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The nucleolus of the assignment game. Structure of the family

F. Javier Martínez-de-Albéniz
Carlos Rafels
Neus Ybern

UNIVERSITAT DE BARCELONA

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Abstract: We show that the family of assignment matrices which give rise to the same nucleolus forms a compact join-semilattice with one maximal element. The above family is in general not a convex set, but path-connected.

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F. Javier Martínez-de-Albéniz
Universitat de Barcelona

Carlos Rafels
Universitat de Barcelona

Neus Ybern
Universitat Politècnica de Catalunya

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1. Introduction

The assignment game (Shapley and Shubik, 1972) is the cooperative viewpoint of a two-sided market. There are two sides of the market, i.e. two disjoint sets of agents, buyers and sellers, who can trade. The profits are collected in a matrix, the assignment matrix. The allocation of the optimal profit should be such that no coalition has incentives to depart from the grand coalition and act on its own. In doing so, a first game-theoretical analysis of cooperation focuses on the core of the game. Shapley and Shubik show that the core of any assignment game is always nonempty. It coincides with the set of solutions of the linear program, dual to the classical optimal assignment problem. A recent survey on assignment games is Núñez and Rafels (2015).

Among other solutions, the nucleolus (Schmeidler, 1969) is a "fair" solution in the general context of cooperative games. It is a unique core-selection that lexicographically minimizes the excesses[1] arranged in a nondecreasing way. The standard procedure for computing the nucleolus proceeds by solving a finite (but large) number of related linear programs. As a solution concept, the nucleolus has been analyzed and computed in many cooperative games. Solymosi and Raghavan (1994) gives an algorithm for the computation of the nucleolus of the assignment game, computed in polynomial time. Recently Martínez-de-Albéniz et al. (2013b) provides a new procedure to compute the nucleolus of the assignment game. An interesting survey on the nucleolus and its computational complexity is given in Greco et al. (2015).

From a geometric point of view, Llerena and Núñez (2011) have characterized the nucleolus of a square assignment game, essential for our purposes. To illustrate

[1] Given a coalition $S \subseteq N$, and an allocation $x \in \mathbb{R}^N$ the excess of a coalition is defined as $e(S,x) := v(S) - \sum_{i \in S} x_i$. Notice they can be considered as complaints.

the geometric characterization, consider the following assignment matrix

$$B = \begin{pmatrix} 8 & 5 \\ 5 & 4 \end{pmatrix}.$$

There are 2 buyers (rows) and 2 sellers (columns). The worth to share is 12, obtained by pairing both sides on the main diagonal. Its nucleolus is $(4,2;4,2) \in \mathbb{R}^2_+ \times \mathbb{R}^2_+$, as we will see later. Consider also matrix

$$C = \begin{pmatrix} 8 & 2 \\ 0 & 4 \end{pmatrix},$$

which has also the same nucleolus. To see it we draw the core[2] of the associated assignment games and their nucleolus. We depict the projection on the buyers' (first) coordinates of the core of both games in Figure 1. The core of the first one $C(w_B)$ is in dark shading, vertices B_1, B_2, B_3 and B_4, and the second one $C(w_C)$ in light shading, vertices C_1, C_2, C_3, C_4 and C_5.

From Llerena and Núñez (2011) the nucleolus of matrix B is the unique core point N such that the distances over some segments to the core's walls are equal: $\overline{A'N} = \overline{NB'}, \overline{C'N} = \overline{ND'}$ and $\overline{EN} = \overline{NF}$. Notice that for matrix C the analogous equalities are $\overline{AN} = \overline{NB}, \overline{CN} = \overline{ND}$ and $\overline{EN} = \overline{NF}$.

From the above geometric illustration we may expect large sets of assignment matrices sharing a given vector as their nucleolus.

In this paper we focus on the structure the family of assignment matrices that give rise to the same nucleolus. The main contributions of the paper are the following:

- The family of matrices with the same nucleolus forms a join-semilattice, i.e.

[2] The core is defined later in (1) and (2).

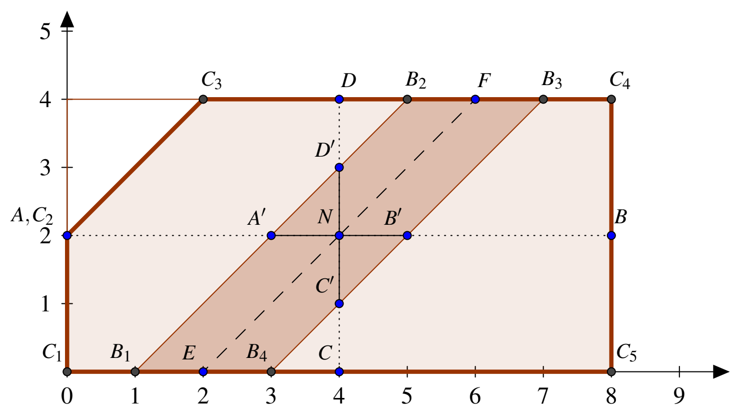

Figure 1: Two cores with the same nucleolus, $(4,2;4,2)$.

closed by entry-wise maximum. The family has a unique maximum element which is always a valuation matrix (Section 4).

- We show that the above family is a path-connected set and give a precise path, connecting any matrix of the family with its maximum element. We also study the cardinality of the family (Section 6).
- The inverse problem is also analyzed and studied, i.e. conditions on a vector to be the nucleolus of some assignment game (Section 7).

2. Preliminaries on the assignment game

An assignment market (M,M',A) is defined to be two disjoint finite sets: M the set of buyers and M' the set of sellers, and a nonnegative matrix $A=(a_{ij})_{i\in M, j\in M'}$ which represents the profit obtained by each mixed-pair $(i,j)\in M\times M'$. To distinguish the j-th seller from the j-th buyer we will write the former as j' when needed. The assignment market is called square whenever $|M|=|M'|$. Usually we denote by $m = |M|$ and $m' = |M'|$. M_m^+ denotes the set of nonnegative square matrices with m rows and columns, and $\mathrm{M}_{m\times m'}^+$ the set of nonnegative matrices with m rows and m' columns.

Recall that $\mathrm{M}_{m\times m'}^+$ forms a lattice with the usual ordering $\leq$ between matrices. The maximum $C = A \vee B$ of two matrices $A, B \in \mathrm{M}_{m\times m'}^+$ is defined entry-wise, i.e. as $c_{ij} = \max\{a_{ij}, b_{ij}\}$ for all $(i, j) \in M \times M'$. Given an ordered subset of matrices $(\mathscr{F}, \leq), \mathscr{F} \subseteq \mathrm{M}_{m\times m'}^+$, we say matrix $C \in \mathscr{F}$ is a maximal (minimal) element of $(\mathscr{F}, \leq)$ if whenever there is a matrix $D \in \mathscr{F}$ with $D \geq (\leq) C$, then $D = C$. Matrix $C \in \mathscr{F}$ is a maximum element of $(\mathscr{F}, \leq)$ if $C \geq D$ for all $D \in \mathscr{F}$.

A *matching* $\mu \subseteq M \times M'$ between M and M' is a bijection from $M_0 \subseteq M$ to $M_0' \subseteq M'$ with $|M_0| = |M_0'| = \min\{|M|, |M'|\}$. We write $(i, j) \in \mu$ as well as $j = \mu(i)$ or $i = \mu^{-1}(j)$. If for some buyer $i \in M$ there is no seller $j \in M'$ satisfying $(i, j) \in \mu$ we say buyer i is unmatched by μ and similarly for sellers. The set of all matchings from M to M' is represented by $\mathscr{M}(M, M')$. A matching $\mu \in \mathscr{M}(M, M')$ is *optimal* for (M, M', A) if $\sum_{(i,j)\in\mu} a_{ij} \geq \sum_{(i,j)\in\mu'} a_{ij}$ for any $\mu' \in \mathscr{M}(M, M')$. We denote by $\mathscr{M}_A^*(M, M')$ the set of all optimal matchings.

Shapley and Shubik (1972) associate any assignment market with a game in coalitional form $(M \cup M', w_A)$ called the *assignment game* in which the worth of a coalition $S \cup T \subseteq M \cup M'$ with $S \subseteq M$ and $T \subseteq M'$ is $w_A(S \cup T) = \max\limits_{\mu\in\mathscr{M}(S,T)} \sum_{(i,j)\in\mu} a_{ij}$, and any coalition formed only by buyers or sellers has a worth of zero.

