Treball final de grau

GRAU DE MATEMÀTIQUES

Facultat de Matemàtiques i Informàtica
Universitat de Barcelona

COMPLETE THEORIES OF BOOLEAN ALGEBRAS

Clara Trullenque Ortiz

Director: Dr. Enrique Casanovas
Realitzat a: Departament de Matemàtiques i Informàtica

Barcelona, June 27, 2018

Abstract

Boolean algebras are well-known mathematical structures. It is known that the theory of the class of these structures is incomplete, so the aim of this work is to present sistematically the different ways to complete it. In order to do this, we must also study the different kinds of Boolean algebras and their properties. Besides completeness, we also study other questions like ω-categoricity and quantifier elimination. The whole subject is studied in the formal language of first order logic.

2010 Mathematics Subject Classification. 03C10, 03C35, 03G05, 03G10, 06E15

Contents

Chapter 1

Introduction

As mathematicians, most of the time we are asked to prove that a property holds in a given structure. However, we do not usually think of this as checking that a sentence belongs to a theory, probably because we are not used to treating particular areas of mathematics, like group theory or euclidean geometry, with the formalism of first order logic. Model theory studies mathematical structures from the perspective of mathematical logic. It can be applied to many different structures and theories, and the present work is an application to Boolean algebras.

Given the theory of a class of structures, in our case Boolean algebras, many questions can be considered: is it complete? Is it κ-categorical for some cardinal number κ? Does it admit quantifier elimination? Can we characterise its models? I believe one of the most the most natural ones is to ask if it is complete, in other words, if given an arbitrary sentence, the sentence or its negation belongs to the theory. A categorical theory determines a structure up to isomorphism. In our case, it is clear that the theory of Boolean algebras cannot be complete or categorical, because there are finite and infinite Boolean algebras. Even in infinite Boolean algebras, there are questions that are left unanswered, like: do they have atoms? As a consequence, the theory of infinite Boolean algebras is still incomplete. Now, I believe that the most important question that needs to be asked and answered here is: now that we know that the theory of Boolean algebras is not complete, how can it be completed? Can we describe the different completions of the theory?

This is a problem that has been studied before. The first mathematician who adressed the problem was Alfred Tarski, who also gave a result on the decidability of models of the elementary theory of Boolean algebras. This was done quite briefly so, later on and independently, Ershov wrote a publication in the Russian journal *Algebra i Logika* ([5]) expanding these results. I have based by work in two more modern versions of this study: the texts by Chang and Keisler ([3]) and Koppelberg ([8]). The first one is a classic model theory manual that dedicates a section to Boolean algebras, because it uses them as an example on how to give all possible completions of a theory using the back-and-forth tecnique. The second one is an extensive work about Boolean algebras, and there is one chapter exclusively dedicated to the completeness of their theories. However, the method they use to prove their final results is a bit different from the one presented in this work: Koppelberg uses Vaught relations, which are basically an adaption of the back-and-forth method for Boolean algebras, and in [3], the final results are proved using Skolem expansions of languages and doing back-and-forth with structures of cardinality 2^{ω}. This tecniques are old-fashioned and hard to understand for someone with little experience working with mathematical logic. For this reason, I have decided to give this problem a more modern approach, using the back-and-forth tecnique with all its generality.

For general notions and results about Boolean algebras, I used the texts by Bell and Machover ([1]) and Halmos and Givant ([6]), and I consulted [9] and [2] to learn about model theory.

Structure of the work

This work is structured as follows:

There is a chapter on preliminaries that presents the fundamentals of first order logic, like semantics. I hope reading this chapter will clear up doubts of some readers, especially if they are not familiar with mathematical logic. This is where I set all the conventions regarding the objects that appear in the rest of the chapters.

In chapter 3 we describe the structure we will be working with: Boolean algebras. In this chapter we present interesting properties of these objects that will give the readers an idea of how they behave. Some of these properties might sound familiar, as one may relate them to an algebraic context.

Chapter 4 is an introduction to model theory. Despite the fact that, in this particular work, we treat model theory mainly as a tool, we manage to see some important results of this field, though we do not give a proof for all of them.

In chapter 5 we study two important examples: atomless Boolean algebras and atomic and infinite Boolean algebras. This chapter is key to understand the applications of model theory in the study of these structures. Some arguments presented in the main proofs of chapter 5 are very similar to the ones in the final results, so studying this particular algebras aside has been very helpful to me and, hopefully, will also be helpful to the reader. Besides, in this chapter we do not only explore the completeness of the algebras, but also *quantifier elimination* and *ω-categoricity.*

Finally, in chapter 6 we prove the main results of this work. We describe *the invariants* of a Boolean algebra, which are pairs of numbers that will be determinant in the classification of the theories of Boolean algebras.

Acknowledgements

For his time and patience, I would like to thank Enrique Casanovas, my advisor. Not only he has oriented me during the process of reading and understanding about model theory and Boolean algebras, but he has done it with great enthusiasm and kindness.

I would also like to express my gratitude to my friends and family, with a special mention to Juan and Nil, who have been by my side during the most stressful moments and given me their inconditional support.

Chapter 2

Preliminaries

In classical logic, we can distinguish propositional and first order logic. Propositional logic is known to be simpler: it describes basic facts, whereas first order logic differenciates between relations, functions and objects, thus it can be used to express a more complex reality. We will mostly use first order logic, but it is important to describe the structure of propositional logic as well.
The *language of propositional logic* consists in a set $\mathbb{P}$ of propositional variables, the connectives $\sqcup, \&, \neg, \rightarrow, \leftrightarrow$ and brackets. Usually, the connective symbols are noted by $\vee, \wedge, \neg, \rightarrow, \leftrightarrow$, but we will use $\sqcup$ and $\&$ instead of $\vee, \wedge$ because otherwise there could be a misunderstanding with the symbols of the language of Boolean algebras.
A *proposition* is a finite sequence of these symbols constructed using the following rules:
If p is a propositional variable, φ, ψ are propositions, $* \in \{\&, \sqcup, \rightarrow, \leftrightarrow, \}$:

$$\frac{}{p} \qquad \frac{\varphi}{\neg\varphi} \qquad \frac{\varphi, \psi}{(\varphi * \psi)}$$

An *interpretation* is a mapping $I : \mathbb{P} \longrightarrow \{0, 1\}$. It is said that $p \in \mathbb{P}$ is *true with* I iff $I(p) = 1$. It is *false with* I otherwise. By recursion, it can be extended to a mapping $I*$, which gives a value within $\{0, 1\}$ to every proposition, in a way that $I*$ preserves the connectives. Every extension $I*$ can be identified with the interpretation I, and that is why, from now on, we will use the symbol $I = I*$.

Let $\Sigma, \varphi \subseteq Prop(\mathbb{P})$. It is said that φ is a logical consequence of Σ iff, for every interpretation I, if $I(\Sigma) \subseteq \{1\}$, then $I(\varphi) = 1$. It is written $\Sigma \models \varphi$. Since we are assuming the completeness theorem, we will write $\Sigma \models \varphi$ or $\Sigma \vdash \varphi$ indistinctly. If φ, ψ are propositions and $\psi \models \varphi$ and $\varphi \models \psi$, then we will say they are equivalent and we will write $\varphi \equiv \psi$.

Now, there are two kind of symbols in first order logic:

- Logical symbols like *connectives* $(\sqcup, \&, \neg, \rightarrow, \leftrightarrow)$, brackets, *quantifiers* $(\forall, \exists)$, $=$, variables $(x, y, z, ...)$.
- Non-logical symbols: relation symbols $(R, S...)$, function symbols $(F, G, H..)$ and constant symbols $(c...)$.

Every relation symbol and every function symbol are associated to a natural number n, which we call their *arity*.

A *language* is a set of non-logical symbols. The set of all relation symbols, function symbols and constant symbols in a language $\mathcal{L}$ is written $\mathcal{R}_{\mathcal{L}}, \mathcal{F}_{\mathcal{L}}$ and $\mathcal{C}_{\mathcal{L}}$, respectively.

A *term* in a language $\mathcal{L}$ is a finite sequence of symbols in $VAR \cup \mathcal{F}_{\mathcal{L}} \cup \mathcal{C}_{\mathcal{L}}$. It is constructed using the following rules:
If $x \in VAR$, $c \in \mathcal{C}_{\mathcal{L}}$,$F \in \mathcal{F}_{\mathcal{L}}$ is an n-ary function symbol and $t_1, ..., t_n$ are terms, then

$$\frac{\quad}{x} \qquad\qquad \frac{\quad}{c} \qquad\qquad \frac{t_1, ..., t_n}{Ft_1...t_n}$$

An *atomic formula* in the language $\mathcal{L}$ is a finite sequence of symbols in $\mathcal{L} \cup VAR \cup \{=\}$. There are two kinds of atomic formulas.

- Equations. They are sequences of the form $t_1 = t_2$, there t_1, t_2 are terms in $\mathcal{L}$.
- Predications. They are sequences of the form $Rt_1...t_n$, where $t_1, ..., t_n$ are terms and $R \in \mathcal{R}_{\mathcal{L}}$ is an n-ary relation symbol.

A *formula* in $\mathcal{L}$ is a finite sequence of symbols in $\mathcal{L} \cup \{=, \forall, \exists, \neg, \&, \sqcup, \rightarrow, \leftrightarrow, (,)\}$ constructed using the following rules
If χ is an atomic formula, φ, ψ are formulas, x is a variable, and $* \in \{\&, \sqcup, \leftrightarrow, \rightarrow\}$

$$\frac{\quad}{\chi} \qquad\qquad \frac{\varphi}{\neg\varphi} \qquad\qquad \frac{\varphi, \psi}{(\varphi * \psi)} \qquad\qquad \frac{\varphi}{\forall x \varphi} \qquad\qquad \frac{\varphi}{\exists x \varphi}$$

It is said that an occurrence of a variable is *free* in a formula if it is not inside the scope of any quantifier of the variable. A formula without free variables is called a *sentence*.

Let $\mathcal{L}$ be a language. An $\mathcal{L}$-*structure* is a pair $\mathcal{M} = (M, I)$ where M is a nonempty set called *universe*, and I is a mapping that sends every symbol in $\mathcal{L}$ to an object in M in the following way:

- $I(c) \in M$ if c is a constant symbol.

- $I(F) : M^n \longrightarrow M$ if F is an n-ary function symbol.
- $I(R) \subseteq M^n$ if R is an n-ary relation symbol.

We will write $\varphi(x_1, ..., x_n)$ for a formula φ whose free variables are in the sequence $x_1, ..., x_n$. It is said that an n-tuple $\overline{a} = (a_1, ..., a_n)$ satisfies a formula $\varphi(\overline{x})$, where $\overline{x} = (x_1, ..., x_n)$, in $\mathcal{M} = (M, I)$ iff it makes it true: $\mathcal{M} \models \varphi(\overline{a})$.

Now that some knowledge about logic has been acquired, there are only a few definitions that need to be clarified.

We will refere to a set X as *denumerable* iff there is a bijection between X and the set of natural numbers ω.

A set of formulas $\Sigma(x_1, ..., x_n)$ in the language $\mathcal{L}$ is said to be *consistent*, or *satisfiable* if there exists an $\mathcal{L}$-structure and an n-tuple in the structure that satisfy every formula in Σ. If such $\mathcal{L}$-structure and tuple cannot exist, it is said that the set is *inconsistent* or *contradictory*. If Σ is a set of sentences, and $\mathcal{M}$ satisfies every sentence in Σ, we say that $\mathcal{M} = (M, I)$ is a *model* of Σ. .

A *theory* of language $\mathcal{L}$ is a consistent set of sentences including all their consequences in the language $\mathcal{L}$. The elements of the theory are called *theorems*. A *complete theory* is a maximal theory with respect to the inclusion relation among theories. If T is a complete theory in the language $\mathcal{L}$, every sentence φ in the language verifies $\varphi \in T$ or $\neg\varphi \in T$, and the converse is also true.

Chapter 3

Boolean algebras

3.1 Lattices

Definition 3.1.1 *A* lattice *is a non empty partially ordered set* $\langle L, \leqslant \rangle$ *in which every pair of elements* x, y, *has a supremum and an infimum. They are unique and the first one is denoted by* $x \vee y$ *and the second one by* $x \wedge y$.

It follows from the definition of supremum and infimum that:

$$x \leqslant y \Leftrightarrow x \wedge y = x \Leftrightarrow x \vee y = y.$$

We can treat both $\vee$ and $\wedge$ as two operations, which we will call *join* and *meet*. Associative, commutative and absortion properties hold in any lattice for both $\vee$ and $\wedge$. In other words:

$$\begin{array}{cc} x \wedge y = y \wedge x, & x \vee y = y \vee x, \\ x \vee (y \vee z) = (x \vee y) \vee z, & x \wedge (y \wedge z) = (x \wedge y) \wedge z, \\ (x \wedge y) \vee y = y, & (x \vee y) \wedge y = y. \end{array}$$

Definition 3.1.2 *It is said that a lattice is* complete *if every subset of* L *has an infimum and a supremum.*

Observation 3.1.3 *If* L *is a complete lattice, the supremum of* L *is its greatest element, and we will denote it by* 1, *and its infimum is its least element, which will be called* 0. *However, it is not necessary for a lattice to be complete in order to have a least and a greatest element. It is also easy to see that* $inf(\emptyset)$ *is the greatest element in* L *and* $sup(\emptyset)$ *is the least one.*

Definition 3.1.4 *It is said that a lattice* L *is* distributive *if the following conditions hold for any* $x, y, z \in L$:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z),$$
$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

Definition 3.1.5 *Let L be a lattice. L is said to be* complemented *if it has a least and a greatest element and, for every $x \in L$, there is an element $y \in L$ so that $x \vee y = 1$ and $x \wedge y = 0$. Such y is unique and we will write y=x^{c}.*

Definition 3.1.6 *A* Boolean Algebra *is a complemented distributive lattice.*

Proposition 3.1.7 *Consider the structure $\mathfrak{B} = \langle B,\wedge,\vee,^{c},0,1\rangle$ where B is a set, $\vee, \wedge$ are binary operations in B, ^{c} is a monary operation in B and $0, 1 \in B$. Assume they verify for any $x, y, z \in B$:*

$$\begin{array}{cc} x \wedge y = y \wedge x, & x \vee y = y \vee x, \\ x \vee (y \vee z) = (x \vee y) \vee z, & x \wedge (y \wedge z) = (x \wedge y) \wedge z, \\ (x \wedge y) \vee y = y, & (x \vee y) \wedge y = y, \\ x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), & x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z), \\ x \wedge x^{c} = 0, & x \vee x^{c} = 1. \end{array}$$

If we define an order in B by $x \leqslant y \Leftrightarrow x \wedge y = x$, then $\langle B, \leqslant\rangle$ is a Boolean Algebra, where $0, 1$ are the least and greatest elements and for every $x \in B$, x^{c} is the complement of x. Also, for any x, y in the Boolean algebra, $x \wedge y$ and $x \vee y$ are the infimum and the supremum of x and y, respectively.

