\equiv\gamma_\mu\nabla^\mu$ and where the covariant derivative acts on the spin-$\frac{3}{2}$ field via both the standard affine and the spinorial connection, as it has both a vector index and spinor components. It is not invariant under a Weyl transformation

$$\psi^\mu(x)\to\Omega^{-5/2}(x)\psi^\mu(x)\ ,\qquad \psi_\mu(x)\to\Omega^{-1/2}(x)\psi_\mu(x)\ . \tag{2.3}$$

Indeed, one can check that

$$\delta_W\mathcal{L}^{\rm SS}_{3/2}(x)=\frac{i}{2}\Omega^{-1}(x)V(x)[\bar\psi^\rho(x)\gamma^\lambda\Omega_{,\rho}(x)\psi_\lambda(x)-\bar\psi^\rho(x)\gamma_\rho\Omega_{,\lambda}(x)\psi^\lambda(x)]\ . \tag{2.4}$$

It is not invariant under gauge transformations of the type

$$\psi^\tau(x)\to\psi^\tau(x)+\epsilon\nabla^\tau\psi(x) \tag{2.5}$$

as it changes according to

$$\delta_g\mathcal{L}^{\rm SS}_{3/2}(x)=i\epsilon V(x)\left[R_{\rho\omega}(x)-\frac{R(x)}{2}g_{\rho\omega}\right]\times[\bar\psi^\rho(x)\gamma^\omega\psi(x)+\bar\psi(x)\gamma^\omega\psi^\rho(x)]\ . \tag{2.6}$$

Gauge invariance only holds for Ricci-flat spaces $R_{\mu\nu}=0$. Of course, quantization requires gauge fixing. They work in the harmonic gauge $\gamma_\mu\psi^\mu(x)=0$. This leads to two Majorana ghosts. These take away four of the eight degrees of freedom of a Majorana ψ^μ field. The remaining two unphysical degrees of freedom enter their calculation through consistency conditions. Their only gauge-independent result is $\alpha+\beta$, the coefficient of the Riemann tensor squared. By choosing the harmonic gauge, their kinetic energy operator squared is of the Gilkey type;[7] this simplifies computations quite a lot.

The gauge-independent part of their result has been obtained simultaneously by other authors for the same Lagrangian density: Perry,[8] who fixes completely the unphysical degrees of freedom working in the gauge $\gamma_\mu\psi^\mu=\nabla_\mu\psi^\mu=0$; Critchley[9] and Yoneya,[10] who work in the harmonic gauge and eliminate the remaining degrees of freedom with the help of a third ghost, the so-called Nielsen-Kallosh ghost.[11]

More recently Fradkin and Tseytlin have again computed all four coefficients,[12] in the harmonic gauge and with three ghosts. They find

$$(\alpha+\beta)_{\rm FT}=-\tfrac{233}{720},\quad \beta_{\rm FT}=\tfrac{229}{720}\ ,\qquad \gamma_{\rm FT}=-\tfrac{74}{180},\quad \delta_{\rm FT}=-\tfrac{1}{9}\ . \tag{2.7}$$

Again the gauge-independent coefficient coincides with all the other values; this is not so for the gauge-dependent ones. Being gauge dependent, this does not seem of much relevance, it only reflects the different treatment of the unphysical degrees of freedom. What seems more important to us is that neither (2.1) nor (2.7) is the anomaly. They are not of type (1.15). They have not been computed for a classically Weyl-invariant Lagrangian.

Weyl invariance implies, in four dimensions, a Lagrangian density

$$\mathcal{L}^W_{3/2}(x)=\frac{i}{2}V(x)\bar\psi^\rho(x)[\nabla\!\!\!/ g_{\rho\tau}+(\tfrac{3}{8}+a)\gamma_\rho\nabla\!\!\!/\gamma_\tau-\tfrac{1}{2}(\gamma_\rho\nabla_\tau+\gamma_\tau\nabla_\rho)]\psi^\tau(x) \tag{2.8}$$

with a arbitrary. Furthermore (2.8) is invariant under gauge transformations of the type

$$\psi^\tau(x)\to\psi^\tau(x)+\epsilon\gamma^\tau\nabla\!\!\!/\psi(x) \tag{2.9}$$

if $a=0$. We thus have a conformal- and gauge-invariant Lagrangian. Quantization of the spin-$\frac{3}{2}$ fields requires fixing the gauge. This is done by taking $a\neq0$, and it is done in a conformal-invariant way. By starting from (2.8) and regularizing dimensionally we make sure that the genuine trace anomaly of type (1.15) will come out. We will also prove that both α and β are gauge independent.