The main goal is to allocate the total worth among the agents, and a prominent solution for cooperative games is the core. Shapley and Shubik (1972) prove that the core of the assignment game is always nonempty. Given an optimal matching $\mu \in \mathscr{M}_A^*(M, M')$, the *core* of the assignment game, $C(w_A)$, can be easily described as the set of nonnegative payoff vectors $(x,y) \in \mathbb{R}_+^M \times \mathbb{R}_+^{M'}$ satisfying

$$x_i + y_j = a_{ij} \text{ for all } (i,j) \in \mu, \tag{1}$$

$$x_i + y_j \geq a_{ij} \text{ for all } (i,j) \in M \times M', \tag{2}$$

and all agents unmatched by μ get a null payoff.

Now we define the nucleolus (Schmeidler, 1969) of an assignment game, taking into account that its core is always nonempty. The excess of a coalition $\emptyset \neq R \subseteq M \cup M'$ with respect to an allocation in the core, $(x,y) \in C(w_A)$, is defined as $e(R,(x,y)) := w_A(R) - \sum_{i \in R \cap M} x_i - \sum_{j \in R \cap M'} y_j$. By the bilateral nature of the market, it is known that the only coalitions that matter are the individual and mixed-pair ones (Núñez, 2004). Given an allocation $(x,y) \in C(w_A)$, define the excess vector $\theta(x,y) = (\theta_k)_{k=1,\ldots,r}$ as the vector of individual and mixed-pair coalitions excesses arranged in a non-increasing order, i.e. $\theta_1 \geq \theta_2 \geq \ldots \geq \theta_r$. Then the *nucleolus* of the game $(M \cup M', w_A)$ is the unique core allocation $\nu(w_A) \in C(w_A)$ which minimizes $\theta(x,y)$ with respect to the lexicographic order[3] over the whole set of core allocations. For ease of notation we will use, for $A \in \mathrm{M}^+_{m \times m'}$, $\nu(A)$ instead of $\nu(w_A)$ if no confusion arises.

We use the characterization of the nucleolus of a square assignment game of Llerena and Núñez (2011), see also Llerena et al. (2015). To introduce this characterization we define the maximum transfer from a coalition to another coalition.

Given any square assignment game $(M \cup M', w_A)$, and two arbitrary coalitions

[3] The *lexicographic order* $\geq_{lex}$ on $\mathbb{R}^d$ is defined in the following way: $x \geq_{lex} y$, where $x,y \in \mathbb{R}^d$, if $x = y$ or if there exists $1 \leq t \leq d$ such that $x_k = y_k$ for all $1 \leq k < t$ and $x_t > y_t$.

$\emptyset \neq S \subseteq M$ and $\emptyset \neq T \subseteq M'$ we define

$$\delta^A_{S,T}(x,y) := \min_{i \in S, j \in M' \setminus T} \left\{ x_i, x_i + y_j - a_{ij} \right\},$$

$$\delta^A_{T,S}(x,y) := \min_{j \in T, i \in M \setminus S} \left\{ y_j, x_i + y_j - a_{ij} \right\},$$

for any core allocation $(x,y) \in C(w_A)$.

Llerena and Núñez (2011) gives a geometric characterization of the nucleolus of a square assignment game. They prove that the nucleolus of a square assignment game is characterized as the unique core allocation $(x,y) \in C(w_A)$ such that

$$\delta^A_{S,T}(x,y) = \delta^A_{T,S}(x,y) \tag{3}$$

for any $\emptyset \neq S \subseteq M$ and $\emptyset \neq T \subseteq M'$ with $|S| = |T|$. In certain cases, the number of equalities can be reduced. Indeed, note that if $T \neq \mu(S)$ for some $\mu \in \mathscr{M}^*_A(M,M')$, then it holds $\delta^A_{S,T}(x,y) = \delta^A_{T,S}(x,y) = 0$. Therefore, for this characterization we only have to check (3) for the cases $T = \mu(S)$ for some optimal matching $\mu \in \mathscr{M}^*_A(M,M')$ and any $\emptyset \neq S \subseteq M$, i.e.

$$\delta^A_{S,\mu(S)}(x,y) = \delta^A_{\mu(S),S}(x,y), \text{ for any } \emptyset \neq S \subseteq M. \tag{4}$$

To analyze the non-square case we can use two different approaches and we will apply any of them.

The first and classical approach consists in adding null rows or columns in order to make the initial matrix square. The added rows or columns correspond to dummy agents and they receive a null payoff at any core allocation and hence also in the nucleolus. At this extended square assignment matrix we apply the previous geometric characterization. Notice that the number of coalitions to be checked grows quickly for each added agent.

To fix our first approach we introduce some notation. Given any arbitrary assignment matrix $A \in \mathrm{M}^+_{m \times m'}$, with $m < m'$ and where $\mu = \{(1,1),(2,2),\ldots,(m,m)\}$

is an optimal matching for A, we define the following square matrix $A^0 \in M^+_{m'}$ obtained from the original matrix A by adding $m'-m$ zero rows, that is $m'-m$ dummy players. Let $M^0 = M \cup \{m+1,\ldots,m'\}$ be the new set of buyers and $A^0 = \left(a^0_{ij}\right)_{1\leq i,j\leq m'}$ where

$$a^0_{ij} = \begin{cases} a_{ij} & \text{if} \quad (i,j) \in M \times M', \\ 0 & \text{if} \quad (i,j) \in (M^0 \backslash M) \times M'. \end{cases} \tag{5}$$

We know that the matching $\mu^0 = \{(1,1),(2,2),\ldots,(m',m')\}$ is optimal for matrix A^0.

The second approach keeps the dimension of the problem as low as we can and it has an interest on its own. Basically it consists in reducing the assignment problem to an appropriate square matrix, dropping out those agents unassigned by an optimal matching, and reassessing the matrix entries. Apart from the dimension issue, the main feature of this approach is that we must not care about the added zero rows or columns when we deal with the matrix.

To introduce the second approach we need some notations. Let $(M,M',A)\,, A \in \mathrm{M}^+_{m\times m'}$ be a non-square assignment market with $m<m'$ and let $\mu \in \mathscr{M}^*_A(M,M')$ be an optimal matching. Define the vector $a^\mu = \left(a^\mu_i\right)_{i\in M} \in \mathbb{R}^M_+$ by

$$a^\mu_i := \max_{j\in M'\backslash\mu(M)} \left\{a_{ij}\right\} \text{ for each buyer } i \in M, \tag{6}$$

and define the square matrix $A^\mu \in \mathrm{M}^+_m$ by

$$a^\mu_{ij} := \max\left\{0, a_{ij} - a^\mu_i\right\}, \text{ for } (i,j) \in M \times \mu(M). \tag{7}$$

Then the relationship between their nucleolus is the following one:

$$\nu_i(A) = \nu_i(A^\mu) + a^\mu_i, \text{ for } i \in M, \tag{8}$$

$$\nu_j(A) = \begin{cases} \nu_j(A^\mu) & \text{for } j \in \mu(M), \text{ and} \\ 0 & \text{for } j \in M'\backslash\mu(M). \end{cases} \tag{9}$$

Moreover the fixed matching μ is also optimal for matrix A^{μ}. A proof of these facts is included in the Appendix.

3. A numerical example

The nucleolus of an assignment game is a geometrical half-way point of a nonempty compact polyhedron, its core. From its description we can conceive its invariance from synchronizing displacements of the core "walls". In other words, the effects of raising or lowering some appropriate entries of the assignment matrix do not change the nucleolus. Our main objective is to analyze the family of assignment matrices that give rise to the same nucleolus. We illustrate our purpose by a 2×2 numerical example.