Proof: First of all, notice that $\langle B, \leqslant\rangle$ is a partially ordered set. Let's prove that $x \wedge y$ is the supremum of x and y. Since $(x \wedge y) \wedge y = x \wedge y$, and $(x \wedge y) \wedge x = x \wedge y$, $x \wedge y \leqslant x$, $x \wedge y \leqslant y$. Now let z be an element in B such that $z \leq x, z \leq y$. Then $z \wedge x = z$, $z \wedge y = z$, and so $z \wedge (x \wedge y) = (z \wedge x) \wedge y = z \wedge x = z$. This means $z \leqslant (x \wedge y)$, thus $x \wedge y$ is the infimum of x,y. We can prove similarly that $x \vee y$ is the supremum of any x, y in B.
The distributivity holds because of the fourth axiom. The only thing left to prove is that this distributive lattice is complemented, but it follows trivially from the fact that 1 is the greatest element in B and 0 is the least one. □

This is a more usual presentation of a Boolean algebra and it is the one we shall use in the future. Nevertheless, there is a third one, which is usually used in a more algebraic context: the Boolean ring.

Definition 3.1.8 *A* Boolean ring *R is a commutative ring in which, for every $x \in R$, $x^2 = x$.*

Proposition 3.1.9 *Let $\mathfrak{B} = (B, \wedge, \vee,^{c}, 0, 1)$ be a Boolean algebra. For any $x, y \in B$, we can define the operations:*

$$x + y = (x \wedge y^{c}) \vee (x^{c} \wedge y),$$
$$x \cdot y = x \wedge y.$$

Then $(B,+,\cdot)$ *is a Boolean ring. Conversely, if* R *is a Boolean ring and we define:*

$$x \wedge y = xy,$$
$$x \vee y = x + y + xy,$$
$$x^{\mathrm{c}} = 1 + x,$$

then $\mathfrak{R} = \langle R,\ \vee, \wedge, ^{\mathrm{c}},\ 0,\ 1\rangle$ *is a Boolean algebra.*

Proof: See [7], chapter 5. □

Definition 3.1.10 *We define as the* trivial algebra *the set* $\{0\}$ *with the usual operations. It is important to keep this definition in mind, since some authors consider* $0 \neq 1$ *as an axiom, and they define as the trivial algebra the set* $\{0,1\}$.

Definition 3.1.11 *Let* $\mathfrak{B} = \langle B, \wedge, \vee, ^c, 0, 1\rangle$ *be a Boolean algebra. It is said that* $\mathfrak{A}$ *is a subalgebra of* $\mathfrak{B}$ *iff* $\mathfrak{A} = \langle A, \wedge\restriction_A, \vee\restriction_A, ^c\restriction_A, 0, 1\rangle$, *where* $A \subseteq B$ *is closed under the operations.*

Definition 3.1.12 *Let* $\mathfrak{B}$ *be a Boolean algebra, and consider* $A \subseteq B$. *There is a minimal set* A' *such that* $A \subseteq A' \subseteq B$ *and* A' *is the universe of a subalgebra of* $\mathfrak{B}$. *In fact,*

$$A' = \bigcap\{C \subseteq B : A \subseteq C,\ 0,1 \in C \textit{ and } C \textit{ is closed under the operations}\}.$$

It is said that the subalgebra of $\mathfrak{B}$ *with universe* A' *is the Boolean algebra generated by* A. *If* $A = \{a_1, ..., a_n\}$, *then we write* $\langle a_1, ..., a_n\rangle$.

3.2 Filters and homomorphisms

Definition 3.2.1 *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be Boolean Algebras. An* homomorphism *between Boolean algebras is an application* $h\colon A \longrightarrow B$ *which verifies*

$$h(x \wedge y) = h(x) \wedge h(y),$$
$$h(x^{\mathrm{c}}) = h(x)^{\mathrm{c}},$$

for every $x, y \in A$.

Observation 3.2.2 *The following propierties hold for any homomorphism* h:

$$h(x \vee y) = h(x) \vee h(y),$$
$$h(x) \leqslant h(y) \textit{ if } x \leqslant y,$$
$$h(0) = 0,$$
$$h(1) = 1.$$

The first property is easy to see because $x \vee y = (x^{\mathrm{c}} \wedge y^{\mathrm{c}})^{c}$. Now, the fact that an homomorphism preserves the order is due to:

$$x \leqslant y \Rightarrow x \wedge y = x \Rightarrow h(x) = h(x \wedge y) = h(x) \wedge h(y) \Rightarrow h(x) \leqslant h(y).$$

The others are a consequence of the first two.

If h is one-to-one and onto, we say h is an isomorphism, and write $\mathfrak{A} \cong \mathfrak{B}$. For the following results, we will write $\mathfrak{B}$ for a Boolean algebra.

Definition 3.2.3 *A* filter *F of $\mathfrak{B}$ is a subset $F \subseteq B$ such that:*

1. $0 \notin F$,
2. *For every $x, y \in F$, $x \wedge y \in F$,*
3. *For every $x \in F$, and for every $y \in B$, $x \leqslant y$ implies $y \in F$.*

If we exclude from this list the condition $0 \notin F$, we can consider B as a filter, which we will call improper filter*.*

Definition 3.2.4 *Let X be a subset of B. It is said that X has the* finite meet property *iff for every $x_1, ..., x_n \in X$, $x_1 \wedge ... \wedge x_n \neq 0$ for any n. In short, we say X has the $f.m.p.$*

Theorem 3.2.5 *A subset $X \subseteq B$ is included in a proper filter iff it has the finite meet property.*

Proof: Assume $X \subseteq F$ where F is a filter. If $x_1, ..., x_n \in F$, then $x_1 \wedge ... \wedge x_n \in F$, but $0 \notin F$, so $x_1 \wedge ... \wedge x_n \neq 0$. Conversely, assume X has de $f.m.p.$ Consider now the set $X^+ = \{y \in B : \exists x_1, ..., x_n \in X \text{ such that } x_1 \wedge ... \wedge x_n \leqslant y\}$. Then X^+ is a filter that includes X. $0 \notin X^+$ because X has de $f.m.p$, and the third condition holds because of the transitivity of $\leqslant$. Now assume $y, z \in X^+$. This means, there are $x_1, ..., x_n, x'_1, ..., x'_n \in X$ such that $x_1 \wedge ... \wedge x_n \leqslant y$ and $x'_1 \wedge ... \wedge x'_n \leqslant z$. Therefore $x_1 \wedge ... \wedge x_n \wedge x_1 \wedge ... \wedge x'_n \leqslant y \wedge z$, and $y \wedge z \in X^+$ as a result. □

Observation 3.2.6 *For each homomorphism of Boolean algebras, $h : \mathfrak{A} \longrightarrow \mathfrak{B}$, $h^{-1}(1)$={ $x \in A$: h(x)=1 } is a filter, and it is called the* hull *of h. This means that to every homomorphism of Boolean algebras can be associated a filter.*

Definition 3.2.7 *An* ideal *of a Boolean algebra is a subset $I \subseteq B$ such that:*

1. $1 \notin I$,
2. *For every $x, y \in I$, $x \vee y \in I$,*
3. *For every $x \in I$ and every $y \in B$, if $y \leqslant x$ then $y \in I$.*

If we exclude the condition $1 \notin I$*, we can consider* B *as an ideal, which will be called the* improper ideal *of* $\mathfrak{B}$*.*

Observation 3.2.8 *Notice that for any ideal* I *of* $\mathfrak{B}$*,* $F=\{x^{c} : x \in I\}$ *is a filter.*

Let us define $x \leftrightarrow y=(x^{c} \vee y) \wedge (y^{c} \vee x)$. Now, remember that every Boolean algebra is also a commutative ring. Ideals in Boolean algebras are also ideals in the Boolean ring associated to the algebras. Since in a commutative ring R, for any ideal I of R, we can consider the ring R/I, as a result of observation 3.2.8, it is natural to think that we could also consider B/F for a filter F in a Boolean algebra with universe B. In fact, if we define the relation $\sim$:

$$x \sim y \Leftrightarrow x \leftrightarrow y \in F,$$

then $\sim$ is a congruence relation with the operations in $\mathfrak{B}$.
As a result, we can consider the quotient B/F, which is a Boolean algebra with the operations $\wedge, \vee, ^{c}$ defined in the natural way.

Observation 3.2.9 *If we consider the natural homomorphism between* $\mathfrak{B}$ *and* $\mathfrak{B}/F$*,*

$$\begin{aligned} h : B &\longrightarrow B/F \\ x &\longrightarrow h(x) = [x]_F, \end{aligned}$$

then h *is an homomorphism and* F *is the hull of* h*.*

Definition 3.2.10 *An* ultrafilter *in a Boolean algebra is a proper filter that is not properly included in any other proper filter.*

Theorem 3.2.11 *Let* F *be a filter in the Boolean algebra* $\mathfrak{B}$*. The following conditions are equivalent:*

1. F *is an ultrafilter,*
2. *For all* $x, y \in B, x \vee y \in F$ *implies* $x \in F$ *or* $y \in F$*,*
3. *For every* $x \in B$*, either* $x \in F$ *or* $x^{c} \in F$*.*

Proof:

- 1$\Rightarrow$ 2: Assume F is an ultrafilter and $x \vee y \in F$. If $x \notin F$, we can define $G = \{z \in B : x \vee z \in F\}$. Its is easy to see G is a filter containing F, so $F = G$. Since $x \vee y \in F$, then $y \in G$ and, as a result, $y \in F$.
- 2 $\Rightarrow$ 3: The third condition follows immediately from the second one because 1=$x \vee x^{c}$ for any $x \in B$.

- $3 \Rightarrow 1$: Assume 3, and let G be a filter such that $F \subseteq G$. If $F \neq G$, then there is $x \in G$ such that $x \notin F$. Hence $x^c \in F$, and from this follows $x^c \in G$ but $0 = x \wedge x^c \in G$! This gives a contradiction, so $F = G$ is proven.

□

Theorem 3.2.12 ***(Ultrafilter theorem)***
For any Boolean algebra $\mathfrak{B}$, each filter F is included in an ultrafilter.

Proof: Let X be the set of all the filters of $\mathfrak{B}$ containing F ($X \neq \emptyset$). If we prove that any chain in X has an upper bound, by Zorn's lemma, we will get that there are maximal elements in X and, as a consequence, that there is an ultrafilter in $\mathfrak{B}$ containing F.
Let $\mathcal{C}$ be chain in X and consider $C = \bigcup \mathcal{C}$. We will now prove C is the upper bound we were looking for. It is obvious that $D \subseteq C$ for every $D \in \mathcal{C}$, so the only thing left to prove is that C is a filter.
For every $x, y \in C$, $x \in D, y \in E$ for some $E, D \in \mathcal{C}$. Since $\mathcal{C}$ is a chain, we can assume $D \subseteq E$, so $x, y \in E$ and $x \wedge y \in E$,thus $x \wedge y \in C$. In addition to this, if $z \in B$ and $x \leq z$, then $z \in D$. Therefore, $z \in C$. Since $0 \notin E$ for any filter E, $0 \notin C$. With all of this, we get that C is a filter, as required. □

Corollary 3.2.13 *Each $x \in B, x \neq 0$ is included in an ultrafilter.*

Proof: This follows immediatly from the fact that every $x \in B, x \neq 0$ is included in a filter and the Ultrafilter theorem. □

Corollary 3.2.14 *For any Boolean algebra $\mathfrak{B}$, and any $x, y \in B$, such that $x \neq y$, either there is an ultrafilter containing x and not y or there is an ultrafilter containing y and not x.*

Proof: For any $x, y \in B$ such that $x \neq y$, $x \nleq y$ or $y \nleq x$. Assume $x \nleq y$, then $x \wedge y^c \neq 0$, thus there is an ultrafilter F in B containing $x \wedge y^c$. This ultrafilter verifies $x \in F$ and $y \notin F$. The other case is proved similarly. □

3.3 Topology

It is easy to see that for any set X, its power set $\mathcal{P}(X)$ is a Boolean algebra with the operations $\cap, \cup, \emptyset, X$.

Definition 3.3.1 *A* field of subsets *of a set X is a subalgebra of $\mathcal{P}(X)$.*

Definition 3.3.2 *We will call **S**$\mathfrak{B}$ to the set of all ultrafilters in the Boolean algebra $\mathfrak{B}$.*

The main purpose of this section is to find out the relation between a Boolean algebra $\mathfrak{B}$ and **S**$\mathfrak{B}$.

Theorem 3.3.3 ***The Stone representation theorem***
Every Boolean algebra $\mathfrak{B}$ is isomorphic to a field of subsets of $\boldsymbol{S}\mathfrak{B}$.