The gauge fixing of (2.8) allows us to quantize the theory, but does not subtract all the unphysical degrees of freedom. We will take them out in the manner of Nielsen and Kallosh, the only conformal-invariant procedure we know.

We already know one result:

$$\alpha_{3/2}+\beta_{3/2}=-\tfrac{233}{720} . \quad (2.10)$$

Indeed, in the harmonic gauge Lagrangians (2.2) and (2.8) coincide and the genuine anomaly calculated from (2.8) does not depend on the way we fix the gauge, as we shall see. Therefore, all that remains is to compute another combination. This is what we will do next.

Let us finish this section recalling that the spin-$\frac{3}{2}$ anomaly is of course unique, and that one can thus use, in principle, any Lagrangian to compute it. However, Lagrangians which are not Weyl invariant have a nonanomalous contribution to the trace, as shown by the results (2.1) and (2.7). Here, by starting from a Weyl-invariant Lagrangian, we will obtain directly the anomaly, as no nonanomalous contribution arises. Once the anomaly is obtained there is no use anymore for Lagrangian (2.8): the Lagrangian (2.2) is likely to be more relevant to physics. Notice furthermore that we are using in (2.8) the same vector-spinor representation used in supergravity, Eq. (2.2), so that although even the "gauge-independent" part of the anomaly, $\alpha+\beta$, depends on the specific field representation,[5] there is no problem for us here in taking over (2.10). This is so because the comments made after (2.10) prove that the coefficient of $\alpha+\beta$, and only this one, is the same for the trace corresponding to the Lagrangian (2.2) (which is not Weyl invariant) as for the anomalous trace (which we are interested in).

III. PRELIMINARY FORMULAS

Consider the first-order differential operator of (2.6):

$$H^{\rho}_{a\tau}\equiv i[\nabla\!\!\!\!/\, g^{\rho}_{\tau}+(\tfrac{3}{8}+a)\gamma^{\rho}\nabla\!\!\!\!/\,\gamma_{\tau}-\tfrac{1}{2}(\gamma^{\rho}\nabla_{\tau}+\gamma_{\tau}\nabla^{\rho})] . \quad (3.1)$$

The effective action, neglecting for the time being ghost fields, is given by

$$e^{iW}\equiv\int\mathcal{D}\psi^{\rho}\exp\left[i\int d^4x\,\mathcal{L}^{W}_{3/2}(x)\right]\propto(\det H_a)^{1/2} , \quad (3.2)$$

where operator notation has been used for (3.1) and where the spin-$\frac{3}{2}$ fields have been taken to be Majorana fields. The Green's function

$$S^{\tau}_{a\lambda}(x,x')=-i\langle T(\psi^{\tau}(x)\bar{\psi}_{\lambda}(x'))\rangle \quad (3.3)$$

satisfies the differential equation

$$H^{\rho}_{a\tau}S^{\tau}_{a\lambda}(x,x')=-V^{-1}(x)\delta(x-x')g^{\rho}{}_{\lambda}(x) . \quad (3.4)$$

The operator formalism is introduced via a Hilbert space with norm

$$\langle x,\tau|x',\lambda\rangle=V^{-1}(x)g^{\tau}{}_{\lambda}(x)\delta(x-x') . \quad (3.5)$$