Example 3.1. *Consider the following assignment matrix*

$$A = \begin{pmatrix} 8 & 6 \\ 6 & 4 \end{pmatrix}.$$

Notice that it has two optimal matchings and $w_A(M\cup M') = 12$*. To draw its core we fix an optimal matching. Let us take* $\mu_1 = \{(1,1),(2,2)\}$ *and we depict the projection of the core on the buyers' coordinates (see Figure 2). The core is given by the segment* A_1A_2*. Its nucleolus is* $\nu(A) = (4,2;4,2)$*, since it is its midpoint.*

Just by looking the geometric interpretation of the nucleolus N*, it is easy to see that matrices* $A_t = \begin{pmatrix} 8 & 6-t \\ 6-t & 4 \end{pmatrix}$ *for* $0\leq t\leq 4$ *share the same nucleolus* $\nu = (4,2;4,2)$ *since the "distances" to the walls of the core are equal:* $\overline{A'N} = \overline{NB'}, \overline{C'N} = \overline{ND'}$ *and* $\overline{A_1N} = \overline{NA_2}$*.*

After $t=4$ *the walls can be moved independently, which adds matrices* $A = \begin{pmatrix} 8 & a_{12} \\ a_{21} & 4 \end{pmatrix}$*. for* $a_{12}, a_{21} \in [0,2]$ *to the family of matrices with the same nucleolus we are dealing with.*

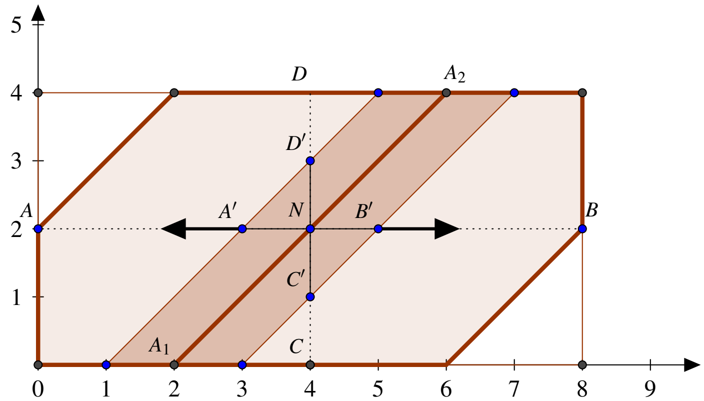

Figure 2: Several cores of matrices $A_t = \left(\begin{smallmatrix} 8 & 6-t \\ 6-t & 4 \end{smallmatrix}\right)$ for $0 \le t \le 4$ with the same nucleolus, $(4,2;4,2)$.

We have just described the family of matrices with the same nucleolus $\nu(A)$ when we fix matching μ_1. See $L_1 \cup L_2$ in Figure 3.

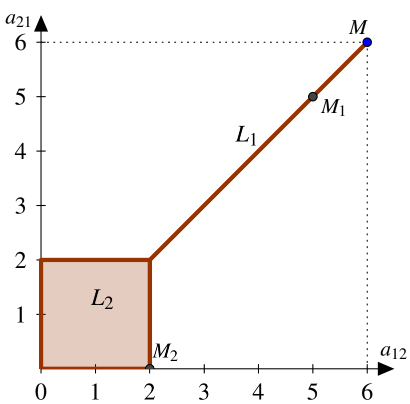

Figure 3: Matrices $\begin{pmatrix} 8 & a_{12} \\ a_{21} & 4 \end{pmatrix}$ with the same nucleolus $(4,2;4,2)$.

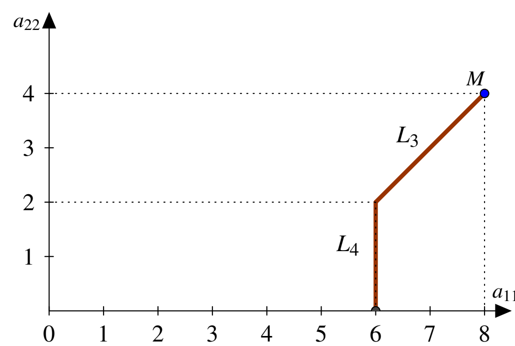

Figure 4: Matrices $\begin{pmatrix} a_{11} & 6 \\ 6 & a_{22} \end{pmatrix}$ with the same nucleolus $(4,2;4,2)$.

To obtain the whole family of matrices with nucleolus $\nu(A)$ we have to repeat the above argument fixing in matrix A the optimal matching $\mu_2 = \{(1,2),(2,1)\}$. This process adds matrices $L_3 \cup L_4$ in Figure 4. Notice that M represents the same *matrix A in Figures 3 and 4. Matrices B and C from the Introduction correspond to points M_1 and M_2 in Figure 3.*

From the previous Example 3.1 we can discuss the structure of the family which will be analyzed in the next sections. Firstly, this family of assignment matrices with the same nucleolus is composed of branches that share a unique maximal element, its maximum, which is matrix A. This family is not a convex set but path-connected. Finally in this case there is no minimum, but two minimal elements.

4. Assignment games with the same nucleolus

We introduce the family of matrices with a given nucleolus. To this end, for an arbitrary assignment matrix $A \in \mathrm{M}^+_{m\times m'}$ we denote by

$$[A]_\nu := \left\{ B \in M^+_{m\times m'} \mid \nu(B) = \nu(A) \right\}$$

the family of matrices that share the same nucleolus than A.

It is clear that matrices with the same nucleolus must have the same worth for the grand coalition even if they do not have any optimal matching in common, see Example 3.1.

We focus now on the structure of this family: it is a nonempty compact join-semilattice[4] with a unique maximal element. Secondly we characterize this maximum and show it is a specific type of assignment matrix, a valuation matrix.

Theorem 4.1. *Let $A \in \mathrm{M}^+_{m\times m'}$ be an assignment matrix. The family $[A]_\nu$ forms a compact join-semilattice with a unique maximal element.*

[4] A family $\mathscr{F} \subseteq \mathrm{M}^+_{m\times m'}$ is a join-semilattice if $A \vee B \in \mathscr{F}$ for all $A, B \in \mathscr{F}$.

Proof. First we prove that this family is a join-semilattice. Let $B, B' \in [A]_\nu$. If $m \neq m'$, we add zero rows or columns to make the matrices square, recall (5). It is known that these rows or columns correspond to dummy players which obtain zero payoff at any core allocation, and also in the nucleolus. Therefore we can assume from now on that matrices are square. We have $B, B' \leq B \vee B'$, and also $C(w_B) \cap C(w_{B'}) \neq \emptyset$, since both games share the nucleolus. We claim

$$C(w_B) \cap C(w_{B'}) = C(w_{B \vee B'}).$$

To see it, take any $(x,y) \in C(w_B) \cap C(w_{B'})$. It is clear $x_i + y_j \geq \max\{b_{ij}, b'_{ij}\}$ for all $(i,j) \in M \times M'$. Then for any optimal matching μ of matrix $B \vee B'$ we have

$$w_{B \vee B'}(M \cup M') = \sum_{(i,j) \in \mu} \max\{b_{ij}, b'_{ij}\} \leq \sum_{(i,j) \in \mu} [x_i + y_j] = w_B(M \cup M') = w_{B'}(M \cup M').$$

As a consequence $w_{B \vee B'}(M \cup M') = w_B(M \cup M') = w_{B'}(M \cup M')$. Now it is easy to see $(x,y) \in C(w_{B \vee B'})$. The other inclusion is straightforward.

Now to see $\nu(B) = \nu(B') = (x,y)$ is the nucleolus of $w_{B \vee B'}$, just note that, for all $\emptyset \neq S \subseteq M$ and $\emptyset \neq T \subseteq M'$ with $|S| = |T|$,

$$\delta_{S,T}^{B \vee B'}(x,y) = \min\left\{ \delta_{S,T}^{B}(x,y), \delta_{S,T}^{B'}(x,y) \right\}, \text{ and}$$
$$\delta_{T,S}^{B \vee B'}(x,y) = \min\left\{ \delta_{T,S}^{B}(x,y), \delta_{T,S}^{B'}(x,y) \right\}.$$

As a consequence, since (x,y) is the nucleolus of w_B and $w_{B'}$, we obtain the equality $\delta_{S,T}^{B \vee B'}(x,y) = \delta_{T,S}^{B \vee B'}(x,y)$, proving that $B \vee B' \in [A]_\nu$.

Now we show that this family is a compact set, and therefore with a unique maximal element. We show that it is bounded and closed. It is bounded since $0 \leq b_{ij} \leq x_i + y_j$ for all $(i,j) \in M \times M'$ and $B \in [A]_\nu$ with $\nu(A) = (x,y)$. It is closed because the functions $\delta_{S,T}^{B}(x,y)$ and $\delta_{T,S}^{B}(x,y)$ are continuous in $B \in \mathrm{M}^{+}_{m \times m'}$ for all $\emptyset \neq S \subseteq M$, $\emptyset \neq T \subseteq M'$ and $|S| = |T|$, and they must satisfy equalities (3). □

In contrast with the previous result, the minimum defined entry-wise of two matrices with the same nucleolus may not have the same nucleolus, see Example 3.1.

Now we introduce a kind of assignment matrices, useful for our purposes. A matrix $A \in \mathrm{M}^{+}_{m\times m'}$ is a *valuation matrix*[5] if for any $i_1, i_2 \in \{1,\dots,m\}$ and $j_1, j_2 \in \{1,\dots,m'\}$ we have $a_{i_1 j_1} + a_{i_2 j_2} = a_{i_1 j_2} + a_{i_2 j_1}$. Clearly this definition is equivalent to see that any 2×2 submatrix has two optimal matchings.