Proof: We shall prove that the isomorphism needed is:

$$\begin{array}{rcl} h : B & \longrightarrow & \mathcal{P}(\mathbf{S}\mathfrak{B}) \\ x & \longrightarrow & h(x) = \{F \in \mathbf{S}\mathfrak{B} : x \in F\} \end{array}$$

Let's first prove it is an homomorphism of Boolean algebras. Let x, y be two elements in B. Then

$$F \in h(x \wedge y) \Leftrightarrow x \wedge y \in F \Leftrightarrow x \in F \text{ and } y \in F \Leftrightarrow F \in h(x) \text{ and } \\ F \in h(y) \Leftrightarrow F \in h(x) \cap h(y).$$

As a result, $h(x \wedge y) = h(x) \cap h(y)$. Similarly,

$$F \in h(x^{\mathrm{c}}) \Leftrightarrow x^{\mathrm{c}} \in F \Leftrightarrow x \notin F \Leftrightarrow F \in \mathbf{S}\mathfrak{B} \backslash h(x) \Leftrightarrow F \in h(x)^{c}.$$

Therefore h is an homomorphism. It is one-to-one because if $x \neq y$, there is an ultrafilter F in $\mathfrak{B}$ containing one and not the other (seen in 3.2.14). As a result, h is an isomorphism from B onto $h(B)$, a subalgebra of $\mathcal{P}(\mathbf{S}\mathfrak{B})$. □

Definition 3.3.4 *A* Boolean space *is a compact Hausdorff topological space which admits a basis of clopen sets.*

Definition 3.3.5 *Given a topological space X, the set of all the clopen subsets of X is called THE* clopen algebra of X. *It is noted by* ***C****(X).*

Lemma 3.3.6 *Let X be a compact topological space. If X admits a basis which is a field of subsets of X, then this basis is* ***C****(X).*

Proof: See [1]. Lemma 4.2. □

Definition 3.3.7 *Let h be the homomorphism defined by $h(x) = \{F \in \boldsymbol{S}\mathfrak{B} : x \in F\}$. Since h(B) is closed under finite intersections, this set forms a basis of a topology in $\boldsymbol{S}\mathfrak{B}$, and we call* Stone Space *of $\mathfrak{B}$ the resultING topological space.*

Theorem 3.3.8 *Let $\mathfrak{B}$ be a Boolean algebra, and $\boldsymbol{S}\mathfrak{B}$ its Stone space. Then $\boldsymbol{S}\mathfrak{B}$ is a Boolean space and $\mathfrak{B} \cong \boldsymbol{C}(\boldsymbol{S}\mathfrak{B})$.*

Proof: In order to prove $\mathbf{S}\mathfrak{B}$ is a Boolean space, we need to see it is Hausdorff, compact, and that it admits a clopen basis.

- Given $F, G \in \mathbf{S}\mathfrak{B}$, if $F \neq G$ then without loss of generality there is $x \in F$ such that $x \notin G$. Since G is an ultrafilter, then $x^{\mathrm{c}} \in G$, so $F \in h(x)$ and $G \in h(x^{\mathrm{c}})$. Now $h(x)$ and $h(x^{\mathrm{c}})$ are both open and disjoint sets of $\mathbf{S}\mathfrak{B}$. It follows that $\mathbf{S}\mathfrak{B}$ is a Hausdorff space.

- To prove the compactness of $\mathbf{S}\mathfrak{B}$, it suffices to see that every basic cover of $\mathbf{S}\mathfrak{B}$ has a finite subcover. Let's assume one doesn't: $\{h(x_i) : i \in I\}$. Then, for any finite subset $I_0 \subseteq I$, $\{h(x_i) : i \in I_0\} \neq \mathbf{S}\mathfrak{B}$. But

  $$h(\bigwedge_{i\in I_0} x_i^{\mathrm{c}}) = \bigcap_{i\in I_0} h(x_i^{\mathrm{c}}) = \bigcap_{i\in I_0}(\mathbf{S}\mathfrak{B}\backslash h(x_i)) = \mathbf{S}\mathfrak{B}\backslash \bigcup_{i\in I_0} h(x_i) \neq \emptyset.$$

  Then $\bigwedge_{i\in I_0} x_i^{\mathrm{c}} \neq 0$ for any finite $I_0 \subseteq I$. This means $\{x_i^{\mathrm{c}} : i \in I\}$ has the finite meet property. Accordingly, there is an ultrafilter F such that $\{x_i^{\mathrm{c}} : i \in I\} \subseteq F$. It follows that $x_i^{\mathrm{c}} \in F$, for every $i \in I$, thus $F \notin \bigcup_{i\in I} h(x_i)$, which contradicts the fact that $\{h(x_i) : i \in I\}$ covers $\mathbf{S}\mathfrak{B}$.

- Since $h(x) = \mathbf{S}\mathfrak{B}\backslash h(x^{\mathrm{c}})$, every element of $\mathbf{S}\mathfrak{B}$ is clopen. It is then clear that the topology of $\mathbf{S}\mathfrak{B}$ admits a clopen basis.

Now we have proven that $\mathbf{S}\mathfrak{B}$ is a Boolean space. The previous lemma gives us $h(B) = \mathbf{C}(\mathbf{S}\mathfrak{B})$ and the Stone Representation theorem proves $\mathfrak{B} \cong h(B)$. As a result, $\mathfrak{B} \cong \mathbf{C}(\mathbf{S}\mathfrak{B})$. $\square$

Notice that we are stating that every Boolean algebra can be identified with the basis of a topological space. As a result, we will get that every algebraic property in $\mathfrak{B}$ can be associated with a topological property in $\mathbf{S}\mathfrak{B}$. However interesting this is, we will not go deeper in this matter, since the main goal of this purpose is only to get acquainted with Boolean algebras.

Theorem 3.3.9 *1. If $\mathfrak{B}$ is a finite Boolean algebra, then $\mathfrak{B} \cong \mathcal{P}(\mathbf{S}\mathfrak{B})$. As a consequence, $|B| = 2^{|\mathbf{S}\mathfrak{B}|}$.*

2. Any two finite Boolean algebras $\mathfrak{A}, \mathfrak{B}$ are isomorphic iff they have the same cardinality.

3. For any n, there exists a Boolean algebra of cardinality n iff $n = 2^m$ for some m.

Proof:

1. If $|B| < \omega$, then $|\mathbf{S}\mathfrak{B}| < \omega$ and, since it is a Hausdorff space, it is discrete. Hence, every subset of $\mathbf{S}\mathfrak{B}$ is clopen and we get $\mathbf{C}(\mathbf{S}\mathfrak{B}) = \mathcal{P}(\mathbf{S}\mathfrak{B})$, so $\mathfrak{B} \cong \mathcal{P}(\mathbf{S}\mathfrak{B})$ because of theorem 3.3.8.

2. This follows immediately from (1): if they have the same cardinality, then $|\mathbf{S}\mathfrak{A}| = |\mathbf{S}\mathfrak{B}|$. As a consequence, both Boolean algebras are isomorphic to a power set algebra of sets with the same cardinality, thus they are isomorphic.

3. If such Boolean algebra exists, it is finite. Therefore, it follows from (1) that it is isomorphic to a power set algebra, so its cardinality must be 2^m. Conversely, if we want a Boolean algebra of cardinality 2^m, then we can take a set of cardinality m, and its power algebra set is a Boolean algebra of the required cardinality.

□

Proposition 3.3.10 *Any equation holds in every Boolean algebra iff it holds in every power set algebra.*

Proof: The first implication is trivial. Now assume an equation holds in every power set algebra. Then it holds in any subalgebra of every power set algebra, *i.e.* in any field of subsets. Since every Boolean algebra is isomorphic to a field of subsets, this expression must also hold in every Boolean algebra. □

3.4 Atoms

Definition 3.4.1 *An* atom *in a Boolean algebra is an element x such that $x \neq 0$ and if y <x, then y=0.*

Observation 3.4.2 *x is an atom iff for every $y \in B$ one and only one of the following condition holds: $x \leqslant y$ or $x \leqslant y^{\mathrm{c}}$.*

Observation 3.4.3 *x is an atom iff $x \neq 0$ and, if $x = y \vee z$ where $y \wedge z = 0$, then $y = 0$ or $z = 0$.*

Observation 3.4.4 *We will define a first order formula $At(x)$ with meaning "x is an atom":*

$$x \neq 0 \ \& \ \neg\exists y(y \neq 0 \ \& \ x \wedge y = y).$$

Definition 3.4.5 *Let $\mathfrak{B}$ be a Boolean algebra. It is said that $\mathfrak{B}$ is* atomic *iff for every $y \in B, y \neq 0$, there is $x \in B$ such that x is an atom and $x \leqslant y$.*

Proposition 3.4.6 *If $\mathfrak{B}$ is an atomic Boolean algebra, every element in B is the join of all the atoms it majorizes.*

Proof: Fix $x \in B$ and let A be the set all the atoms in $\mathfrak{B}$ such that $a \leqslant x$ for every $a \in A$. It is obvious that x is an upper bound for A. Let's prove that, if there is $y \in B$ such that y is also an upper bound for A, then $x \leqslant y$. If we assume $x \nleqslant y$, we get $x \wedge y^{\mathrm{c}} \neq 0$, thus there must be an atom $b \in B$ such that $b \leqslant x \wedge y^{\mathrm{c}}$. But $x \wedge y^{\mathrm{c}} \leqslant x$, so $b \in A$, and from this follows that b is below y. Hence, $b \leqslant (x \wedge y^{\mathrm{c}}) \wedge y = x \wedge (y^{\mathrm{c}} \wedge y) = 0$! This gives a contradiction, because if $b = 0$, it cannot be an atom, so we can conclude that $\bigvee A$, exists and $b = \bigvee A$, as required. □

Definition 3.4.7 *A Boolean algebra is said to be* atomless *iff it contains no atoms.*

Observation 3.4.8 *Every non-trivial atomless Boolean algebra is infinite.*

Let's see some examples of atomic and atomless Boolean algebras.

1. Let $\mathcal{L}$ be the language of Propositional logic, and Prop(X) the set of all formulas in $\mathcal{L}$. For any $\phi, \psi \in$Prop(X), we define the equivalence relation $\phi \sim \psi$ iff $\phi \equiv \psi$. Let $|\phi| = \{\psi \in$Prop(x) $: \phi \sim \psi\}$ be the $\sim$-class of ϕ. Then, we can consider the set of all $\sim$- classes B, and, if we define $\wedge, \vee, ^{c}, 0, 1$ in B by:

$$\begin{gathered} |\phi| \wedge |\phi| = |\phi \;\&\; \psi|, \\ |\phi| \vee |\phi| = |\phi \sqcup \psi|, \\ |\phi|^{c} = |\neg\phi|, \\ 1 = |\chi| \text{ for any } \chi \in \text{Prop(X) such that} \vdash \chi, \\ 0 = |\chi| \text{ for any } \chi \in \text{Prop(X) such that} \vdash \neg\chi, \end{gathered}$$

   then $\langle B, \wedge, \vee, ^{c}, 0, 1\rangle$ is a Boolean algebra. Moreover, this Boolean algebra is called the *Lindenbaum-Tarski Boolean algebra* and it is atomless. The order is defined by $|\varphi| \leqslant |\psi| \Leftrightarrow \varphi \models \psi$. To see that this algebra is atomless, it suffices to see that for every $|\varphi| > 0$ there is $|\psi| \neq 0$ such that $|\psi| < |\varphi|$. We only have to take $\psi = p \wedge \varphi$, where p is a propositional variable that does not appear in φ.

2. Let X be an infinite set. Now consider the filter $F = \{A \in \mathcal{P}(X) : |A^{c}| < \omega\}$ in the power set Boolean algebra. Then the quotient $\mathcal{P}(X)/F$ is an atomless Boolean algebra because $[A]_F \neq 0$ iff A is not in the ideal associated to F according to 3.2.8, *i.e.*, A is an infinite set. In this case, there exist $B, C \in \mathcal{P}(X)$, both infinite and disjoint such that $A = B \cup C$ and, as a consequence, $[A]_F = [B]_F \dot{\cup} [C]_F$. This means $[B]_F, [C]_F \subseteq [A]_F$, so $[A]_F$ cannot be an atom, because if they are infinite, $[B]_F, [C]_F \neq 0$.

Observation 3.4.9 *It has been proved in the previous section that every finite Boolean algebra is isomorphic to a power set algebra. It is easy to see that all power set Boolean algebras are atomic and complete Boolean algebras. As a consequence, every finite Boolean algebra is atomic and complete.*

Let's see the converse:

Theorem 3.4.10 *Let $\mathfrak{B}$ be a Boolean algebra. If $\mathfrak{B}$ is atomic and complete, then $\mathfrak{B} \cong \mathcal{P}(X)$ for some set X.*

Proof: Let $A = \{a \in B : a$ is an atom$\}$. Claim: $\mathfrak{B} \cong \mathcal{P}(A)$.
We shall prove the homomorphism needed is:

$$\begin{aligned} f : B &\longrightarrow \mathcal{P}(A) \\ x &\longrightarrow f(x) = \{a \in A : a \leqslant x\}. \end{aligned}$$

In order to prove f is an isomorphism from B onto $\mathcal{P}(A)$ we need to prove f is a bijective homomorphism:

- Let x, y be any two elements in B, and $a \in B$ be an atom.
  $a \in f(x \wedge y)$ iff $a \leqslant x \wedge y$. But $x \wedge y \leqslant x, x \wedge y \leqslant y$, so $a \leqslant x, y$, thus get $a \in f(x), f(y)$. This is equivalent to $a \in f(x) \cap f(y)$.
  $a \in f(x^{\mathrm{c}})$ iff $a \leqslant x^{\mathrm{c}}$ iff $a \nleqslant x$ iff $a \notin f(x)$ iff $a \in \mathcal{P}(A) \backslash f(x)$
  Accordingly, f is an homomorphism

- Let x, y be two different elements in B. Then $x \nleqslant y$ or $y \nleqslant x$. If we assume the latter one, then $x^{\mathrm{c}} \wedge y \neq 0$ and, since $\mathfrak{B}$ is atomic, there is an atom $a \in B$ such that $a \leqslant x^{\mathrm{c}} \wedge y \leqslant x^{\mathrm{c}}, y$. It follows $a \in f(x^{\mathrm{c}}), f(y)$, thus $a \notin f(x), a \in f(y)$. It results $f(x) \neq f(y)$, and f is one-to-one.