Then

$$S^{\tau}_{a\lambda}(x,x')\equiv\langle x,\tau|S_a|x',\lambda\rangle \quad (3.6)$$

and (3.4) reads

$$H_aS_a=-\mathbb{1} . \quad (3.7)$$

From (3.2) one readily obtains

$$W=\frac{i}{2}\mathrm{tr}\ln(-S_a)$$
$$=\frac{i}{2}\sum_{\mu}\int d^4x\,V(x)\mathrm{Tr}\langle x,\mu|\ln(-S_a)|x,\mu\rangle , \quad (3.8)$$

where Tr means the trace over spinor indices. Recall furthermore that

$$-S=H^{-1}=i\int_0^{\infty}ds\,e^{-isH} \quad (3.9)$$

so that, for a constant,

$$\ln(-S)=-\ln H=\int_0^{\infty}\frac{ds}{s}e^{-iHs} . \quad (3.10)$$

Thus the whole H-dependent contribution to the effective action is given by

$$W=\frac{i}{2}\mathrm{tr}\int_0^{\infty}\frac{ds}{s}e^{-iH_as} . \quad (3.11)$$

Let us now prove that W does not depend on a, following steps similar to the ones first performed by Brown and Cassidy.[2] Because of gauge invariance, it follows immediately from (3.1) that

$$\gamma\nabla\!\!\!\!/\,\gamma H_a=a\gamma\nabla\!\!\!\!/\,\gamma H_1=ia(\gamma\nabla\!\!\!\!/\,\gamma)^2 , \quad (3.12)$$

so that

$$\gamma\nabla\!\!\!\!/\,\gamma H^n_a=a^n\gamma\nabla\!\!\!\!/\,\gamma H^n_1=(ia)^n(\gamma\nabla\!\!\!\!/\,\gamma)^{n+1} . \quad (3.13)$$

A variation of W with respect to a gives

$$\delta W=i\frac{\delta a}{2}\mathrm{tr}\int_0^{\infty}ds\,\gamma\nabla\!\!\!\!/\,\gamma e^{-iH_as}$$
$$=i\frac{\delta a}{2}\mathrm{tr}\int_0^{\infty}ds\,\gamma\nabla\!\!\!\!/\,\gamma e^{a\gamma\nabla\!\!\!\!/\,\gamma s}$$
$$=i\frac{\delta a}{2a}\mathrm{tr}\int_0^{\infty}ds\frac{\partial}{\partial s}e^{a\gamma\nabla\!\!\!\!/\,\gamma s} , \quad (3.14)$$

which is a constant and can thus be neglected. This proof is formal and requires, in principle, both ultraviolet (UV) and infrared (IR) regulators. We will use a mass as the IR regulator. The anomaly of course does not depend on the IR behavior of the theory and introducing a mass allows one to work in momentum space. Thus we will have

$$H^{\rho}_{\tau}(m)\equiv H^{\rho}_{a\tau}-mg^{\rho}_{\tau} . \quad (3.15)$$

We will furthermore bosonize our differential operator by squaring it as it is usually done for fermions, and work with

$$K^{\rho}_{\tau}\equiv H^{\rho}_{\omega}(m)H^{\omega}_{\tau}(-m) . \quad (3.16)$$

In flat space, one finds

$$K^{\rho}_{\tau}(g^{\mu\nu}=\eta^{\mu\nu})=\eta^{\rho}_{\tau}(-\Box-m^2)+(\tfrac{5}{16}-4a^2)\gamma^{\rho}\Box\gamma_{\tau}$$
$$+\tfrac{1}{2}(\partial^{\rho}\partial_{\tau}-\Sigma^{\rho}_{\nu}\partial^{\nu}\partial_{\tau}-\partial^{\rho}\partial_{\nu}\Sigma^{\nu}_{\tau}) \quad (3.17)$$

with $\Sigma^{\alpha\beta}=\frac{1}{4}[\gamma^{\alpha},\gamma^{\beta}]$. The corresponding momentum-space Green's function, which we will call $\tilde{G}$, satisfies

$$(k^2-m^2)\tilde{G}^{\rho}_{\ \alpha}-(\tfrac{5}{16}-4a^2)k^2\gamma^{\rho}\gamma_{\omega}\tilde{G}^{\omega}_{\ \alpha}$$
$$-\tfrac{1}{2}(2k^{\rho}k_{\omega}-\tfrac{1}{2}\gamma^{\rho}k\!\!\!/\,k_{\omega}-\tfrac{1}{2}k^{\rho}k\!\!\!/\,\gamma_{\omega})\tilde{G}^{\omega}_{\ \alpha}=-\eta^{\rho}_{\alpha} . \quad (3.18)$$