Obviously, any fully-optimal[6] square matrix is a valuation matrix, and for square matrices the converse also holds. This characterization fails for non-square matrices as the following matrix shows:

$$D = \left(\begin{array}{ccc|cc} 3 & 6 & 8 & 1 & 0 \\ 4 & 7 & 9 & 2 & 1 \\ 6 & 9 & 11 & 4 & 3 \end{array}\right). \tag{10}$$

This is a valuation matrix, but clearly not all matchings are optimal.

Finally we want to point out two general properties for non-square valuation matrices. Let $A \in \mathrm{M}^{+}_{m\times m'}$ be an non-square valuation matrix with $m < m'$ and $\mu \in \mathscr{M}^*_A(M,M')$ any optimal matching, Then:

(i) The square submatrix $A_{M\times\mu(M)}$ is fully-optimal. Its worth is $w_A(M\cup M')$.

(ii) The entries of matrix A satisfy $a_{ij_1} \geq a_{ij_2}$ for all $i \in M$, $j_1 \in \mu(M)$ and $j_2 \in M' \setminus \mu(M)$.

Theorem 4.2. *Let $A \in \mathrm{M}^{+}_{m\times m'}$ be an assignment matrix. The maximal element of the family $[A]_\nu$ is a valuation matrix. In the square case, $m = m'$, the maximal element is the unique valuation matrix of the family.*

[5] Following Topkis (1998), a function is a valuation if it is submodular and supermodular.

[6] $A \in \mathrm{M}^{+}_{m\times m'}$ is a fully-optimal matrix if all matchings are optimal, i.e. $\mathscr{M}^*_A(M,M') = \mathscr{M}(M,M')$

Proof. Let $\nu(A) = (x,y) \in \mathbb{R}_+^M \times \mathbb{R}_+^{M'}$ be the nucleolus of matrix $A \in \mathrm{M}^+_{m\times m'}$, where we assume without loss of generality that $m \le m'$ and $\mu = \{(1,1),(2,2),\ldots,(m,m)\}$ is an optimal matching for A.

Define now matrix $\overline{A} \in \mathrm{M}^+_{m\times m'}$ as follows

$$\overline{a}_{ij} = \begin{cases} x_i + y_j, & \text{for } 1 \le i,j \le m, \\ x_i - \min\limits_{j=1,\ldots,m} \{y_j\} & \text{otherwise.} \end{cases}$$

We claim:

(i) $\overline{a}_{ij} \ge 0$, for all $(i,j) \in M \times M'$,

(ii) $\overline{A} \in [A]_\nu$, i.e. $\nu(\overline{A}) = \nu(A)$,

(iii) $\overline{A}$ is the maximum of the family $[A]_\nu$, and clearly a valuation matrix.

To prove claim (i), let A^0, M^0, and μ^0 the notation introduced in (5) to make square the initial matrix A. We denote by $\left(x^0,y^0\right) \in \mathbb{R}_+^{M^0} \times \mathbb{R}_+^{M'}$ the vector defined by $x_k^0 = x_k$ if $k \in M$ and $x_k^0 = 0$ if $k \in M^0 \backslash M$ and $y_k^0 = y_k$ if $k \in M'$.

We know $\nu\left(A^0\right) = \left(x^0,y^0\right)$ and then $\delta^{A^0}_{M,\mu^0(M)}(x^0,y^0) = \delta^{A^0}_{\mu^0(M),M}(x^0,y^0)$, but

$$\delta^{A^0}_{\mu^0(M),M}(x^0,y^0) = \min_{i\in M^0\backslash M,\, j\in\mu(M)} \{y_j^0, x_i^0 + y_j^0 - a_{ij}^0\} = \min_{j\in\mu(M)} \{y_j\}, \text{ and}$$

$$\delta^{A^0}_{M,\mu^0(M)}(x^0,y^0) = \min_{i\in M,\, j\in M'\backslash\mu(M)} \{x_i^0, x_i^0 + y_j^0 - a_{ij}^0\} =$$

$$= \begin{cases} \min\limits_{i\in M}\{x_i\} & \text{if } m = m', \\ \min\limits_{i\in M,\, j\in M'\backslash\mu(M)} \{x_i - a_{ij}\} & \text{if } m < m'. \end{cases}$$

where we have used that $y_j^0 = y_j = 0$ for $j \in M' \setminus \mu(M)$ and $x_i^0 = x_i \ge x_i - a_{ij}$ for $i \in M$ and $j \in M' \setminus \mu(M)$.

From the above equality we easily deduce $x_i \ge \min\limits_{j\in\mu(M)} \{y_j\}$ for $i \in M$ which proves our first claim.

We prove claim (ii), $\nu(\overline{A}) = (x,y)$, by proving its equivalent form, $\nu((\overline{A})^0) = (x^0,y^0)$. Notice that $\mu^0 = \{(1,1),(2,2),\ldots,(m',m')\}$ is optimal for matrix $(\overline{A})^0$. To prove $\nu((\overline{A})^0) = (x^0,y^0)$, we distinguish several cases depending on an arbitrary coalition $S \subseteq M^0, S \neq \emptyset$:

Case 1: $S \cap (M^0 \setminus M) \neq \emptyset$. We obtain $\delta^{(\overline{A})^0}_{S,\mu^0(S)}(x^0,y^0) = 0$ and $\delta^{(\overline{A})^0}_{\mu^0(S),S}(x^0,y^0) = 0$, since $x^0_i = 0$ for all $i \in S \cap (M^0 \setminus M)$ for the first equality and there exists $j \in \mu^0(S) \cap (M' \setminus \mu(M))$, which implies $y_j = 0$ for the second.

Case 2: $S \subseteq M, S \neq M$. We obtain $\delta^{(\overline{A})^0}_{S,\mu^0(S)}(x^0,y^0) = 0$, since there exists $j \in \mu(M) \setminus \mu(S)$ and then $x_i + y_j = \overline{a}_{ij}$ for all $i \in S$. Similarly $\delta^{(\overline{A})^0}_{\mu^0(S),S}(x^0,y^0) = 0$.

Case 3: $S = M$. We have

$$\delta^{(\overline{A})^0}_{\mu^0(M),M}(x^0,y^0) = \min_{i\in M^0\setminus M, j\in\mu(M)}\{y^0_j, x^0_i + y^0_j - \overline{a}^0_{ij}\} = \min_{j\in\mu(M)}\{y_j\}, \text{ and}$$

$$\delta^{(\overline{A})^0}_{M,\mu^0(M)}(x^0,y^0) = \min_{i\in M, j\in M'\setminus\mu(M)}\{x^0_i, x^0_i + y^0_j - \overline{a}^0_{ij}\} =$$

$$= \begin{cases} \min\limits_{i\in M}\{x_i\}, \text{ if } m = m', \\ \min\limits_{i\in M, j\in M'\setminus\mu(M)}\{x_i - \overline{a}_{ij}\} = \min\limits_{i\in M}\{x_i - (x_i - \min\limits_{j\in\mu(M)}\{y_j\})\} = \min\limits_{j\in\mu(M)}\{y_j\}, \text{ if } m < m'. \end{cases}$$

Now, for $m < m'$ they trivially coincide and for $m = m'$, the square case, they coincide since $\nu(A) = (x,y)$, and then $\delta^A_{M,\mu(M)}(x,y) = \min_{i\in M}\{x_i\} = \delta^A_{\mu(M),M}(x,y) = \min_{j\in\mu(M)}\{y_j\}$.

Therefore, we have proved the second claim.

To prove claim (iii), let $B \in [A]_\nu$ be an arbitrary matrix of the family. We can assume that $\mu = \{(1,1),(2,2),\ldots,(m,m)\}$ is optimal for matrix B, since in other case, we consider matrix $B \vee A$ as a new matrix B, see Theorem 4.1 and notice $B \leq B \vee A$. Recall $\nu(B) = \nu(A) = (x,y)$. Clearly $\overline{a}_{ij} = x_i + y_j \geq b_{ij}$ for $1 \leq i,j \leq m$.