- The only thing left to prove is f is surjective. We take $X \in \mathcal{P}(A)$ and, since $\mathfrak{B}$ is complete, we can consider $x = sup(X)$. We claim $X = f(x)$. Trivially, $X \subseteq f(x)$, because for every $a \in X, a \leqslant sup(X) = x$. Now, consider $a \in A \backslash X$. For any $b \in X$, $a \wedge b = 0$, because a is an atom and $a \wedge b \leqslant a$. Hence $b \leqslant a^{\mathrm{c}}$ for any $b \in X$, so $x \leqslant a^{\mathrm{c}}$, which means $a \nleqslant x$. From this follows $a \notin f(x)$ and, as a consequence, $f(x) \subseteq X$.

$\square$

Chapter 4

Model theory

Now that we have become familiar with Boolean algebras and their properties, it is time we learn about concepts like partial isomorphisms, which are a tool that will allow us to go deep in the particularities of these structures.

4.1 Partial isomorphisms

Definition 4.1.1 *Let $\mathcal{M}$,$\mathcal{N}$ be structures of language $\mathcal{L}$. Consider now a one-to-one mapping f: $M \longrightarrow N$ such that dom(f) $\subseteq M$ and rec(f) $\subseteq N$. It is said that f is a* partial isomorphism *between $\mathcal{M}$ and $\mathcal{N}$ iff:*

1. *For every n-ary relation symbol $R \in \mathcal{L}$, for any n-tuple $\overline{a} \in dom(f)$, $\overline{a} \in R^{\mathcal{M}}$ iff $f(\overline{a}) \in R^{\mathcal{N}}$.*
2. *For every n-ary function symbol $F \in \mathcal{L}$, for any n-tuple $\overline{a} \in dom(f)$, and element $b \in dom(f)$, $F^{\mathcal{M}}(\overline{a}) = b$ iff $F^{\mathcal{N}}(f(\overline{a})) = f(b)$.*
3. *For every constant symbol $c \in \mathcal{L}$, $a \in dom(f)$, $c^{\mathcal{M}} = a$ iff $c^{\mathcal{N}} = f(a)$.*

Definition 4.1.2 *Given two $\mathcal{L}$-structures $\mathcal{M}$,$\mathcal{N}$, we denote by $S_0(\mathcal{M},\mathcal{N})$ the set of all partial isomorphisms between $\mathcal{M}$ and $\mathcal{N}$.*

Observation 4.1.3 *The empty set is a partial isomorphism for any two $\mathcal{L}$-structures, $\mathcal{M}$,$\mathcal{N}$. In other words, $\emptyset \in S_0(\mathcal{M},\mathcal{N})$.*

Definition 4.1.4 *Let n be a natural number and $\mathcal{M}$,$\mathcal{N}$ be two $\mathcal{L}$-structures. Now we shall define the set $S_n(\mathcal{M},\mathcal{N})$ of n-isomorphisms between $\mathcal{M}$ and $\mathcal{N}$ by recursion:*
The case $n = 0$ has been defined previously. Now, assuming we have already constructed $S_n(\mathcal{M},\mathcal{N})$, a partial isomorphism f is in $S_{n+1}(\mathcal{M},\mathcal{N})$ iff it satisfies the back-and-forth conditions*:*

- forth condition*: for every $a \in M$ there is $g \in S_n(\mathcal{M},\mathcal{N})$ such that $a \in dom(g)$ and g is an extension of f.*

- back condition*: for every $b \in N$ there is $g \in S_n(\mathcal{M},\mathcal{N})$ such that $b \in rec(g)$ and g is an extension of f.*

Proposition 4.1.5 *Consider n, m such that $0 \leqslant n \leqslant m$. Then $S_m(\mathcal{M},\mathcal{N}) \subseteq S_n(\mathcal{M},\mathcal{N})$.*

Proof: Notice that it suffices to see $S_{n+1}(\mathcal{M},\mathcal{N}) \subseteq S_n(\mathcal{M},\mathcal{N})$ for any n. We shall prove this by induction:

- If $n = 0$, then clearly $S_1(\mathcal{M},\mathcal{N}) \subseteq S_0(\mathcal{M},\mathcal{N})$ because every element in $S_1(\mathcal{M},\mathcal{N})$ is a partial isomorphism.

- Assume now $S_n(\mathcal{M},\mathcal{N}) \subseteq S_{n-1}(\mathcal{M},\mathcal{N})$ and take $f \in S_{n+1}(\mathcal{M},\mathcal{N})$. Then, for each $a \in M$, there is $g \in S_n(\mathcal{M},\mathcal{N})$ such that $a \in$dom(g) and g extends f. Now, by the induction hypothesis, $g \in S_n(\mathcal{M},\mathcal{N})$ implies $g \in S_{n-1}(\mathcal{M},\mathcal{N})$: this means f satisfies the forth condition for n, thus $f \in S_n(\mathcal{M},\mathcal{N})$. The back condition is proved similarly.

□

Observation 4.1.6 *The previous proposition states that:*

$$...S_{n+1}(\mathcal{M},\mathcal{N}) \subseteq S_n(\mathcal{M},\mathcal{N}) \subseteq ... \subseteq S_1(\mathcal{M},\mathcal{N}) \subseteq S_0(\mathcal{M},\mathcal{N}).$$

Definition 4.1.7 *It is said that f is an ω-*isomorphism *iff $f \in S_n(\mathcal{M},\mathcal{N})$ for any $n \geqslant 0$. The set of all ω-isomorphisms between $\mathcal{M}$ and $\mathcal{N}$ is denoted by $S_\omega(\mathcal{M},\mathcal{N})$.*

Observation 4.1.8 *Notice that, if $S_\omega(\mathcal{M},\mathcal{N}) = \emptyset$, then $S_n(\mathcal{M},\mathcal{N}) = \emptyset$ for some n.*

Proof: This is because if $S_n(\mathcal{M},\mathcal{N}) \neq \emptyset$ for every n, then $\emptyset \in S_n(\mathcal{M},\mathcal{N})$, thus $\emptyset \in \bigcap_n S_n(\mathcal{M},\mathcal{N}) = S_\omega(\mathcal{M},\mathcal{N})$. □

Definition 4.1.9 *Let $\mathcal{M},\mathcal{N}$ be two $\mathcal{L}$-structures. It is said that they are* elementarily equivalent*, and it is noted by $\mathcal{M} \equiv \mathcal{N}$, if for every sentence φ, $\mathcal{M} \models \varphi$ iff $\mathcal{N} \models \varphi$.*

Now, there is no reason to stop at ω! We can define $S_\alpha(\mathcal{M},\mathcal{N})$ for any ordinal α as follows:

Definition 4.1.10
- *If α is a limit ordinal, $f \in S_\alpha(\mathcal{M},\mathcal{N})$ iff $f \in S_\beta(\mathcal{M},\mathcal{N})$ for every $\beta < \alpha$.*
- *If $\alpha = \beta + 1$ for some ordinal β, then $f \in S_\alpha(\mathcal{M},\mathcal{N})$ iff for any $a \in M, b \in N$, there are $g, h \in S_\beta(\mathcal{M},\mathcal{N})$ such that $f \subseteq g, h$ and $a \in dom(g)$, $b \in rec(h)$.*

Definition 4.1.11 *$S_\infty(\mathcal{M},\mathcal{N}) = \bigcap_\alpha S(\mathcal{M},\mathcal{N})$.*

Definition 4.1.12 *It is said that two $\mathcal{L}$-structures $\mathcal{M},\mathcal{N}$ are* partially isomorphic *via I iff $I \neq \emptyset$ is a collection of partial isomorphisms between $\mathcal{M}$ and $\mathcal{N}$ such that:*

1. *For any $f \in I, a \in M$ there is $g \in I$ such that g extends f and $a \in dom(g)$.*
2. *For any $f \in I, b \in N$ there is $g \in I$ such that g extends f and $b \in rec(g)$.*

It is noted by I: $\mathcal{M} \cong_p \mathcal{N}$.

Observation 4.1.13 *$S_\infty(\mathcal{M},\mathcal{N}) = S_\alpha(\mathcal{M},\mathcal{N})$ for some ordinal α.*

Proof: Otherwise, the chain

$$... \subsetneq S_{\alpha+1}(\mathcal{M},\mathcal{N}) \subsetneq S_\alpha(\mathcal{M},\mathcal{N}) \subsetneq ... \subsetneq S_0(\mathcal{M},\mathcal{N})$$

would never end, and this is a contradiction with the axiom of replacement and the fact that the family of all ordinals is not a set but a proper class. □

Proposition 4.1.14 *$I : \mathcal{M} \cong_p \mathcal{N}$ for some set I iff $S_\infty(\mathcal{M},\mathcal{N}) \neq \emptyset$.*

Proof: $\Rightarrow$ Assume $I : \mathcal{M} \cong_p \mathcal{N}$, and we will show that, for all ordinals $\alpha, I \subseteq S_\alpha(\mathcal{M},\mathcal{N})$. We will prove this by induction:

- The case $\alpha = 0$ is trivial.
- If $\alpha = \beta + 1$, we want to prove that for every $f \in I$, $f \in S_{\beta+1}(\mathcal{M},\mathcal{N})$. Assume the condition is true for β, and take $a \in M$. We know that there is $g \in I$ such that $f \subseteq g$ and $a \in dom(g)$. By hypothesis, $g \in S_\beta(\mathcal{M},\mathcal{N})$, so $f \in S_{\beta+1}(\mathcal{M},\mathcal{N}) = S_\alpha(\mathcal{M},\mathcal{N})$, as required.
- If α is a limit, then $S_\alpha(\mathcal{M},\mathcal{N}) = \bigcap_{\beta<\alpha} S_\beta(\mathcal{M},\mathcal{N})$. The induction hypothesis is that for every $\beta < \alpha, I \subseteq S_\beta(\mathcal{M},\mathcal{N})$, so the fact that $I \subseteq S_\alpha(\mathcal{M},\mathcal{N})$ follows immediately.

$\Leftarrow$ Assume now $S_\infty(\mathcal{M},\mathcal{N}) = I \neq \emptyset$. We will show $I : \mathcal{M} \cong_p \mathcal{N}$. Fix the minimal ordinal α such that $S_\alpha(\mathcal{M},\mathcal{N}) = S_\infty(\mathcal{M},\mathcal{N})$, and also fix $f \in I, a \in M$. Then $f \in S_{\alpha+1}$, so there is $g \in S_\alpha(\mathcal{M},\mathcal{N}) = I$ such that $f \subseteq g$ and $a \in dom(g)$. From this follows the fact that the forth condition holds, and the back condition is proved similarly. □

Proposition 4.1.15 *1. $\mathcal{M} \cong \mathcal{N} \Rightarrow I : \mathcal{M} \cong_p \mathcal{N}$ for some set I.*

2. Let $\mathcal{M}, \mathcal{N}$ be two countable $\mathcal{L}$-structures such that $\mathcal{M} \cong_p \mathcal{N}$. Then $\mathcal{M} \cong \mathcal{N}$.

Proof:

1. If f is an isomorphism between $\mathcal{M}$ and $\mathcal{N}$, then the set $I = \{f\}$ is a set of partial isomorphisms between $\mathcal{M}$ and $\mathcal{N}$, thus $I : \mathcal{M} \cong_p \mathcal{N}$.

2. This is because, if both structures are countable, we can enumerate them in the following way: $M = \{a_i : i \in \omega\}$ and $N = \{b_i : i \in \omega\}$. We need to construct inductively an ascending chain of partial isomorphisms. Take $f_0 \in I$ arbitrary. Then, assuming we have already constructed f_n, we can take $f'_n \in I$ such that $f_n \subseteq f'_n$ and $a_n \in dom(f'_n)$. Now have $f_{n+1} \in I$ such that $f'_n \subseteq f_{n+1}$ with $b_n \in rec(f_{n+1})$. Now that we have f_n defined for every $n \in \omega$, we can take $f = \bigcup_n f_n$, and this is a mapping such that $dom(f) = M$ and $rec(f) = N$: it is an isomorphism between $\mathcal{M}$ and $\mathcal{N}$.

□

Definition 4.1.16 *Let f be a partial isomorphism between $\mathcal{M}$ and $\mathcal{N}$ such that $f \in S_n(\mathcal{M}, \mathcal{N})$ but $f \notin S_{n+1}(\mathcal{M}, \mathcal{N})$ for some n. Then, we say the* Fraïssé rank of *f is n. However, if $f \in S_\omega(\mathcal{M}, \mathcal{N})$, then the Fraïssé rank of f is equal or greater than ω.*

Definition 4.1.17 *Two n-tuples $\overline{a} = (a_1, ..., a_n)$ and $\overline{b} = (b_1, ..., b_n)$ are said to be* m -equivalent *iff there is an m-isomorphism f such that f(a_i)=b_i for every $i, 1 \leqslant i \leqslant n$. It is written $(\overline{a}, \mathcal{M}) \sim_m (\overline{b}, \mathcal{N})$. If there is an ω-isomorphism between them, they are said to be* ω-equivalent*.*

The term can also be applied to structures and not only to tuples: it is said $\mathcal{M}$ and $\mathcal{N}$ are $n - equivalent$ iff there is a $n - isomorphism$ between them. This can also be seen as if we were considering $\emptyset$ as a tuple.