This gives immediately

$$\tilde{G}_{\alpha}^{\ \rho}=\frac{-1}{k^2-m^2}\left[\eta_{\alpha}^{\rho}+\frac{4}{k^2-4m^2}k^{\rho}k_{\alpha}-\frac{1}{k^2-4m^2}(\gamma^{\rho}\not{k}k_{\alpha}+k^{\rho}\not{k}\gamma_{\alpha})+\frac{k^2[k^2-(5-64a^2)m^2]}{4(k^2-4m^2)(16a^2k^2-m^2)}\gamma^{\rho}\gamma_{\alpha}\right]. \tag{3.19}$$

No gauge fixing can avoid having a pole at $k^2=4m^2$, besides the canonical one at $k^2=m^2$. This is due to our differential operator K not being of the Gilkey type. This is different from what happened to the photon field, where for the Feynman gauge we recover a Gilkey-type Laplace-Beltrami operator. We will choose a gauge for which there are at least no other poles. Two such gauges exist: $|a|=\frac{1}{4}$ and $|a|=\frac{1}{8}$. We will use the first one.

Recall now the DeWitt-Schwinger proper-time representation of the Green's function:[13]

$$G_{\nu}^{\mu}(x,x')=-i\Delta^{1/2}(x,x')\frac{1}{(4\pi)^{n/2}}\int_0^{\infty}ids\frac{1}{(is)^{n/2}}\exp\left[-im^2s+\frac{\sigma(x,x')}{2is}\right]F_{\nu}^{\mu}(x,x';is). \tag{3.20}$$

When $x'\to x$, $\Delta(x,x')\to[-g(x)]^{-1/2}$, $G(x,x')\to 0$ and the UV divergences appear as divergences around $s=0$ in the proper-time integral of (3.20). Having dimensionally regularized, one can take the $x'\to x$ limit getting

$$G_{\nu}^{\mu}(x,x)=-i[-g(x)]^{-1/2}\frac{1}{(4\pi)^{n/2}}\times\int_0^{\infty}\frac{i\,ds}{(is)^{n/2}}\exp(-im^2s)F_{\nu}^{\mu}(x,x;is). \tag{3.21}$$

Recall that $F_{\nu}^{\mu}(x,x;is)$ has the adiabatic (small-proper-time) expansion

$$F_{\nu}^{\mu}(x,x;is)=\sum_{j=0}^{\infty}a_{j\nu}^{\mu}(x)(is)^j. \tag{3.22}$$

The coefficient which leads to the anomaly is a_2 ("the magical a_2 coefficient," see Ref. 14 for its history). How do we compute $F_{\nu}^{\mu}(x,x;is)$? This is done by introducing

$$\mathcal{G}_{\nu}^{\mu}(x,x')=[-g(x)]^{1/4}G_{\nu}^{\mu}(x,x')[-g(x')]^{1/4} \tag{3.23}$$

and its Fourier transform

$$\mathcal{G}_{\nu}^{\mu}(x,x')=(2\pi)^{-n}\int d^nk\, e^{-ik\cdot y}\tilde{\mathcal{G}}_{\nu}^{\mu}(k), \tag{3.24}$$

where y^{μ} are the normal coordinates of x, the origin being at x', and $k\cdot y=k_{\alpha}\eta^{\alpha\beta}y_{\beta}$, so that one works in a localized momentum space. $\mathcal{G}_{\nu}^{\mu}$ is obtained from the corresponding Green's-function equation in momentum space and one then readily obtains $a_2(x)$ using the above formulas.

That the anomaly is basically $a_2(x)$ has been proven very many times, so here we will only comment on what "basically" means. First, Brown and Cassidy showed that in dimensional regularization there is a further contribution to the anomaly for the spin-1 case. This was shortly reviewed at the end of Sec. I. However, this problem only affects the $\Box R$ term and we are not going to compute this one. Furthermore, $a_2(x)$ is *a priori* gauge dependent, while the anomaly is not. There is no contradiction in this statement, if the gauge dependence is carried by the $\Box R$ term. Endo showed that this is so for the spin-1 field,[4] and that the a dependence is of type ln$a$. His steps can be repeated here with the same results. Again, all this is irrelevant to our computation as we will not compute the $\Box R$ term. It will nevertheless be a nontrivial check of our computation to see the logarithmic terms cancel out at the end.