If $m = m'$ we are done, and $B \leq \overline{A}$. Otherwise, $m < m'$. Consider matrix B^0, see

(5). We know $\nu(B^0) = (x^0, y^0)$. Then

$$\delta^{B^0}_{\mu^0(M),M}(x^0,y^0) = \min_{i\in M^0\setminus M,\, j\in\mu(M)} \{y^0_j, x^0_i + y^0_j - b^0_{ij}\} = \min_{j\in\mu(M)} \{y_j\}, \quad \text{and}$$

$$\delta^{B^0}_{M,\mu^0(M)}(x^0,y^0) = \min_{i\in M,\, j\in M'\setminus\mu(M)} \{x^0_i, x^0_i + y^0_j - b^0_{ij}\} = \min_{i\in M,\, j\in M'\setminus\mu(M)} \{x_i - b_{ij}\}.$$

We obtain for all $i \in M$ and $j \in M' \setminus \mu(M)$, $x_i - b_{ij} \geq \min_{i\in M,\, j\in M'\setminus\mu(M)} \{x_i - b_{ij}\} = \min_{j\in\mu(M)} \{y_j\}$, or equivalently

$$\overline{a}_{ij} = x_i - \min_{j\in\mu(M)} \{y_j\} \geq b_{ij} \quad \text{for all } i \in M, j \in M' \setminus \mu(M).$$

This ends our third claim, and proves the maximality of matrix $\overline{A}$ since we have seen that in the non-square case, $B \leq \overline{A}$.

The fact that $\overline{A}$ is a valuation matrix is left to the reader. Moreover in the square case any valuation matrix of the family (with the same nucleolus) is fully-optimal, and then it must coincide with matrix $\overline{A}$. □

From the statement of Theorem 4.2 we expect several valuation matrices if the initial assignment matrix is not square. In (10) we have introduced matrix $D \in M^+_{3\times 5}$ which is an example of such a situation. By (4), (8) and (9) it is easy to check that the nucleolus of matrix D is $\nu(D) = (2,3,5;1,4,6,0,0)$ and the maximum matrix of $[D]_\nu$ is given by the valuation matrix

$$\overline{D} = \left(\begin{array}{ccc|cc} 3 & 6 & 8 & 1 & 1 \\ 4 & 7 & 9 & 2 & 2 \\ 6 & 9 & 11 & 4 & 4 \end{array}\right),$$

which is strictly greater than the valuation matrix D. Both valuation matrices share the same nucleolus.

In the proof of Theorem 4.2 we have found the expression of the maximum element of family $[A]_\nu$, with $\nu(A) = (x,y)$. It is matrix $\overline{A} \in \mathrm{M}^+_{m\times m'}$ as follows

$$\overline{a}_{ij} = \begin{cases} x_i + y_j, & \text{for } (i,j) \in M \times \mu(M), \\ x_i - \min\limits_{j\in\mu(M)} \{y_j\} & \text{for } (i,j) \in M \times (M' \setminus \mu(M)), \end{cases} \tag{11}$$

where $\mu \in \mathscr{M}^*_A(M,M')$ is an optimal matching. A close look at (11) could raise expectations of different maximum matrices $\overline{A}$ depending on the chosen optimal matching μ, but this is not the case, as the reader can check.

5. The 2×2 case

As an application of the above results we reveal how to describe the whole family $[A]_\nu$, when we deal with 2×2 assignment matrices.

Consider an arbitrary 2×2 assignment market represented by matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

From now on and without loss of generality, we assume the following normalization conditions:

$$\begin{aligned} & a_{11} + a_{22} \geq a_{12} + a_{21}, \\ & a_{11} \geq a_{22}, \quad a_{12} \geq a_{21}. \end{aligned} \tag{12}$$

These conditions mean that the main diagonal of matrix A is an optimal matching and it is sorted from highest to lowest. Sectors are interchangeable so that entries of matrix A outside the main diagonal are ordered, following (12).

We assume that matrix entries a_{11} and a_{22} are fixed, and depict in Figure 5 any arbitrary matrix A satisfying (12), depending on matrix entries a_{12} and a_{21}. Notice that conditions (12) force the range of variables a_{12} and a_{21} to belong to the triangle with vertices A, B and C in Figure 5.

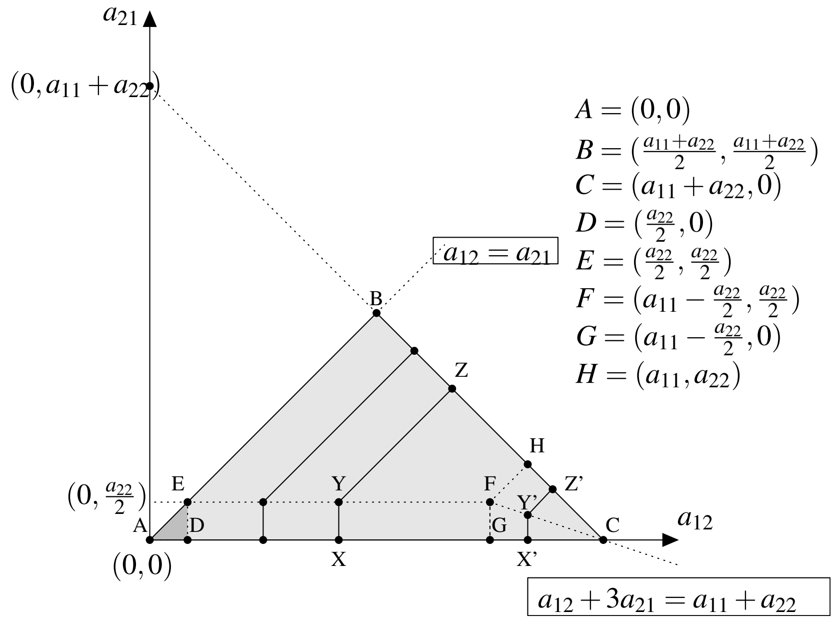

Figure 5: The regions for families with the same nucleolus

Now whenever we have a matrix A with the normalization conditions (12) we depict it in the above Figure 5. In this way we easily describe the family $[A]_v$. The analysis is divided in three regions.

Region 1: Matrix A belongs to triangle ADE or segment EB. All these matrices share the same nucleolus, precisely the equal division between optimal matched pairs. This region corresponds to the symmetric case, $a_{12}=a_{21}$, and matrices where a_{12} and a_{21} are "small" with respect the optimal entries: $a_{21}\leq a_{12}\leq \frac{a_{22}}{2}$. Notice because of the normalization conditions, we have a symmetric region outside triangle ABC with the same nucleolus. Therefore the family is composed of a square and a segment, see e.g. Figure 3. Outside this region the family is composed of two segments.

Region 2: Matrix A belongs to the region limited by D,E,B,H,F,G. In this case, a generic family is given in Figure 5 by segments $[X,Y] \cup [Y,Z]$. One of them, the vertical one $[X,Y]$, increases entry a_{21} from zero up to $\frac{a_{22}}{2}$ and the other $[Y,Z]$ has a 45° slope.

Region 3: Matrix A belongs to the region limited by G,F,H,C. In this case, a generic family is given in Figure 5 by segments $[X',Y'] \cup [Y',Z']$. One of them, the vertical one $[X',Y']$, increases smaller entry a_{21} from zero up to the straight line FC with equation $a_{12} + 3a_{21} = a_{11} + a_{22}$ and the other $[Y',Z']$ has a 45° slope.

Matrices given by points Z or Z', on the segment $[B,C]$ in Figure 5, correspond with the unique valuation matrix $\overline{A}$ of each family $[A]_\nu$. Recall that in general there are two branches of each family, depending on the chosen optimal matching of $\overline{A}$.

In Table 1 we show buyers' coordinates of the nucleolus of an assignment matrix satisfying the normalization conditions (12). In it we denote by d^A the difference between the main diagonal and the secondary diagonal, i.e. $d^A = a_{11} + a_{22} - a_{12} - a_{21}$. Recall that the whole nucleolus $\nu(A) = (u_1^*, u_2^*; v_1^*, v_2^*)$ is obtained by $v_i^* = a_{ii} - u_i^*$ for $i = 1,2$. A proof of the facts given in Table 1 can be checked in Martínez-de-Albéniz et al. (2013a), computed under the previous normalization conditions (12).

The above formulas for the nucleolus allow us to obtain the valuation matrix of the family, given by (11). Then once reached matrix $\overline{A}$ to describe the whole family $[A]_\nu$ we have to repeat the analysis rearranging conveniently the entries of $\overline{A}$, now for the other optimal matching, given by the secondary diagonal.