Definition 4.1.18 *Let $\mathcal{M}, \mathcal{N}$ be $\mathcal{L}$-structures. It is said that $\mathcal{M}$ is an* extension *of $\mathcal{N}$, or $\mathcal{N} \subseteq \mathcal{M}$, iff:*

1. *$N \subseteq M$.*
2. *For every constant symbol c, $c^{\mathcal{M}} = c^{\mathcal{N}}$.*
3. *For every n-ary function symbol F, every n-tuple $\overline{a} \in N$, $F^{\mathcal{M}}(\overline{a}) = F^{\mathcal{N}}(\overline{a})$.*
4. *For every n-ary relation symbol R, $R^{\mathcal{N}} = R^{\mathcal{M}} \cap N^n$.*

An extension is said to be elementary *iff for every $\varphi(x_1, ..., x_n)$, and every $a_1, ..., a_n \in M$, $\mathcal{M} \models \varphi(a_1, ..., a_n)$ iff $\mathcal{N} \models \varphi(a_1, ..., a_n)$. This is noted by $\mathcal{M} \preceq \mathcal{N}$.*

Observation 4.1.19 *If $\mathcal{M} \preceq \mathcal{N}$, then $\mathcal{M} \equiv \mathcal{N}$.*

Definition 4.1.20 *Let $\mathcal{L}$ be a language. Then we define the* quantifier rank *of a formula as follows:*

- *For every variable x, $QR(x) = 0$.*
- *For every constant c, $QR(c) = 1$.*
- *For every m-ary function symbol F, $QR(Ft_1...t_m) = 1 + \Sigma_{1 \leqslant i \leqslant m} QR(t_i)$.*
- *For every equation $t_1 = t_2$, where t_1, t_2 are terms in $\mathcal{L}$, $QR(t_1 = t_2) = QR(t_1) + QR(t_2) - 1$*
- *For every m-ary relation symbol R, $QR(Rt_1...t_m) = \Sigma_{1 \leqslant i \leqslant m} QR(t_i)$.*
- *If φ is a formula in $\mathcal{L}$, then $QR(\neg\varphi) = QR(\varphi)$.*
- *If φ, ψ are formulas in $\mathcal{L}$, then $QR(\varphi \sqcup \psi) = QR(\varphi \& \psi) = max\{QR(\varphi), QR(\psi)\}$.*
- *If φ is a formula in $\mathcal{L}$ and x is a variable, $QR(\forall x \varphi) = QR(\exists x \varphi) = QR(\varphi) + 1$.*

Theorem 4.1.21 ***Fraïssé's theorem***
Let $\mathcal{M}$ and $\mathcal{N}$ be two $\mathcal{L}$-structures. Then, the tuples $\overline{a} = (a_1, ..., a_n) \in M^n$ and $\overline{b} = (b_1, ..., b_n) \in N^n$ are m-equivalent iff they satisfy the same formulas ($\overline{a}$ in $\mathcal{M}$ and $\overline{b}$ in $\mathcal{N}$) with quantifier rank at most m.

Proof: See [4]. Lemma 3.2. □

Corollary 4.1.22 *Any two $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$ are elementarily equivalent iff $S_\omega(\mathcal{M}, \mathcal{N}) \neq \emptyset$.*

4.2 Complete and ω-categorical theories. Quantifier elimination

Let T be a complete theory in the language $\mathcal{L}$ and $\mathcal{M}$ a model of T. Fix $n \geqslant 1$.

Definition 4.2.1 *An n-*type over $\emptyset$ *is a set of formulas $\Sigma(x_1, ..., x_n) = \{\varphi_i(x_1, ..., x_n) : i \in I\}$ such that $T \cup \Sigma$ is consistent (or Σ is consistent with T).*

Observation 4.2.2 *An n-type Σ is complete iff it is maximal with respect to the inclusion among n-types. An equivalent condition is that, for any formula of $\mathcal{L}$, $\varphi(x_1, ..., x_n)$, $\varphi(x_1, ..., x_n) \in \Sigma$ or $\neg\varphi(x_1, ..., x_n) \in \Sigma$.*

Consider now $A \subseteq M$, a set of parameters. It is clear that the language $\mathcal{L}$ can be expanded to form $\mathcal{L}(A)$=$\mathcal{L} \cup \{c_a : a \in A\}$. This is useful whenever we want to name the elements in A, because c_a are not in the language $\mathcal{L}$. Similarly, the model $\mathcal{M}$ is expanded to the $\mathcal{L}(A)$-structure $(\mathcal{M}a)_{a\in A}$, where $a = c_a^{(\mathcal{M}a)_{a\in A}}$. We will note $T(A) = Th((\mathcal{M}a)_{a\in A})$.

Definition 4.2.3 *An n-*type over A *is an n-type of $T(A)$ over $\emptyset$.*

Observation 4.2.4 *An n-type over A will have in its formulas constants to refer to the elements in A. They are often written like $c_a = a$.*

Observation 4.2.5 *A complete n-type over A is defined in the natural way. We define the set $S_n(A)$ as $S_n(A) = \{p(x_1, ..., x_n) : p(x_1, ..., x_n)$ is a complete n-type over $A\}$*

Let $\Sigma(x_1, ..., x_n)$ be an n-type over A.

Definition 4.2.6 *An n-tuple $(a_1, ..., a_n) \in M^n$ realizes Σ if $\mathcal{M} \models \varphi(a_1, ..., a_n)$ for every $\varphi \in \Sigma$.*

Definition 4.2.7 *Fix $(a_1, ..., a_n) \in M^n$. The* type *of $(a_1, ..., a_n)$ over A is the set $\{\varphi(x_1, ..., x_n) \in \mathcal{L}(A) : \mathcal{M} \models \varphi(a_1, ..., a_n)\}$. It is noted $tp(a_1, ..., a_n/A)$.*

Observation 4.2.8 *$tp(a_1, ..., a_n/A)$ is an n-type over A for any $a_1, ..., a_n \in M$. Furthermore, $(a_1, ..., a_n)$ realizes $tp(a_1, ..., a_n/A)$ in $\mathcal{M}$.*

Definition 4.2.9 *Let f be a mapping between two $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$. This mapping is said to be* elementary *iff, for any $a_1, ..., a_n \in dom(f), \varphi = \varphi(x_1, ..., x_n)$, $\mathcal{M} \models \varphi(a_1, ..., a_n)$ iff $\mathcal{N} \models \varphi(f(a_1), ..., f(a_n))$.*

Definition 4.2.10 *Let $\mathcal{M}$ be a structure in the language $\mathcal{L}$. We can define $\mathcal{L}(\mathcal{M})$ as the language obtained by adding to $\mathcal{L}$ a constant symbol c_a for every element a of M. The structure $\mathcal{M}$ can be viewed as an $\mathcal{L}(\mathcal{M})$-structure in which the symbols in $\mathcal{L}$ are interpreted as before, and each new constant c_a is interpreted as the element a. The* elementary diagram *of $\mathcal{M}$ is the set of all $\mathcal{L}(\mathcal{M})$ sentences that are true in $\mathcal{M}$. It is written $Diag_{el}(\mathcal{M})$.*

Proposition 4.2.11 *In general, it is not true that every n-type over A is realized in $\mathcal{M}$, but every n-type over A is realized in some elementary extension of $\mathcal{M}$.*

Proof: Consider an $\mathcal{L}$-structure $\mathcal{M}$, $A \subseteq M$ and p, a type over A. Let's define $T = p \cup Diag_{el}(\mathcal{M})$, and we will show that it is consistent. In order to see this, we will show that any finite subset of T is consistent. A finite subset of T can be reduced to $\Delta = \{\varphi_1, ..., \varphi_k, \psi_1, ..., \psi_n\}$, where $\varphi_i \in Diag_{el}(\mathcal{M})$ for every $i = 1, ..., k$ and $\psi_j \in p$ for every $j = 1, ..., n$. Since p is a type over A, it is known that there is an $\mathcal{L}(A)$-structure $\mathcal{N}$ such that $\mathcal{N} \models T(A) \cup p$. In particular, $\mathcal{N} \models \psi_j$ for every $j = 1, .., n$. Let's see that $\mathcal{N} \models \varphi_i$, too.
We can name $\psi = \psi_1 \wedge ... \wedge \psi_k$, and write ψ as an $\mathcal{L}(A)$-formula by $\psi(\overline{b})$, where $\overline{b}$ is a tuple of parameters that are in $M \backslash A$. Since $\mathcal{M} \models \psi(\overline{b})$, then $\mathcal{M} \models \exists x \psi(\overline{x})$ and, by the choice of $\mathcal{N}$, there must be a tuple $\overline{c} \in N$ such that $\mathcal{N} \models \psi(\overline{c})$. Now, by interpreting the $\mathcal{L}_{\mathcal{M}}$-symbols $\overline{b}$ as the elements $\overline{c}$, we can consider $\mathcal{N}$ as an $\mathcal{L}_{\mathcal{M}}$-structure which satisfies: $\mathcal{N} \models \varphi_1 \wedge ... \wedge \varphi_k \wedge \psi_1 \wedge ... \wedge \psi_n$. Hence, T is consistent, so we know that there is an $\mathcal{L}_{\mathcal{M}}$-structure $\mathcal{M}'$ which satisfies T. $\mathcal{M}'$ interprets a symbol for each element of $\mathcal{M}$, so $\mathcal{M}$ is naturally embedded into $\mathcal{M}'$. Furthermore, since $\mathcal{M}'$ satisfies $Diag_{el}(\mathcal{M})$, this embedding is elementary, thus we get $\mathcal{M} \preceq \mathcal{M}'$, as required. □

Definition 4.2.12 *Let T be a complete theory, and $\mathcal{M}$ a model of T. $\mathcal{M}$ is said to be* ω-saturated *iff, for every finite $A \subseteq M$, every $p \in S_1(A)$ is realized in $\mathcal{M}$.*

Observation 4.2.13 *It is a fact that not only the types in $S_1(A)$ are realized in an ω-saturated structure: for every $n \in \omega$, the types in $S_n(A)$ are realized in $\mathcal{M}$, too. Besides, if $\mathcal{M}$ is ω-saturated, every n-type over any $A \subseteq M$ is realized in $\mathcal{M}$, not only the complete ones.*

Observation 4.2.14 *T is a complete theory iff every pair of models of T are elementarily equivalent.*

Definition 4.2.15 *A theory T is* ω-categorical *iff it has a denumerable model and any two denumerable models $\mathcal{M}, \mathcal{N}$ of T are isomorphic.*

Proposition 4.2.16 *Every structure $\mathcal{M}$ has an elementary extension $\mathcal{N}$ that is ω-saturated.*

Proof: We will not give every precise detail of this proof, but a sketch. For further information, see [9], theorem 5.1. We will construct an *elementary chain*: a chain of structures such that $\mathcal{M}_i \preceq \mathcal{M}_{i+1}$ for every $i \in \mathbb{N}$.
Take $\mathcal{M}_0 = \mathcal{M}$ and, assuming we have $\mathcal{M}_n$, we will take $\mathcal{M}_{n+1}$ to be a structure such that, for any finite $A \subseteq M_n$ and every $p \in S_1(A)$, there is some $a \in M_{n+1}$ such that a realizes p in $\mathcal{M}_{n+1}$. We know that such $\mathcal{M}_{n+1}$ exists because every type is realized in some elementary extension of its structure. Then, if we consider $\mathcal{N} = \bigcup_n \mathcal{M}_n$, $\mathcal{M}_n \preceq \mathcal{N}$ for every $n \in \mathbb{N}$, and it is ω-saturated. □

Proposition 4.2.17 *Let T be a theory in the language $\mathcal{L}$, and let $\Sigma(x_1, ..., x_n) \neq \emptyset$ be a set of formulas such that any two n-tuples of models of T are ω-equivalent iff they satisfy the same formulas in Σ. We will write $\overline{x} = (x_1, ..., x_n)$. Then, for every $\varphi(\overline{x})$, there is a Boolean combination of formulas in Σ, $\psi(\overline{x})$, such that $T \models \forall \overline{x}(\psi(\overline{x}) \leftrightarrow \varphi(\overline{x}))$.*

Proof: See [9]. Theorem 5.2. □

Observation 4.2.18 *The converse is immediate.*

Definition 4.2.19 *Under the conditions of the latter proposition, if we can take as Σ the set of atomic formulas, we say T admits* quantifier elimination.

Theorem 4.2.20 *Let T be a theory. Then*

1. *T is complete iff for any $\mathcal{M}, \mathcal{N}$ ω-saturated models of T there exists a set I such that $I : \mathcal{M} \cong_p \mathcal{N}$.*
2. *T is complete and admits quantifier elimination iff for any $\mathcal{M}, \mathcal{N}$ ω-saturated models of T, $I : \mathcal{M} \cong_p \mathcal{N}$ where I is the set of all partial isomorphisms between $\mathcal{M}$ and $\mathcal{N}$ whose domain is a finitely generated substructure of $\mathcal{M}$.*

Now assume the language $\mathcal{L}$ of the theory is denumerable. Then:

3. *T is complete and ω-categorical iff for any $\mathcal{M}, \mathcal{N}$ models of T there exists a set I such that $I : \mathcal{M} \cong_p \mathcal{N}$.*
4. *T is complete, ω-categorical and admits quantifier elimination iff for any $\mathcal{M}, \mathcal{N}$ models of T, $I : \mathcal{M} \cong_p \mathcal{N}$ where I is the set of all partial isomorphisms between $\mathcal{M}$ and $\mathcal{N}$ whose domain is a finitely generated substructure of $\mathcal{M}$.*

The proof of this theorem requires many previous results, so we will only give a sketch of the proof of (1) and, if the readers want further information, they may consult [2].