Once $a_2(x)$ is known, the anomaly will be given by[1]

$$\langle T^{\mu}_{\text{ren}\ \mu}\rangle=-\frac{1}{16\pi^2}a_2(x). \tag{3.25}$$

Here the minus sign corresponding to fermions is included in the definition of $a_2(x)$.

IV. THE COMPUTATION

We will compute $a_2(x)$ for a specific gravitational background, for which important simplifications occur. The computation is still quite cumbersome, so that only some intermediate steps will be given. (The whole computation has been done twice, by hand and with an algebraic program.)

We have chosen a maximally symmetric space.[15] It is known that they are uniquely specified by a constant curvature K. Then

$$R_{\mu\nu\lambda\sigma}=K(g_{\nu\lambda}g_{\mu\sigma}-g_{\nu\sigma}g_{\mu\lambda}),\quad R_{\mu\nu}=-3Kg_{\mu\nu},\quad R=-12K. \tag{4.1}$$

We will use the following normal coordinates:

$$g_{\mu\nu}(y)=\eta_{\mu\nu}+Ky_{\mu}y_{\nu}(1+Ky^2)+O(K^3). \tag{4.2}$$

The following useful expressions can then be obtained:

$$\Gamma^{\mu}_{\rho\sigma}=Ky^{\mu}(\eta_{\rho\sigma}+Ky_{\rho}y_{\sigma})\ ,$$
$$-g=1+Ky^2(1+Ky^2)\ , \tag{4.3}$$
$$V^{\beta}_{\mu}=\eta^{\beta}_{\mu}+\frac{K}{2}y^{\beta}y_{\mu}+\tfrac{3}{8}K^2y^2y^{\beta}y_{\mu}\ ,$$
$$\Gamma_{\nu}=\frac{K}{2}\Sigma^{\gamma}_{\nu}y_{\gamma}\left[1+\frac{K}{4}y^2\right],\quad \gamma^{\rho}\equiv V^{\rho}_{\beta}\gamma^{\beta}\ .$$

They are valid up to order K^2 included. Recall that the covariant derivative acts on ψ^{ρ} according to

$$\nabla_{\mu}\psi^{\rho}=(\partial_{\mu}+\Gamma_{\mu})\psi^{\rho}+\Gamma^{\rho}_{\mu\nu}\psi^{\nu}\ . \tag{4.4}$$

In this space

$$\Box R=F=0\ , \tag{4.5}$$

so that

$$\beta=\frac{a_2}{G} \tag{4.6}$$

with

$$G=24K^2\ . \tag{4.7}$$

Our starting equation is

$$K^{\rho}_{\tau}G^{\tau}_{\nu}(x,x')=-\frac{g^{\rho}{}_{\nu}}{\sqrt{-g}}\delta(x-x')\ . \tag{4.8}$$

In momentum space, and expanding in K, one can write

$$\tilde{\mathcal{G}}^{\mu}_{\nu}(k)\equiv\tilde{\mathcal{G}}^{\mu}_{\nu}(0)+K\tilde{\mathcal{G}}^{\mu}_{\nu}(1)+K^2\tilde{\mathcal{G}}^{\mu}_{\nu}(2)\ , \tag{4.9}$$
$$\tilde{K}^{\rho}_{\tau}\left[\frac{\partial}{\partial k}\right]\equiv\tilde{K}^{\rho}_{\tau}(0)+K\tilde{K}^{\rho}_{\tau}(1)+K^2\tilde{K}^{\rho}_{\tau}(2)\ .$$