6. About the cardinality of the family

The family of matrices with the same nucleolus is not in general a convex set. To see it just consider appropriate matrices of Example 3.1 and their midpoint, but as the reader must suspect, there is a path linking any two matrices of the family,

Table 1: Nucleolus formulas of an arbitrary 2×2 assignment matrix satisfying normalization conditions (12)

Region			$\mathbf{u_1^*}$	$\mathbf{u_2^*}$
$A\ D\ E$	$a_{21} \le \min\left\{\frac{a_{22}}{2}, \frac{d^A}{2}\right\}$	$a_{12} \le \frac{a_{22}}{2}$	$\frac{a_{11}}{2}$	$\frac{a_{22}}{2}$
$D\ G\ F\ E$		$\frac{a_{22}}{2} < a_{12} \le a_{11} - \frac{a_{22}}{2}$	$\frac{a_{11}}{2} + \frac{a_{12}}{2} - \frac{a_{22}}{4}$	$\frac{a_{22}}{2}$
$G\ C\ F$		$a_{11} - \frac{a_{22}}{2} < a_{12}$	$a_{11} - \frac{a_{21}}{3} - \frac{d^A}{3}$	$\frac{a_{21}}{3} + \frac{d^A}{3}$
$E\ F\ H\ B$	$a_{21} > \min\left\{\frac{a_{22}}{2}, \frac{d^A}{2}\right\}$	$a_{21} \ge a_{12} + a_{22} - a_{11}$	$\frac{a_{11}}{2} + \frac{a_{12}}{2} - \frac{a_{21}}{2}$	$\frac{a_{22}}{2}$
$F\ C\ H$		$a_{21} < a_{12} + a_{22} - a_{11}$	$a_{11} - \frac{a_{21}}{2} - \frac{d^A}{4}$	$\frac{a_{21}}{2} + \frac{d^A}{4}$

maybe passing through its maximum.

Now we prove an interesting property. There is a continuous piecewise linear path (maybe not unique) between any matrix in $[A]_v$ and its maximum element $\overline{A}$. From here it is clear that the family $[A]_v$ is a path-connected set.

Theorem 6.1. *Let $A \in \mathrm{M}^+_{m\times m'}$ be an assignment matrix, and $\overline{A} \in [A]_v$ the maximal element of the family. Then for any $B \in [A]_v$ there exists an increasing piecewise linear path[7] from B to $\overline{A}$ inside $[A]_v$. As a consequence, the family $[A]_v$ is a path-connected set. In particular, for any $B \in [A]_v, B \neq \overline{A}$, there exists $C \in [A]_v$ with*

[7] A path in $\mathscr{X} \subseteq M^+_{m\times m'}$ from A to B, $A, B \in \mathscr{X}$, is a continuous function f from the unit interval $I = [0,1]$ to $\mathscr{X}$, i.e. $f : [0,1] \to \mathscr{X}$, with $f(0) = A$ and $f(1) = B$. Moreover a subset $\mathscr{X} \subseteq M^+_{m\times m'}$ is path-connected if for any two elements $A, B \in \mathscr{X}$ there exists a path from A to B entirely contained in $\mathscr{X}$.

$B < C < \overline{A}$.[8]

Proof. First we analyze the square case, $m = m'$. We can assume $|M| = |M'| \geq 2$. Let it be $B \in [A]_\nu$, and $\nu(A) = \nu(B) = (x, y) \in \mathbb{R}_+^M \times \mathbb{R}_+^{M'}$. Let us define the set formed by the distances that appear in the geometric characterization of the nucleolus, see (3), except for the grand coalition,

$$\Delta(B) = \left\{ \delta_{S,T}^B(x,y) \,|\, S \subseteq M, T \subseteq M', |S| = |T|, S \neq \emptyset, M, \text{ and } T \neq \emptyset, M' \right\}.$$

These elements are used for the characterization of the nucleolus and correspond to the minimum of some numbers. The elements of $\Delta(B)$ can be ordered increasingly:

$$0 = \delta_0^B < \delta_1^B < \ldots < \delta_{r_*}^B,$$

and then $\Delta(B) = \{\delta_0^B, \delta_1^B, \ldots, \delta_{r_*}^B\}$.

From these parameters we can define a new matrix B^0 with the same nucleolus. We set $b_{ij}^0 = b_{ij}$ if $x_i + y_j - b_{ij} \in \Delta(B)$, and we raise the worth of entry b_{ij} to b_{ij}^0 in such a way that $x_i + y_j - b_{ij}^0$ equals the closest one-below element of $\Delta(B)$, that is, if $\delta_k^B < x_i + y_j - b_{ij} < \delta_{k+1}^B$ for some k, then $b_{ij}^0 = x_i + y_j - \delta_k^B$.

It is clear that matrix B^0 has the same nucleolus as matrix B since the equalities of the geometric characterization of the nucleolus haven't changed and therefore $B^0 \in [A]_\nu$. Moreover $\Delta(B) = \Delta(B^0)$. We may choose increasing linear paths from B to B^0, one for each entry to raise. Notice that since we are moving up the entries that do not determine the distances of $\Delta(B)$, all matrices on these paths will preserve the original nucleolus.

Now we have a matrix $B^0 \in [A]_\nu$ such that $x_i + y_j - b_{ij}^0 \in \Delta(B^0)$ for all $(i, j) \in M \times M'$. Moreover if $\delta_{S,T}^{B^0}(x,y) = \delta_{r_*}^B$, for some $S \subset M$ and $T \subset M'$ with $|S| = |T| \neq$

[8] If $B, C \in \mathrm{M}_{m\times m'}^+$, $B < C$ if and only if $B \leq C$ and $B \neq C$.

m, then we have, for all $i \in S$ and $j \notin T$, $x_i + y_j - b^0_{ij} = \delta^B_{r_*}$. We finish the proof in the square case by raising the entries of matrix B^0 iteratively up to get $\overline{A}$.

Firstly, notice that if $r_* = 0$, that is $\Delta(B) = \Delta(B^0) = \{0\}$. Then matrix B^0 coincides with the valuation matrix of the family $\overline{A}$ since then $x_i + y_j = b^0_j$ for all $(i,j) \in M \times M'$, see (11) and recall $m = m'$.

Otherwise, $r_* > 0$. In this case, for all $(i,j) \in M \times M'$ such that $x_i + y_j - b^0_{ij} = \delta^{B^0}_{r_*}$ raise linearly and simultaneously b^0_{ij} to b^1_{ij} defined by the equality $x_i + y_j - b^1_{ij} = \delta^{B^0}_{r_*-1}$. We obtain a new matrix $B^1 \in [A]_\nu$, defined for all $i \in M$ and $j \in M'$ by

$$b^1_{ij} = \begin{cases} x_i + y_j - \delta^{B^0}_{r_*-1} & \text{if} \quad x_i + y_j - b^0_{ij} = \delta^{B^0}_{r_*}, \\ b^0_{ij} & \text{otherwise.} \end{cases}$$

It is easy to see that $\Delta(B^1) \subseteq \Delta(B^0)$, and $\Delta(B^1) \neq \Delta(B^0)$. This means we have reduced the set of distances related with the nucleolus. Once again by (3) it is easy to see that $\nu(B^1) = (x,y)$ or equivalently $B^1 \in [A]_\nu$.

Now, in a finite number of steps, proceed sequentially raising all entries until for all $(i,j) \in M \times M'$ we have $x_i + y_j - b^{r_*}_{ij} = 0$. That is, matrix B^{r_*} coincides with matrix $\overline{A}$ for the square case. In it all matchings are optimal.

For the non-square case, we assume $|M| < |M'|$. Let $B \in [A]_\nu$, and let $\mu \in \mathcal{M}^*_B(M,M')$ be an optimal matching.

Notice first that matrix B can be modified without changing its nucleolus in the following way:

(i) for all $(i,j) \in M \times \mu(M)$ if $b_{ij} < b^\mu_i$ then raise these entries to $b^\mu_i = \max\limits_{j \in M' \setminus \mu(M)} \{b_{ij}\}$, see (6);

(ii) for all $(i,j) \in M \times (M' \setminus \mu(M))$ raise entries b_{ij} to b^μ_i, and we do not modify the rest of entries.

This new matrix, denoted by $\widetilde{B}$ has the same nucleolus and then $\widetilde{B} \in [A]_\nu$.

Indeed, matrix $\widetilde{B}$ has also μ as an optimal matching and then by definition it has the same same square matrix $B^{\mu} \in M_m^+$, i.e. $(\widetilde{B})^{\mu} = B^{\mu}$, see (7). It is easy to see that the relationships between matrices $\widetilde{B}$ and $(\widetilde{B})^{\mu}$ are

$$\widetilde{b}_{ij}^{\mu} = \widetilde{b}_{ij} - b_i^{\mu} \text{ for all } (i,j) \in M \times \mu(M). \tag{13}$$

From (8) and (9) applied to matrix $\widetilde{B}$ we know $\nu\left(\widetilde{B}\right) = \nu(B) = (x,y) \in \mathbb{R}_+^M \times \mathbb{R}_+^{M'}$ or equivalently $\nu\left((\widetilde{B})^{\mu}\right) = (x',y') \in \mathbb{R}_+^M \times \mathbb{R}_+^{\mu(M)}$, with $x_i' = x_i - b_i^{\mu}$ for $i \in M$, and $y_j' = y_j$ for $j \in \mu(M)$.