Lemma 4.2.21 *Let $\mathcal{M}, \mathcal{N}$ be two ω-saturated $\mathcal{L}$-structures. If $\mathcal{M} \equiv \mathcal{N}$ then $I : \mathcal{M} \cong_p \mathcal{N}$ for some set I.*

Proof: Assume $\mathcal{M} \equiv \mathcal{N}$ and consider $I = \{f : f \in S_0(\mathcal{M}, \mathcal{N})$ elementary with a finite domain $\}$. Claim: $I : \mathcal{M} \cong_p \mathcal{N}$.
First notice that $I \neq \emptyset$ because, since $\mathcal{M} \equiv \mathcal{N}, \emptyset \in I$. Now, take $f \in I$, $a \in M$. We can name $A = dom(f) \subseteq M$, and we know it is finite, thus $p(x) = tp(a/A)$ is a type over A. Then $q(x) = p(x)^f = \{\varphi(x, f(a_1), ..., f(a_n)) : \varphi(x, a_1, ..., a_n) \in p(x)\}$ is a type over $B = rec(f)$ and, because of the ω-saturation of $\mathcal{N}$, there is $b \in N$

such that b realizes $q(x)$. As a consequence, $g = f \cup \{(a, b)\}$ extends f and verifies the forth condition. For symmetry, it will also verify the back condition. $\square$

It is obvious that the converse is true, because every ∞-isomorphism is an ω-isomorphism.

Proposition 4.2.22 *Let T be a theory. T is complete iff for any $\mathcal{M}, \mathcal{N}$ ω-saturated models of T there exists a set I such that $I : \mathcal{M} \cong_p \mathcal{N}$.*

Proof: Assume $\mathcal{M}, \mathcal{N}$ are two ω-saturated models of the complete theory T. Hence $\mathcal{M} \equiv \mathcal{N}$ and, because of the previous lemma, there is $I \neq \emptyset$ such that $I : \mathcal{M} \cong_p \mathcal{N}$. Conversely, let us take $\mathcal{M}, \mathcal{N}$, two models of T. In order to see T is complete, it suffices to see that they are elementarily equivalent. Now, consider $\mathcal{M}', \mathcal{N}'$ two ω-saturated models of T such that $\mathcal{M} \preceq \mathcal{M}'$ and $\mathcal{N} \preceq \mathcal{N}'$. Then $\mathcal{M} \equiv \mathcal{M}' \equiv \mathcal{N}' \equiv \mathcal{N}$ by hypothesis, and we can conclude that T is complete. $\square$

Now that we have proven (1), one of the implications in (3) is also easy to see: if $\mathcal{L}$ is a denumerable language and for any $\mathcal{M}, \mathcal{N}$ models of T there exists a set I such that $I : \mathcal{M} \cong_p \mathcal{N}$, then T is complete and ω-categorical. It is complete because any ω-saturated models of T are partially isomorphic via some set I, and it is ω-categorical because of proposition 4.1.15.

Chapter 5

Some examples

5.1 Finitely generated Boolean algebras

The purpose of this chapter is to study the theories of atomless Boolean algebras and of infinite atomic Boolean algebras. This will give us some clues on how to complete the theory of Boolean algebras in particular cases and will serve as an example of how to prove quantifier elimination.

Definition 5.1.1 *The set 2^n is the set of all mappings from $\{0, 1, ..., n-1\}$ to $\{0, 1\}$.*

It is obvious that 2^n is also a natural number, but the context is enough to distinguish both different meanings.

Definition 5.1.2 *Let $\mathfrak{A}$ be a Boolean algebra, $\varepsilon \in 2^n$ and $\overline{a} = (a_1, ..., a_n)$ be an n-tuple in A. Then $\varepsilon\overline{a} = \varepsilon(0)a_1 \wedge ... \wedge \varepsilon(n-1)a_n$, where $0a_i := a_i^c$ and $1a_i := a_i$. $\varepsilon\overline{a}$ is called a* bit *of $\overline{a}$.*

Observation 5.1.3 *Let $\mathfrak{A} = \langle a_1, ..., a_n \rangle$ be a Boolean algebra. For every a in $\mathfrak{A}$, $a \neq 0$, there exist $\varepsilon_1, ..., \varepsilon_k \in 2^n$ such that $a = \varepsilon_1\overline{a} \vee ... \vee \varepsilon_k\overline{a}$ where $\overline{a} = (a_1, ..., a_n)$.*

Proof: This follows immediately from the fact that every element in A can be written in its disjunctive normal form. □

Observation 5.1.4 *Any finitely generated Boolean algebra is finite.*

Proof: The proof follows trivially from 5.1.3. □

Observation 5.1.5 *$\varepsilon\overline{a}$ is equal to 0 or it is an atom.*

Proof: Let's assume $\varepsilon\overline{a}$ is neither 0 or an atom. Then there is b in $\mathfrak{A}$ such that $0 < b < \varepsilon\overline{a}$, and $b = \varepsilon_1\overline{a} \vee ... \vee \varepsilon_k\overline{a}$ for some $\varepsilon_1, ..., \varepsilon_k$ because of 5.1.3. Since $b \neq 0$,

$\varepsilon_i\overline{a} \neq 0$ for some $i \in \{1,...,k\}$, and $\varepsilon_i\overline{a} \leqslant b \leqslant \varepsilon\overline{a}$. Then $\varepsilon_i = \varepsilon$ follows from $\varepsilon_i\overline{a} \leqslant \varepsilon\overline{a}$, because if there was j such that $\varepsilon_i(j) \neq \varepsilon(j)$, for example, $\varepsilon(j) = 0$ and $\varepsilon_i(j) = 1$, we could consider $\varepsilon_i\overline{a} = \varepsilon_i\overline{a} \wedge a_j \leq \varepsilon\overline{a} \wedge a_j = 0$, due to $a_j^{c} \leqslant \varepsilon\overline{a}$. In this case, we would get $\varepsilon_i\overline{a} = 0$, which gives us a contradiction! □

Observation 5.1.6 *For any atom x in $\mathfrak{A} = \langle a_1, ..., a_n\rangle$, x=$\varepsilon\overline{a}$ for some $\varepsilon \in 2^n$.*

Proof: Let x be an atom. By observation 5.1.3, $x = \varepsilon_1\overline{a} \vee ... \vee \varepsilon_k\overline{a}$. Notice that for every $i = 1, ..., k, \varepsilon_i\overline{a} \leqslant x$ so, since x is an atom, for every $i = 1, ...k, \varepsilon_i\overline{a} = 0$ or $\varepsilon_i\overline{a} = x$. We know $\varepsilon_i\overline{a} \neq 0$ for some i, because otherwise we would get $x = 0$, thus $\varepsilon_i\overline{a} = x$, as required. □

Lemma 5.1.7 *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two finite Boolean algebras with atoms $a_1, .., a_n$ and $b_1, ..., b_n$ $(a_i \neq a_j, b_i \neq b_j$ if $i \neq j)$. There exists $f : \mathfrak{A} \longmapsto \mathfrak{B}$ an isomorphism such that $f(a_i) = b_i$ for every $i = 1, ..., n$.*

Proof: Notice that, since $\mathfrak{A}, \mathfrak{B}$ are finite, they are both atomic, and so they are generated by their atoms: $\mathfrak{A} = \langle a_1, ..., a_n\rangle$ and $\mathfrak{B} = \langle b_1, ..., b_n\rangle$. Consider now:

$$\begin{array}{rcl} f : A & \longrightarrow & B \\ a = \bigvee_{1\leqslant i\leqslant l} a_i & \longrightarrow & f(a) = \bigvee_{1\leqslant i\leqslant l} b_i \end{array}$$

It is clear that this mapping verifies $f(a_i) = b_i$ for every $i = 1, ..., n$. Let's see that it is an isomorphism between Boolean algebras:

- It is easy to see that f preserves $\vee$, because if x majorizes $a_1, ..., a_k$ and y majorizes $a'_1, ..., a'_k$, then $x \vee y$ majorizes $a_1, ..., a_k, a'_1, ..., a'_k$.
- $f(a^{c}) = \bigvee_{j\in J} b_j$, where $a_j \leqslant a^{c}$ for every $j \in J$. Then, since $1_{\mathfrak{B}} = \bigvee_{j\in J} b_j \vee \bigvee_{j\notin J} b_j$ and $1_{\mathfrak{A}} = a^{c} \vee a = \bigvee_{j\in J} a_j \vee \bigvee_{j\notin J} a_j$, $f(a)^c = (\bigvee_{j\notin J} b_j)^c = \bigvee_{j\in J} b_j = f(a^{c})$.
- It is surjective because $\mathfrak{B}$ is atomic.
- It is one-to-one because if two elements are different in an atomic algebra, they majorize different atoms.

□

Proposition 5.1.8 *Let $\mathfrak{A} = \langle a_1, ..., a_n\rangle$ and $\mathfrak{B} = \langle b_1, ..., b_n\rangle$. Let's assume that for every $\varepsilon \in 2^n, \varepsilon\overline{a} = 0 \Leftrightarrow \varepsilon\overline{b} = 0$. Then there is an isomorphism $f : \mathfrak{A} \longmapsto \mathfrak{B}$ such that $f(a_i) = b_i$ for every $i = 1, ..., n$.*

Proof: This follows from the fact that the atoms in $\mathfrak{A}$ are $\varepsilon\overline{a}$, with $\varepsilon \in 2^n$ such that $\varepsilon\overline{a} \neq 0$, and the same for the atoms in $\mathfrak{B}$. As a consequence, $\mathfrak{A}$ and $\mathfrak{B}$ have the same number of atoms and there is f such that $f(\varepsilon_i\overline{a}) = \varepsilon_i\overline{b}$ because of the previous lemma. Now we need to prove that $f(a_i) = b_i$, but this results from the fact that the atoms majorized by a_i are the elements $\varepsilon\overline{a} \neq 0$ such that $\varepsilon(i-1) = 1$, because then $f(\varepsilon\overline{a}) = \varepsilon\overline{b}$, and the atoms below $\varepsilon\overline{b}$ also verify $\varepsilon(i-1) = 1$, and these are the ones that are are majorized by b_i. This added to the fact that an homomorphism between Boolean algebras preserves $\vee$, gives us $f(a_i) = b_i$. □

Observation 5.1.9 *The converse of 5.1.8 is also true.*

5.2 Atomless Boolean algebras

The axioms of the theory of atomless Boolean algebras are the universal quantification of the following formulas:

$$\begin{array}{cc} x \wedge y = y \wedge x, & x \vee y = y \vee x, \\ x \vee (y \vee z) = (x \vee y) \vee z, & x \wedge (y \wedge z) = (x \wedge y) \wedge z, \\ (x \wedge y) \vee y = y, & (x \vee y) \wedge y = y, \\ x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), & x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z), \\ x \wedge x^{\mathrm{c}} = 0, & x \vee x^{\mathrm{c}} = 1, \\ \neg At(x), & 0 \neq 1. \end{array}$$

Recall that $At(x)$ is $x \neq 0 \ \& \ \neg\exists y(y \neq 0 \ \& \ x \wedge y = y)$. We only add the axiom $0 \neq 1$ in order to exclude the trivial algebra as a model of the theory.

Theorem 5.2.1 *The theory of atomless Boolean algebras is complete, ω-categorical and admits quantifier elimination in the language $\mathcal{L} = \{\wedge, \vee, ^{\mathrm{c}}, 0, 1\}$.*

Proof: Due to theorem 4.2.20, it suffices to see that for any $\mathfrak{A}, \mathfrak{B}$ models of T, $I : \mathfrak{A} \cong_p \mathfrak{B}$, where I is the set of all partial isomorphisms from $\mathfrak{A}$ to $\mathfrak{B}$ whose domain is a finitely generated substructure of $\mathfrak{A}$. It is obvious that $I \neq \emptyset$, because we can consider the algebra $\{0, 1\}$ and the mapping f that sends $0_{\mathfrak{A}}$ to $0_{\mathfrak{B}}$ and $0_{\mathfrak{A}}$ to $1_{\mathfrak{B}}$.
Let us take $f \in I$, an isomorphism between $\mathfrak{A}_0 = \langle a_1, ..., a_n \rangle$ and $\mathfrak{B}_0 = \langle b_1, ..., b_n \rangle$, subalgebras of $\mathfrak{A}$ and $\mathfrak{B}$, respectively, in a way such that $f(a_i) = b_i$ for every $i = 1, ..., n$. The only thing we need to prove is that the back-and-forth conditions hold. Furthermore, we only need to prove the forth condition because the back condition will also hold for symmetry.
Consider $a \in A$. What we need to find is $g \in I$ such that $f \subseteq g$ and g is an isomorphism between $\langle a_1, ..., a_n, a \rangle$ and $\langle b_1, ..., b_n, b \rangle$ for some $b \in B$. Now, because of proposition 5.1.8, it suffices to see that for every $\varepsilon \in 2^{n+1}$, $\varepsilon\overline{a}' = 0$ iff $\varepsilon\overline{b}' = 0$ (where $\overline{a}' = (a_1, ..., a_n, a)$, and $\overline{b}' = (b_1, ..., b_n, b)$). This is equivalent to see that, for every $\varepsilon \in 2^n$,

$$\varepsilon\overline{a} \wedge a = 0 \Leftrightarrow \varepsilon\overline{b} \wedge b = 0,$$
$$\varepsilon\overline{a} \wedge a^{\mathrm{c}} = 0 \Leftrightarrow \varepsilon\overline{b} \wedge b^{\mathrm{c}} = 0,$$

where $\overline{a} = (a_1, ..., a_n)$ and $\overline{b} = (b_1, ..., b_n)$. Since $\mathfrak{A}_0 \cong \mathfrak{B}_0$, we know $\varepsilon\overline{a} = 0 \Leftrightarrow \varepsilon\overline{b} = 0$. As a consequence, we only need to find b that satisfies the previous conditions for every $\varepsilon \in 2^n$ that verifies $\varepsilon\overline{a} \neq 0$, because otherwise the result is immediate. Let $\varepsilon_1, ..., \varepsilon_l$ be the only elements ε in 2^n such that $\varepsilon\overline{a} \neq 0$. Now, for every i, $1 \leqslant i \leqslant l$, we define c_i in the following way:

1. If $\varepsilon_i\overline{a} \wedge a = 0$, we will take $c_i = 0$.

2. If $\varepsilon_i\overline{a} \wedge a^{\mathrm{c}} = 0$, we will take $c_i = \varepsilon_i\overline{b}$.

3. If none of the previous is true we will do the following: since $\mathfrak{B}$ is an atomless Boolean algebra and $\varepsilon_i\overline{b} \neq 0$, $\varepsilon_i\overline{b} = x \vee y$ for some $x, y \neq 0$, such that $x \wedge y = 0$. Then we take $c_i = x$.