Obviously

$$\tilde{K}^{\rho}_{\lambda}(0)\tilde{\mathcal{G}}^{\lambda}_{\alpha}(0)=-\eta^{\rho}_{\alpha}\ , \tag{4.10}$$

which is precisely Eq. (3.18). Notice that $\tilde{K}^{\rho}_{\lambda}(0)$ does not have derivatives. Also

$$\tilde{\mathcal{G}}^{\lambda}_{\alpha}(1)=\tilde{\mathcal{G}}^{\lambda}_{\rho}(0)\tilde{K}^{\rho}_{\lambda}(1)\tilde{\mathcal{G}}^{\lambda}_{\alpha}(0) \tag{4.11}$$

and

$$\mathrm{Tr}\tilde{\mathcal{G}}^{\alpha}_{\alpha}(2)=\mathrm{Tr}\tilde{\mathcal{G}}^{\alpha}_{\rho}(0)\tilde{K}^{\rho}_{\lambda}(2)\tilde{\mathcal{G}}^{\lambda}_{\alpha}(0)+\mathrm{Tr}\tilde{\mathcal{G}}^{\alpha}_{\rho}(0)\tilde{K}^{\rho}_{\lambda}(1)\tilde{\mathcal{G}}^{\lambda}_{\alpha}(1)\ . \tag{4.12}$$

An intermediate result is

$$\begin{aligned}\tilde{\mathcal{G}}^{\lambda}_{\alpha}(1)=&\ \eta^{\lambda}_{\alpha}(-\tfrac{2}{3}P^2-4P^3-8P^4+9PQ-\tfrac{28}{3}Q^2)+\gamma^{\lambda}\gamma_{\alpha}(-\tfrac{1}{2}P^2+\tfrac{1}{3}P^3+\tfrac{2}{3}P^4-\tfrac{5}{2}PQ+2Q^2-\tfrac{16}{3}Q^3-\tfrac{128}{3}Q^4)\\&+4k^{\lambda}k_{\alpha}(-\tfrac{4}{3}P^3+\tfrac{8}{3}P^4-4P^2Q-7PQ^2+\tfrac{28}{3}Q^3-\tfrac{128}{3}Q^4)+\gamma^{\lambda}\not{k}k_{\alpha}(\tfrac{4}{3}P^3-\tfrac{8}{3}P^4+2P^2Q+5PQ^2-\tfrac{16}{3}Q^3+\tfrac{128}{3}Q^4)\\&+k^{\lambda}\not{k}\gamma_{\alpha}(\tfrac{7}{3}P^3-\tfrac{8}{3}P^4+4P^2Q+7PQ^2-\tfrac{28}{3}Q^3+\tfrac{128}{3}Q^4)\ ,\end{aligned} \tag{4.13}$$

where

$$P\equiv\frac{1}{k^2-m^2},\quad Q\equiv\frac{1}{k^2-4m^2} \tag{4.14}$$

and powers of m^2 are understood so as to have the right dimensions. Furthermore,

$$\mathrm{Tr}\tilde{\mathcal{G}}^{\alpha}_{\rho}(0)\tilde{K}^{\rho}_{\lambda}(2)\tilde{\mathcal{G}}^{\lambda}_{\alpha}(0)=-48P^3-42P^4-54P^5-54P^6-\tfrac{536}{3}PQ^2+\tfrac{608}{3}Q^3-224Q^4+13\,824Q^5+55\,296Q^6 \tag{4.15}$$

and

$$\begin{aligned}\mathrm{Tr}\tilde{\mathcal{G}}^{\alpha}_{\rho}(0)\overleftarrow{\tilde{K}}{}^{\rho}_{\lambda}(1)\tilde{\mathcal{G}}^{\lambda}_{\alpha}(1)=&\ \tfrac{6839}{27}P^3+\tfrac{845}{3}P^4+\tfrac{1057}{3}P^5+777P^6+768P^7-\tfrac{4352}{27}P^2Q-\tfrac{12\,584}{27}PQ^2+\tfrac{13\,040}{27}Q^3\\&+\tfrac{3008}{3}Q^4+\tfrac{20\,224}{3}Q^5+56\,320Q^6+262\,144Q^7\ ,\end{aligned} \tag{4.16}$$

where the $\tilde{K}^{\rho}_{\lambda}(1)$ derivatives act to the left, as a k integration is performed anyhow afterwards. Putting (4.15) and (4.16) together one gets

$$\begin{aligned}\mathrm{Tr}\tilde{\mathcal{G}}^{\alpha}_{\alpha}(2)=&\ \tfrac{5543}{27}P^3+\tfrac{719}{3}m^2P^4+\tfrac{895}{3}m^4P^5+723m^6P^6+768m^8P^7-\tfrac{4352}{27}P^2Q-\tfrac{17\,408}{27}PQ^2+\tfrac{18\,512}{27}Q^3\\&+\tfrac{2336}{3}m^2Q^4+\frac{61\,696}{3}m^4Q^5+111\,616m^6Q^6+262\,144m^8Q^7\ .\end{aligned} \tag{4.17}$$

Notice that the coefficient of PQ^2 is exactly 4 times the one of P^2Q, which does not happen for (4.15) and (4.16). This is the reason for the cancellation of a ln4 term which appears after the k integration is performed. As $\Box R=0$ in our space-time, no logarithms are expected to appear.