We can apply the previous procedure for square matrices to obtain an increasing piecewise linear path from $(\widetilde{B})^{\mu}$ to its maximum matrix in $[(\widetilde{B})^{\mu}]_{\nu}$. This path, applied to matrix $\widetilde{B}_{M\times\mu(M)}$, see (13), induces a path from $\widetilde{B}_{M\times\mu(M)}$ to $\overline{A}_{M\times\mu(M)}$, where $\overline{A}$ denotes the maximum element of the family $[A]_{\nu}$.

Moreover, for $(i,j) \in M \times (M' \setminus \mu(M))$ recall by (11) that $\overline{a}_{ij} = x_i - \min_{j\in\mu(M)} \{y_j\}$. From the equality $\delta_{M,\mu(M)}^{(\widetilde{B})^{\mu}}(x',y') = \delta_{\mu(M),M}^{(\widetilde{B})^{\mu}}(x',y')$ we know that $\min_{i\in M}\{x_i'\} = \min_{j\in\mu(M)}\{y_j'\} = \min_{j\in\mu(M)}\{y_j\}$, and then for some $i_* \in M$ we have $x_{i_*}' = x_{i_*} - b_{i_*}^{\mu} = \min_{j\in\mu(M)}\{y_j\}$. That is, for $i_* \in M$ we have $\overline{a}_{i_*j} = b_{i_*}^{\mu}$ for all $j \in M' \setminus \mu(M)$. For any $i \neq i_*, i \in M$ such that $x_i' > \min_{i\in M}\{x_i'\}$ or equivalently $x_i' = x_i - b_i^{\mu} > \min_{i\in M}\{x_i'\} = \min_{j\in\mu(M)}\{y_j\}$, that is $b_i^{\mu} < x_i - \min_{j\in\mu(M)}\{y_j\}$, we can raise at the same time entries $\widetilde{b}_{ij} = b_i^{\mu}$ to $\overline{a}_{ij} = x_i - \min_{j\in\mu(M)}\{y_j\}$ for all $j \in M' \setminus \mu(M)$ without changing the nucleolus, as the reader can check applying (8) and (9). This ends the proof. □

There is a continuum of elements in any family $[A]_{\nu}$, $A \in \mathrm{M}_m^+$, except for the null matrices and 2×2 assignment matrices

$$\begin{pmatrix} k & k \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ k & k \end{pmatrix}, \begin{pmatrix} k & 0 \\ k & 0 \end{pmatrix}, \begin{pmatrix} 0 & k \\ 0 & k \end{pmatrix}, \quad k > 0.$$

In these special cases the family $[A]_{\nu}$ reduces to a singleton. The null case is obvious and the case $m=2$ is checked easily from the description of the family in the 2×2 case.

In the case $m\geq 3$, notice that if matrix $A\in \mathrm{M}_m^+$ is not the maximum element of the family Theorem 6.1 gives a continuum of elements of the family. It only rests to analyze the case $A=\overline{A}$. In this case we know, since we are in the square case, that $x_i+y_j=a_{ij}$ for all $(i,j)\in M\times M'$, where $\nu(A)=(x,y)$. Clearly all matchings are optimal. In this case let $a_{i^*j^*}$ be an arbitrary positive entry of matrix A. It exists since we are not in the null case. Define matrix $B\in \mathrm{M}_m^+$ by lowering this entry to zero, i.e.

$$b_{ij}=\begin{cases} a_{ij} & \text{if} \quad (i,j)\neq (i^*,j^*),\\ 0 & \text{if} \quad (i,j)=(i^*,j^*).\end{cases}$$

We leave some details to the reader to check that matrix $B\in [A]_\nu$ and $B\neq A$ by using (3). Once again by Theorem 6.1 the continuum of elements of $[A]_\nu$ is guaranteed.

7. The inverse problem

In this section we study the conditions to ensure that a given vector is the nucleolus of some assignment game.

Firstly notice that not any vector is a candidate to be a nucleolus. For instance, the vector $(3,2,1,4)\in\mathbb{R}_+^2\times\mathbb{R}_+^2$ can never be the nucleolus of any 2×2 assignment game. For any candidate $(x,y)\in\mathbb{R}_+^M\times\mathbb{R}_+^{M'}$ with $|M|=|M'|$, to be the nucleolus of an assignment game with matrix $A\in\mathrm{M}_m^+$, by (3) it must satisfy

$$\delta^A_{M,M'}(x,y)=\min_{i\in M}\{x_i\}=\min_{j\in M'}\{y_j\}=\delta^A_{M',M}(x,y). \tag{14}$$

In our case $\min\{x_1,x_2\}=2\neq 1=\min\{y_1,y_2\}$.

Moreover, let us see that condition (14) turns out to be a simple characterization of it. To see the characterization, just define the square matrix $V=(v_{ij})_{1\leq i,j\leq m}$

defined by $v_{ij} := x_i + y_j$, for all $(i, j) \in M \times M'$ being $(x, y) \in \mathbb{R}_+^M \times \mathbb{R}_+^{M'}$ with $|M| = |M'|$ and $\min_{i\in M} \{x_i\} = \min_{j\in M'} \{y_j\}$. Indeed, any matching is optimal in V and the vector $(x, y) \in C(w_V)$. Therefore $\delta^V_{S,T}(x, y) = \delta^V_{T,S}(x, y) = 0$ for all $\emptyset \neq S \subseteq M$ and $\emptyset \neq T \subseteq M'$ with $|S| = |T|$, and $S \neq M$. Moreover $\delta^V_{M,M'}(x, y) = \delta^V_{M',M}(x, y)$ by assumption. Hence we have $\nu(V) = (x, y)$. Summarizing we have the following result.

Theorem 7.1 (Condition for the nucleolus in the square case). *Let $(x, y) \in \mathbb{R}_+^M \times \mathbb{R}_+^{M'}$ be a vector, with $|M| = |M'|$. The following statements are equivalent:*

1. *There exists a matrix $A \in \mathrm{M}_m^+$, such that $\nu(A) = (x, y)$,*

2. $\min_{i\in M} \{x_i\} = \min_{j\in M'} \{y_j\}$.

To analyze the non-square case we use the approach given in (8) and (9). Since it is well known that the nucleolus of a non-square assignment game gives zero payoff to all non-optimally assigned players, then a candidate vector must assign zero to some players. The next result is the precise necessary and sufficient condition. Its proof is in the Appendix.

Theorem 7.2 (Condition for the nucleolus in the non-square case). *Let $(x, y) \in \mathbb{R}_+^M \times \mathbb{R}_+^{M'}$ be a vector, with $|M| < |M'|$, and let $Z_0 = \{j \in M' \mid y_j = 0\}$. The following statements are equivalent:*

1. *There exists a matrix $A \in \mathrm{M}_{m\times m'}^+$, such that $\nu(A) = (x, y)$,*

2. (a) *There exists $Z_0' \subseteq Z_0$ with $|Z_0'| = |M'| - |M|$, and*

   (b) $\min_{i\in M} \{x_i\} \geq \min_{j\in M'\setminus Z_0'} \{y_j\}$.

Notice that from Theorem 7.1 and 7.2, the vector $(3, 2, 1, 4) \in \mathbb{R}_+^2 \times \mathbb{R}_+^2$ can never be the nucleolus of any 2×2 assignment game, but the vector $(3, 2, 1, 4, 0) \in \mathbb{R}_+^2 \times \mathbb{R}_+^3$ is the nucleolus of some assignment game. Indeed, matrix $V = \left(\begin{array}{cc|c} 2 & 7 & 2 \\ 3 & 5 & 1 \end{array}\right)$ has the desired nucleolus.

8. Appendix

Proof of (8) and (9)

Proof. Let $A \in \mathrm{M}^{+}_{m\times m'}, m < m'$ and let $\mu \in \mathscr{M}^{*}_{A}(M, M')$. Without loss of generality, we can assume that $\mu = \{(1,1),(2,2),\ldots,(m,m)\}$ is an optimal matching of matrix A.