Claim: we can take $b = \bigvee_{1\leqslant i\leqslant l} c_i$. Fix i such that $1 \leqslant i \leqslant l$.

- Assume $\varepsilon_i\overline{a} \wedge a = 0$. We know that $\varepsilon_i\overline{b} \wedge b = \varepsilon_i\overline{b} \wedge (\bigvee_{1\leqslant j\leqslant l} c_j) = \bigvee_{1\leqslant j\leqslant l}(\varepsilon_i\overline{b} \wedge c_j)$. In order to see that this expression equals 0, we will show that $\varepsilon_i\overline{b} \wedge c_j = 0$ for every j. This is true for $j = i$, because we have chosen $c_i = 0$. For $i \neq j$, it suffices to realise $c_j \leqslant \varepsilon_j\overline{b}$ implies $\varepsilon_i\overline{b} \wedge c_j \leqslant \varepsilon_j\overline{b} \wedge \varepsilon_i\overline{b} = 0$.

- Now assume $\varepsilon_i\overline{a} \wedge a \neq 0$. Notice that this is only possible in the second or the third case of the definition of c_i, and that we want to see $\varepsilon_i\overline{b} \wedge b \neq 0$. But $\varepsilon_i\overline{b} \wedge b = \bigvee_{1\leqslant j\leqslant l}(\varepsilon_i\overline{b} \wedge c_j)$, thus if we prove $\varepsilon_i\overline{b} \wedge c_j \neq 0$ for some j, we will get what was required. Now, remember that in case (2) $c_i = \varepsilon_i\overline{b}$, and in case (3), $c_i = x_i \leqslant \varepsilon_i\overline{b}$, so $c_i \wedge \varepsilon_i\overline{b} = c_i \neq 0$ follows.

- Assume $\varepsilon_i\overline{a} \wedge a^{\mathrm{c}} = 0$. We want to show $\varepsilon_i\overline{b} \wedge b^{\mathrm{c}} = 0$, but $\varepsilon_i\overline{b} \wedge b^{\mathrm{c}} = \epsilon_i\overline{b} \wedge (\bigvee_{1\leqslant j\leqslant l} c_j)^{\mathrm{c}} = \varepsilon_i\overline{b} \wedge \bigwedge_{1\leqslant j\leqslant l} c_j^{\mathrm{c}} = (\varepsilon_i\overline{b} \wedge c_i^{\mathrm{c}}) \wedge \bigwedge_{1\leqslant j\leqslant l, i\neq j} c_j^{\mathrm{c}} = (\varepsilon_i\overline{b} \wedge \varepsilon_i\overline{b}^{\mathrm{C}}) \wedge \bigwedge_{1\leqslant j\leqslant l, i\neq j} c_j^{\mathrm{c}} = 0$, using that the only case possible is the second one and this means $c_i = \varepsilon_i\overline{b}$.

- Assume now $\varepsilon_i\overline{a} \wedge a^{\mathrm{c}} \neq 0$. This is only possible in cases (1) or (3).
  Now, in case (1), we have already proved that $\varepsilon_i\overline{a} \wedge a = 0$ implies $\varepsilon_i\overline{b} \wedge b = 0$, thus it is not possible that $\varepsilon_i\overline{b} \wedge b^{\mathrm{c}} = 0$, because if both of them were equal to 0, we would get $\varepsilon_i\overline{b} = 0$. As a consequence, $\varepsilon_i\overline{b} \wedge b^{\mathrm{c}} \neq 0$.
  If case (3) were to happen, then $\varepsilon_i\overline{b} \wedge (\bigvee_{1\leqslant j\leqslant l} c_j)^{\mathrm{c}} = \bigvee_{1\leqslant j\leqslant l}(\varepsilon_i\overline{b} \wedge (c_j^{\mathrm{c}} \wedge \varepsilon_j\overline{b}))$. Remember in case (3), $c_i = x_{ji} \leqslant \varepsilon_i\overline{b}$ and $c_i^{\mathrm{c}} \wedge \varepsilon_i\overline{b} = y_i \leqslant \varepsilon_i\overline{b}$. As a consequence, $0 \neq y_i = \varepsilon_i\overline{b} \wedge (c_j^{\mathrm{c}} \wedge \varepsilon_j\overline{b}) \leqslant \bigvee_{1\leqslant j\leqslant l}(\varepsilon_i\overline{b} \wedge (c_j^{\mathrm{c}} \wedge \varepsilon_j\overline{b}))$. Hence, $\bigvee_{1\leqslant j\leqslant l}(\varepsilon_i\overline{b} \wedge (c_j^{\mathrm{c}} \wedge \varepsilon_j\overline{b})) = \varepsilon_i\overline{b} \wedge b^{\mathrm{c}} \neq 0$, as we wanted to prove.

$\square$

5.3 Atomic and infinite Boolean algebras

The axioms of the theory of atomic and infinite Boolean algebras are the universal quantification of the following formulas:

$$\begin{array}{cc}
x \wedge y = y \wedge x, & x \vee y = y \vee x, \\
x \vee (y \vee z) = (x \vee y) \vee z, & x \wedge (y \wedge z) = (x \wedge y) \wedge z, \\
(x \wedge y) \vee y = y, & (x \vee y) \wedge y = y, \\
x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), & x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z), \\
x \wedge x^{\mathrm{c}} = 0, & x \vee x^{\mathrm{c}} = 1, \\
\forall x (x \neq 0 \rightarrow \exists y (y \wedge x = y \ \&\ At(y))), & \{\exists x_1, ..., x_n (\&_{1 \leqslant i < j \leqslant n} \neg x_i = x_j)\}_{n < \omega}.
\end{array}$$

We will expand the usual language in order to prove that the theory is complete and admits quantifier elimination. We will add the predicates $A_m(x)$ for every $m \in \omega$, which mean "x is greater that, at least, m atoms":

$$\forall x (A_m(x) \Leftrightarrow \exists y_1, ..., y_m (\&_{1 \leqslant i < j \leqslant m}\ y_i \neq y_j\ \&_{1 \leqslant i \leqslant m}\ (y_i \leqslant x \ \&\ At(y_i)))$$

Observation 5.3.1 *If we prove T' is complete, where T' is the theory of atomic and infinite Boolean algebras expanded using the language with the predicates A_m containing the correspondent definitions, then the usual theory of these algebras is complete too.*

Proof: This is because, for any sentence φ in the language $\mathcal{L}$, $T' \models \varphi$ or $T' \models \neg\varphi$ because of its completeness. Since $T \models \varphi$ iff $T' \models \varphi$, we get T is complete, too. □

Observation 5.3.2 *Isomorphisms between structures must preserve the predicates defined in the language. For this reason, if we expand the language adding $A_m(x)$ and f is an isomorphism between structures in this language, it must verify $A_m(x) \Leftrightarrow A_m(f(x))$ for every $m \in \omega$.*

Lemma 5.3.3 *In an ω-saturated Boolean algebra $\mathfrak{B}$, if a is an element greater than infinitely many atoms, then there exists $b \in B$ such that $a \wedge b$ and $a \wedge b^{\mathrm{c}}$ are both greater than infinitely many atoms.*

Proof: Consider the set $\Gamma(x) = \{A_m(x \wedge a)\}_{m \in \omega} \cup \{A_m(a \wedge x^{\mathrm{c}})\}_{m \in \omega}$. We claim that $\Gamma(x)$ is a type. In order to see this, we only need to see that it is consistent, *i.e.* that every finite subset of $\Gamma(x)$ is consistent. Any finite subset of $\Gamma(x)$ is of the form $\Gamma_0(x) \subseteq \{A_1(x \wedge a), ..., A_m(x \wedge a), A_1(a \wedge x^{\mathrm{c}}), ..., A_m(a \wedge x^{\mathrm{c}})\}$ for some $m \in \omega$. It is known that $\mathfrak{A} \models A_{2m}(a)$, thus we can take $a_1, ..., a_{2m}$ different atoms below a. Then, taking $b = a_1 \vee ... \vee a_m$, it is easy to see that $a_1, ..., a_m \leqslant a \wedge b$ and $a_{m+1}, ..., a_{2m} \leqslant a \wedge b^{\mathrm{c}}$. As a consequence, $\Gamma_0(x)$ is consistent, and so is $\Gamma(x)$. Now that we know that $\Gamma(x)$ is a type, we get that there must be $b \in B$ such that $A_m(b \wedge a)$ and $A_m(a \wedge b^{\mathrm{c}})$ for every $m \in \omega$, because since $\mathfrak{B}$ is ω-saturated, it must realize Γ. □

Theorem 5.3.4 *The theory of infinite and atomic Boolean algebras is complete and admits quantifier elimination in the language $\{\wedge, \vee, ^{\mathrm{c}}, 0, 1\} \cup \{A_m\}_{m \in \omega}$.*

Proof: Due to theorem 4.2.20, it suffices to see that, for any $\mathfrak{A}, \mathfrak{B}$ ω-saturated models of the theory, $I : \mathfrak{A} \cong_p \mathfrak{B}$, where I is the set of all partial isomorphisms

from $\mathfrak{A}$ to $\mathfrak{B}$ that have a finitely generated substructure of $\mathfrak{A}$ as its domain. Notice that $I \neq \emptyset$ because both models admit the Boolean algebra $\{0,1\}$ as a subalgebra, thus we can take f, the mapping that sends $0_{\mathfrak{A}}$ to $0_{\mathfrak{B}}$ and $1_{\mathfrak{A}}$ to $1_{\mathfrak{B}}$. We know it preserves the predicates because both $\mathfrak{A}$ and $\mathfrak{B}$ are infinite, thus they must have infinitely many atoms and we get $A_m(1_{\mathfrak{A}})$ and $A_m(1_{\mathfrak{B}})$.
Take $f \in I$ such that f is an isomorphism between the subalgebras $\mathfrak{A}_0 = \langle a_1, ..., a_n \rangle$ of $\mathfrak{A}$ and $\mathfrak{B}_0 = \langle b_1, ..., b_n \rangle$ of $\mathfrak{B}$. We also take $a \in A$. For the back-and-forth conditions to hold, we only need to find $b \in B$ such that for every $\varepsilon \in 2^n$:

$$A_m(\varepsilon\overline{a} \wedge a) \Leftrightarrow A_m(\varepsilon\overline{b} \wedge b) \text{ for each } m,$$
$$A_m(\varepsilon\overline{a} \wedge a^{\mathrm{c}}) \Leftrightarrow A_m(\varepsilon\overline{b} \wedge b^{\mathrm{c}}) \text{ for each } m.$$

First notice that, if these conditions hold, then:

$$\varepsilon\overline{a} \wedge a = 0 \Leftrightarrow \varepsilon\overline{b} \wedge b = 0,$$
$$\varepsilon\overline{a} \wedge a^{\mathrm{c}} = 0 \Leftrightarrow \varepsilon\overline{b} \wedge b^{\mathrm{c}} = 0,$$
$$A_m(x) \Leftrightarrow A_m(f'(x)) \text{ for any } x \in \langle a_1, ..., a_n, a \rangle,$$

where f' extends f and is an isomorphism between $\langle a_1, ..., a_n, a \rangle$ and $\langle b_1, ..., b_n, b \rangle$. This is because, since we are working with atomic algebras, $\varepsilon\overline{a} \wedge a = 0$ iff $\neg A_1(\varepsilon\overline{a} \wedge a)$ iff $\neg A_1(\varepsilon\overline{b} \wedge b)$ iff $\varepsilon\overline{b} \wedge b = 0$, and the same argument can be made for $\varepsilon\overline{a} \wedge a^{\mathrm{c}}$ and $\varepsilon\overline{b} \wedge b^{\mathrm{c}}$. Let's see that the third condition also holds. If $\overline{a}' = (a_1, ..., a_n, a)$ and $\overline{b}' = (b_1, ..., b_n, b)$, then we know that any x in $\langle a_1, ..., a_n, a \rangle$ can be expressed like $x = \varepsilon_1\overline{a}' \vee ... \vee \varepsilon_k\overline{a}'$ for some $\varepsilon_1, ..., \varepsilon_k \in 2^{n+1}$. Since $\varepsilon_i\overline{a}' \wedge \varepsilon_j\overline{a}' = 0$ if $i \neq j$, $A_m(\varepsilon_1\overline{a}' \vee ... \vee \varepsilon_k\overline{a}')$ iff $A_{m_i}(\varepsilon_i\overline{a}')$ for some m_i such that $m = \Sigma_{1 \leqslant i \leqslant k} m_i$, and this will happen iff $A_{m_i}(\varepsilon_i\overline{b}')$ for some m_i such that $m = \Sigma_{1 \leqslant i \leqslant k} m_i$ or, equivalently, $A_m(\varepsilon_1\overline{b}' \vee ... \vee \varepsilon_k\overline{b}')$. Notice that we have just proven that $A_m(x)$ iff $A_m(f'(x))$.