From (3.21)–(3.24) we finally have

$$K^2\int d^4k\ \mathrm{Tr}\tilde{\mathcal{G}}^{\alpha}_{\alpha}(2)=a_2\pi^2\int_0^{\infty}ds\ e^{-im^2s}\ , \tag{4.18}$$

where $a_2\equiv\mathrm{Tr}a^{\mu}_{2\mu}(0)$. Thus

$$a_2=i\frac{K^2m^2}{\pi^2}\int d^4k\ \mathrm{Tr}\tilde{\mathcal{G}}^{\alpha}_{\alpha}(2)\ . \tag{4.19}$$

This leads, after performing the k integration, to

$$a_2=\tfrac{104}{15}K^2\ . \tag{4.20}$$

There are now the following factors to be taken into account: a factor $\frac{1}{2}$ because we have squared the differential operator and a factor -1 because we are dealing with fermions. Finally, three Majorana ghosts have to be subtracted: from (1.18) and (4.6) this gives

$$a_2^{(3/2)}=(-\tfrac{52}{15}+3\times\tfrac{11}{30})K^2=-\tfrac{71}{30}K^2 \ . \tag{4.21}$$

Thus, from (4.6),

$$\beta_{3/2}=-\tfrac{71}{720} \ , \tag{4.22}$$

and, from (2.10),

$$\alpha_{3/2}=-\tfrac{9}{40} \ . \tag{4.23}$$

V. CONCLUSIONS

We have computed the spin-$\frac{3}{2}$ gravitational trace anomaly. It is independent of the gauge parameter and the result is

$$\alpha_{3/2}=-\tfrac{9}{40}, \ \ \beta_{3/2}=-\tfrac{71}{720} \ . \tag{5.1}$$

There is no doubt that anomaly cancellation, in spite of some remarkable divergent views,[16] plays an utmost role in modern quantum field and string theory. The constancy in sign of the spin-0, -$\frac{1}{2}$, and -1 α and β values (1.17)–(1.19) did not allow for cancellation. Indeed, arguments based on positivity of the two-graviton Green's function were put forward which made anomaly cancellation unlikely.[17] They do not seem to apply to spin-$\frac{3}{2}$ fields, and they in fact did not bother any of the authors of the previous spin-$\frac{3}{2}$ anomaly computations. Our result opens this possibility again. Nothing can be concluded, however, until the genuine spin-2 anomaly is computed. Again, this does not seem to have been done, up to now.

Let us mention finally that we are not claiming that spin-$\frac{3}{2}$ particles should be described by the Lagrangian (2.8), but that this is the Lagrangian which allows an unambiguous computation of the trace anomaly; spin-$\frac{3}{2}$ particles might be more adequately described by other Lagrangians, which then, however, lead to nonanomalous contributions to the trace of the stress tensor. Another issue, which we have not addressed here, is whether spin-$\frac{3}{2}$ particles should be described by fields in a representation different from the vector-spinor one used here.

ACKNOWLEDGMENTS

We thank J. L. Petersen and P. DiVecchia for reading the manuscript and N. K. Nielsen for helpful remarks. This work was written while one of us (R.T.) was visiting Nordisk Institut for Teoretisk Atomfysik (NORDITA), København, Denmark; he acknowledges its hospitality and financial support by both NORDITA and Comissió Interdepartamental de Recerca i Innovació Tecnològica, Catalonia, Spain. This work has been supported by Comisión Asesora de Investigación Científica y Técnica Contract No. AE87-0016-3. J.T. thanks the Ministerio de Educación y Ciencia, Spain for financial support.

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