We first prove that matching μ is also an optimal matching of matrix A^{μ}, defined by (6) and (7). To see it, consider any allocation $(x,y) \in C(w_A)$. Clearly $x_i \geq a^{\mu}_i = \max\limits_{j\in M'\setminus\mu(M)} \{a_{ij}\}$ for all $i \in M$. Then, as $x_i - a^{\mu}_i \geq 0$ for all $i \in M$, we obtain $(x_i - a^{\mu}_i) + y_j \geq 0$, for all $(i,j) \in M \times \mu(M)$. Moreover, for all $(i,j) \in M \times \mu(M)$, we also have $(x_i - a^{\mu}_i) + y_j \geq a_{ij} - a^{\mu}_i$. From both inequalities we have

$$(x_i - a^{\mu}_i) + y_j \geq a^{\mu}_{ij} \ \text{ for all } \ (i,j) \in M \times \mu(M).$$

Since $\mu = \{(1,1),(2,2),\ldots,(m,m)\}$ is an optimal matching for A, then $a_{ii} \geq a^{\mu}_i$ for all $i \in M$, and we obtain $(x_i - a^{\mu}_i) + y_i = a_{ii} - a^{\mu}_i = a^{\mu}_{ii}$, for all $i \in M$.

Now we can prove that μ is also optimal for matrix A^{μ}. To see it for any other matching $\mu' \in \mathscr{M}(M, \mu(M))$, we have

$$\sum_{i=1}^{m} a^{\mu}_{ii} = \sum_{i=1}^{m} (x_i - a^{\mu}_i) + y_i = \sum_{(i,j)\in\mu'} (x_i - a^{\mu}_i) + y_j \geq \sum_{(i,j)\in\mu'} a^{\mu}_{ij}.$$

Let A^0, M^0 and μ^0 the notation introduced in (5) to make square the non-square initial matrix A. We know that matching $\mu^0 = \{(1,1),(2,2),\ldots,(m',m')\}$ is optimal for matrix A^0, i.e. $\mu^0 \in \mathscr{M}^{*}_{A^0}(M^0, M')$.

For each vector $(x,y) \in \mathbb{R}^{M}_{+} \times \mathbb{R}^{M'}_{+}$ we denote by $(x^0, y^0) \in \mathbb{R}^{M^0}_{+} \times \mathbb{R}^{M'}_{+}$ the vector defined by $x^0_k = x_k$ if $k \in M$ and $x^0_k = 0$ if $k \in M^0 \setminus M$ and $y^0_k = y_k$ if $k \in M'$. It is easy to prove that vector $(x,y) \in C(w_A)$ if and only if $(x^0, y^0) \in C(w_{A^0})$. Moreover $(x,y) \in C(w_A)$ if and only if $(x', y') \in C(w_{A^{\mu}})$, where $x'_i = x_i - a^{\mu}_i$ for $i \in M$, and $y'_j = y_j$ for $j \in \mu(M)$. The proof of these facts is left to the reader.

Let us denote $\nu(A) = (x,y)$. We have to show $\nu(A^{\mu}) = (x',y')$. To this end, take $\mu = \{(1,1),(2,2),\ldots,(m,m)\}$ an optimal matching of A, and also of A^{μ}. Recall that $\mu^0 = \{(1,1),(2,2),\ldots,(m',m')\}$ is optimal for A^0.

We claim that for any $S \subseteq M,\ S \neq \emptyset$ we have

$$\delta^{A^{\mu}}_{S,\mu(S)}\left(x',y'\right) = \delta^{A^{0}}_{S,\mu(S)}\left(x^0,y^0\right) \quad \text{and} \quad \delta^{A^{\mu}}_{\mu(S),S}\left(x',y'\right) = \delta^{A^{0}}_{\mu(S),S}\left(x^0,y^0\right).$$

We only prove the first equality, and the second is similar. To see it, notice that

$$\begin{aligned}
\delta^{A^{0}}_{S,\mu(S)}\left(x^0,y^0\right) &= \min_{i\in S\ j\in M'\setminus\mu(S)} \left\{x_i, x_i + y_j - a_{ij}\right\} = \\
&= \min_{i\in S\ j\in\mu(M)\setminus\mu(S)} \left\{x_i, x_i + y_j - a_{ij}, x_i - a_i^{\mu}\right\} = \\
&= \min_{i\in S\ j\in\mu(M)\setminus\mu(S)} \left\{x_i - a_i^{\mu}, x_i + y_j - a_{ij}\right\} = \\
&= \min_{i\in S\ j\in\mu(M)\setminus\mu(S)} \left\{x_i - a_i^{\mu}, (x_i - a_i^{\mu}) + y_j - (a_{ij} - a_i^{\mu})\right\} = \\
&= \min_{i\in S\ j\in\mu(M)\setminus\mu(S)} \left\{x_i - a_i^{\mu}, (x_i - a_i^{\mu}) + y_j - a_{ij}^{\mu}\right\} = \\
&= \delta^{A^{\mu}}_{S,\mu(S)}\left(x',y'\right).
\end{aligned}$$

where the second equality comes from the fact that for all $j \in M' \setminus \mu(M)$ we have $y_j = 0$, the third equality since for all $i \in S$, $x_i \geq x_i - a_i^{\mu}$ and the fifth one comes from the fact that whenever $a_{ij} - a_i^{\mu} < 0$ we have $(x_i - a_i^{\mu}) + y_j - (a_{ij} - a_i^{\mu}) > (x_i - a_i^{\mu})$, which allows us to introduce the term a_{ij}^{μ}.

We finish the proof by recalling that $\nu(A^0) = (x^0, y^0)$.

□

Proof of Theorem 7.2

Proof. 1. $\longrightarrow$ 2. Let $A \in \mathrm{M}^{+}_{m\times m'}, m < m'$ be a matrix and let $\nu(A) = (x,y)$ be its nucleolus.

Let $\mu \in \mathscr{M}_A^*(M,M')$ be an optimal matching. Clearly, non-assigned sellers by μ get zero payoffs in the nucleolus. Therefore, let Z_0' be the set of non-assigned sellers by μ, i.e. $Z_0' = M' \setminus \mu(M)$.

Now apply (8) and (9) and $\nu(A^\mu) = (x', y')$, with $x_i' = x_i - a_i^\mu$ for $i \in M$, and $y_j' = y_j$ for $j \in \mu(M)$ where vector $a^\mu = \left(a_i^\mu\right)_{i\in M}$ and matrix A^μ are defined as in (6) and (7). Then, applying Theorem 7.1, $\min_{i\in M}\{x_i\} \geq \min_{i\in M}\left\{x_i - a_i^\mu\right\} = \min_{j\in M'\setminus Z_0'}\left\{y_j\right\}$. This is condition 2.

2. $\longrightarrow$ 1. We define matrix $V \in \mathrm{M}^+_{m\times m'}$ by

$$v_{ij} := \begin{cases} x_i + y_j & \text{if} \quad i \in M, \text{ and } j \in M' \setminus Z_0', \\ x_i - \min\limits_{j\in M'\setminus Z_0'}\left\{y_j\right\} & \text{if} \quad i \in M, \text{ and } j \in Z_0'. \end{cases}$$

Note that any matching between M and $M' \setminus Z_0'$ is optimal for V, i.e. $\mathscr{M}(M, M' \setminus Z_0') \subseteq \mathscr{M}_V^*(M,M')$. This matrix $V \in \mathrm{M}^+_{m\times m'}$ is, in fact, a valuation matrix and its proof is left to the reader.

We must prove now that vector (x,y) is the nucleolus of this matrix V. From (8) and (9), $(x,y) = \nu(V)$ if and only if $\nu(V^\mu) = (x',y')$, with $x_i' = x_i - v_i^\mu$ for $i \in M$, and $y_j' = y_j$ for $j \in \mu(M)$, for some $\mu \in \mathscr{M}(M, M' \setminus Z_0')$. Indeed, all of them are optimal.

By (6), $v_i^\mu = x_i - \min_{j\in M'\setminus Z_0'}\left\{y_j\right\}$ for all $i \in M$ and then $x_i' = x_i - v_i^\mu = \min_{j\in M'\setminus Z_0'}\left\{y_j\right\}$ for all $i \in M$. By its definition and the above equalities matrix V^μ satisfies, for all $(i,j) \in M \times (M' \setminus Z_0')$,

$$v_{ij}^\mu = \max\left\{0, y_j + \min_{j\in M'\setminus Z_0'}\left\{y_j\right\}\right\} = y_j + \min_{j\in M'\setminus Z_0'}\left\{y_j\right\} = y_j' + x_i'.$$

Since $\min_{i\in M}\{x_i'\} = \min_{i\in M}\left\{\min_{j\in M'\setminus Z_0'}\left\{y_j\right\}\right\} = \min_{j\in M'\setminus Z_0'}\left\{y_j'\right\}$ and V^μ is a square valuation matrix, by (3) we obtain $\nu(V^\mu) = (x',y')$.

$\square$

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