Let's fix $\varepsilon_i \in 2^n$ such that $\varepsilon_i\overline{a} \neq 0$ and consider the cases:

1. If $A_m(\varepsilon_i\overline{a})$ and $\neg A_{m+1}(\varepsilon_i\overline{a})$, the same is true for $\varepsilon_i\overline{b}$. In addition, we also know $\varepsilon_i\overline{a} \wedge a$ and $\varepsilon_i\overline{a} \wedge a^{\mathrm{c}}$ are greater than p and q atoms, respectively, where $m = p + q$. In this case, we take c_i as the supremum of the p atoms $x_1, ..., x_p$ such that $x_j \leqslant \varepsilon_i\overline{b}$ for every $j \leqslant p$. In the particular case that $\varepsilon_i\overline{a} \wedge a = 0$ then we take $c_i = 0$, and if $\varepsilon_i\overline{a} \wedge a^{\mathrm{c}} = 0$, then $c_i = \varepsilon_i\overline{b}$.

2. $\varepsilon_i\overline{a}$ is greater than infinitely many atoms, $\varepsilon_i\overline{a} \wedge a$ is greater than p atoms and $\varepsilon_i\overline{a} \wedge a^{\mathrm{c}}$ is greater than infinitely many atoms. Since, in this case, $\varepsilon_i\overline{b}$ is greater than infinitely many atoms, we can take c_i to be the supremum of p of them: $c_i = \bigvee_{1 \leqslant j \leqslant p} x_j$ where $x_j \in D_i$ for every j, $x_j \neq x_k$ if $j \neq k$, and $D_i = \{x : x$ is an atom and $x \leqslant \varepsilon_i\overline{b}\}$.

3. $\varepsilon_i\overline{a}$ is greater than infinitely many atoms, $\varepsilon_i\overline{a} \wedge a^{\mathrm{c}}$ is greater than p atoms and $\varepsilon_i\overline{a} \wedge a$ is greater than infinitely many atoms. Then $c_i = \varepsilon_i\overline{b} \wedge (\bigwedge_{1 \leqslant j \leqslant p} x_j^{\mathrm{c}})$ where $x_j \in D_i$ are different atoms for every j.

4. $\varepsilon_i\overline{a}$ is greater than infinitely many atoms, and so are $\varepsilon_i\overline{a} \wedge a$ and $\varepsilon_i\overline{a} \wedge a^{\mathrm{c}}$. Then we can split $\varepsilon_i\overline{b}$ in two parts such that they are both greater than infinitely many atoms, because of the previous lemma. In this case we can take c_i to be one of this parts.

Claim: we can take $b = \bigvee_{1\leqslant j\leqslant l} c_j$.
Assume $A_m(\varepsilon_i\overline{a} \wedge a)$. Then, by the choice of c_i, we know that c_i is the supremum of, at least, m different atoms $x_1, ..., x_m$ such that $x_j \leqslant \varepsilon_i\overline{b}$ for every $j = 1, ..., m$. Then $x_1, ..., x_m \leqslant \varepsilon_i\overline{b} \wedge c_i \leqslant \varepsilon_i\overline{b} \wedge (\bigvee_{1\leqslant j\leqslant l} c_j) = \varepsilon_i\overline{b} \wedge b$, thus we get $A_m(\varepsilon_i\overline{b} \wedge b)$. Conversely, if $A_m(\varepsilon\overline{b} \wedge b)$, then $A_m(\bigvee_{1\leqslant j\leqslant l}(\varepsilon_j\overline{b} \wedge c_j))$, and this is equivalent to $A_{m_j}(\bigvee_{1\leqslant j\leqslant l}(\varepsilon_i\overline{b} \wedge c_j))$ for some m_j such that $m = \Sigma_{1\leqslant j\leqslant l} m_j$, because $(\varepsilon_i\overline{b} \wedge c_j) \wedge (\varepsilon_i\overline{b} \wedge c_k) = 0$ if $i \neq k$. But $\varepsilon_i\overline{b} \wedge c_j = 0$ if $i \neq j$, thus $m_j = 0$ for any $j \neq i$, so $A_m(\varepsilon_i\overline{b} \wedge c_i)$. As a consequence, $A_m(\varepsilon_i\overline{a} \wedge a)$.

An analogous argument can be made to prove that $A_m(\varepsilon_i\overline{a} \wedge a^{\mathrm{c}})$ implies $A_m(\varepsilon_i\overline{b} \wedge b^{\mathrm{c}})$. Now, if we assume $A_m(\varepsilon_i\overline{b} \wedge b^{\mathrm{c}})$ then, since $b^{\mathrm{c}} = \bigvee_{1\leqslant j\leqslant l}(\varepsilon_j\overline{b} \wedge c_j^{\mathrm{c}})$, we get that $A_m(\bigvee_{1\leqslant j\leqslant l} \varepsilon_i\overline{b} \wedge (\varepsilon_j\overline{b} \wedge c_j^{\mathrm{c}}))$, thus $A_m(\varepsilon_i\overline{b} \wedge c_i^{\mathrm{c}})$, because if $i \neq j$, $\varepsilon_i\overline{b} \wedge \varepsilon_j\overline{b} \wedge c_j^{\mathrm{c}} = 0$. As a consequence, we get $A_m(\varepsilon_i\overline{a} \wedge a^{\mathrm{c}})$, as required.

$\square$

Chapter 6

Classification of the complete theories of Boolean algebras

6.1 Atomic elements

Let $\mathfrak{B}$=⟨B, $\wedge, \vee,^c$, 0, 1⟩ a Boolean algebra.

Definition 6.1.1 *Let a be an element in B. Then $B|a = \{x \in B : x \leqslant a\}$.*

Observation 6.1.2 *If we define $x^a = x^c \wedge a$, then $\mathfrak{B}|a$=⟨$B|a$, $\wedge, \vee,^a$, 0, a ⟩ is a Boolean algebra, where $\wedge$ and $\vee$ are the restrictions of $\wedge, \vee$ in the universe $B|a$.*

Definition 6.1.3 *Let $\mathfrak{A}, \mathfrak{B}$ be Boolean algebras defined on the sets A, B, respectively, with the usual operations. Then we can define operations in $A \times B$ in the following way:*
For any $(x, y), (u, v) \in A \times B$,

$$\begin{aligned} (x, y) \wedge_{A\times B} (u, v) &= (x \wedge u, y \wedge v), \\ (x, y) \vee_{A\times B} (u, v) &= (x \vee u, y \vee v), \\ (x, y)^{c_{A\times B}} &= (x^c, y^c). \end{aligned}$$

Observation 6.1.4 $\mathfrak{A} \times \mathfrak{B} = \langle A \times B, \wedge_{A\times B}, \vee_{A\times B},^{c_{A\times B}}, (0, 0), (1, 1)\rangle$ *is a Boolean algebra. It is called* product algebra of $\mathfrak{A}$ and $\mathfrak{B}$.

Observation 6.1.5 *Fix $a \in B$. Then for all $x \in B$ there are $y, z \in B$ such that $x = y \vee z$, where $y \in B|a$ and $z \in B|a^c$. In fact, $y = x \wedge a$ and $z = x \wedge a^c$: they are unique. If we consider the mapping:*

$$\begin{aligned} f : \mathfrak{B} &\longrightarrow \mathfrak{B}|a \times \mathfrak{B}|a^c \\ x &\longrightarrow f(x) = (x \wedge a, x \wedge a^c) \end{aligned}$$

we get that f is an isomorphism and, as a consequence, $\mathfrak{B} \cong \mathfrak{B}|a \times \mathfrak{B}|a^c$

Definition 6.1.6 *An element $a \in B$ is said to be* atomic *iff $\mathfrak{B}|a$ is an atomic Boolean algebra. An element $a \in B$ is said to be* atomless *iff $\mathfrak{B}|a$ is atomless.*

Proposition 6.1.7 *For any $a \in B$,*

1. *$\mathfrak{B}$ is atomic iff $\mathfrak{B}|a, \mathfrak{B}|a^c$ are atomic.*
2. *$\mathfrak{B}$ is atomless iff $\mathfrak{B}|a, \mathfrak{B}|a^c$ are atomless.*

Proof:

1. We will prove a more general property: $\mathfrak{A}, \mathfrak{B}$ are atomic iff $\mathfrak{A} \times \mathfrak{B}$ is atomic. Assume that $\mathfrak{A}, \mathfrak{B}$ are atomic and take any $(x, y) \in A \times B$, $(x, y) \neq (0, 0)$. If $x \neq 0$ then there is $a \in A$ such that a is an atom and $a \leqslant x$, thus $(a, 0) \leqslant (x, y)$. Notice that $(a, 0)$ is an atom in $\mathfrak{A} \times \mathfrak{B}$. Now, if $x = 0$, then $y \neq 0$ and, since $\mathfrak{B}$ is atomic, there is $b \in B$ such that $b \leqslant y$ and b is an atom. This results in $(0, b) \leqslant (x, y)$, and we know that $(0, b)$ is an atom. In any of the two possible cases, we have found an atom $z \in A \times B$ such that $z \leqslant (x, y)$, so $\mathfrak{A} \times \mathfrak{B}$ must be atomic.

   Conversely, assume for all $(x, y) \in \mathfrak{A} \times \mathfrak{B}$ such that $(x, y) \neq (0, 0)$ there exists an atom (a, b) such that $(a, b) \leqslant (x, y)$. Then, for every $x \in A$, $x \neq 0$ there is an atom $a \in A$ such that $a \leqslant x$. In order to see this, it suffices to take the first component of the atom $(a, b) \in A \times B$ such that $(a, b) \leqslant (x, 0)$. This shows that $\mathfrak{A}$ is atomic, and a similar argument can be made to see that $\mathfrak{B}$ is atomic.

2. Let's prove that $\mathfrak{A}, \mathfrak{B}$ are atomless iff $\mathfrak{A} \times \mathfrak{B}$ is atomless.
   If $\mathfrak{A}, \mathfrak{B}$ are atomless and (a, b) is an atom in $\mathfrak{A} \times \mathfrak{B}$, then either a is an atom in $\mathfrak{A}$ and b is 0, or b is an atom in $\mathfrak{B}$ and $a = 0$. The fact that one of them must be 0 for (a, b) to be an atom is trivial, because otherwise $(a, 0) < (a, b)$ or $(0, b) < (a, b)$, depending on the case. Now, assume $a = 0$. Then b must be an atom, because otherwise there would be $y \in B$ such that $y < b$ and $(0, y) < (0, b) = (a, b)$! This is a contradiction with the fact that $\mathfrak{B}$ is atomless, thus $\mathfrak{A} \times \mathfrak{B}$ must be atomless too. A similar argument can be made if $b = 0$

   Conversely, if $\mathfrak{A} \times \mathfrak{B}$ are atomless and a is an atom in $\mathfrak{A}$, then $(a, 0)$ would be an atom in $\mathfrak{A} \times \mathfrak{B}$! This implies that there cannot be any atom in $\mathfrak{A}$: it must be atomless. We can use a similar argument to prove that if b is an atom in $\mathfrak{B}$, $(0, b)$ is an atom in $\mathfrak{A} \times \mathfrak{B}$, so $\mathfrak{B}$ must be atomless too.

$\square$

Definition 6.1.8 *I($\mathfrak{B}$)=$\{x \in B : x = y \vee z$ for some y, z such that y is atomic and z is atomless$\}$.*

Observation 6.1.9 *It is easy to see that $I(\mathfrak{B})$ is an ideal of $\mathfrak{B}$, maybe an improper one.*

Proposition 6.1.10 *If $\mathfrak{B}$ is atomic or has a finitely many atoms, then $I(\mathfrak{B}) = \mathfrak{B}$.*

Proof: If $\mathfrak{B}$ is atomic, any element x in $\mathfrak{B}$ is atomic. Then $x = x \vee 0$, where x is atomic and 0 is atomless. As a consequence, $I(\mathfrak{B}) = \mathfrak{B}$.
If $\mathfrak{B}$ has a finite number of atoms and we take $x \in B$, then $x = y \vee z$, where $y = \bigvee_{1 \leqslant i \leqslant n} y_i$, and $\{y_1, ..., y_n\}$ is the set of atoms in $\mathfrak{B}$ such that $y_i \leqslant x$ for every i, and $z = x \wedge y^c$. Note that y is atomic and z is atomless, so the result is proven. □

6.2 Invariants

The main purpose of this and the next section is to find a uniform way to complete the theory of Boolean algebras so that every possible completion appears. In order to do this, we need to define some concepts.
For $\mathfrak{B}$ a Boolean algebra, we will now define by induction a sequence of homomorphisms, ideals and quotient algebras:
Let x be an element in B.Then

$$x^{(0)} = x,$$
$$I^{(0)} = \{0\},$$
$$\mathfrak{B}^{(0)} = \mathfrak{B},$$

and, for any $k < \omega$,

$$x^{(k+1)} = x^{(k)}/I(\mathfrak{B}^{(k)}),$$
$$I^{(k+1)} = \{x \in B : x^{(k+1)} = 0\},$$
$$\mathfrak{B}^{(k+1)} = \mathfrak{B}^{(k)}/I(\mathfrak{B}^{(k)}).$$

Notice that for any $k < \omega$,

- The set $I^{(k)}$ is an ideal of $\mathfrak{B}$.
- $\{(x, x^{(k)}) : x \in B\}$ is an homomorphism from $\mathfrak{B}$ onto $\mathfrak{B}^{(k)}$. The kernel of this homomorphism is $I^{(k)}$. As a consequence,

$$\mathfrak{B}^{(k)} \cong \mathfrak{B}/I^{(k)}.$$

Observation 6.2.1 *Due to proposition 6.1.10, if $\mathfrak{B}^{(k)}$ is atomic or has a finite number of atoms, $\mathfrak{B}^{(k+1)}$ is trivial.*

Proposition 6.2.2 *There are formulas $\varphi_k(x), \psi_k(x), \rho_k(x), \eta_{k,l}(x), \sigma_{k,l}(x)$ for every $k, l < \omega$ such that, for any $a \in B$:*

1. *$\mathfrak{B} \models \varphi_k(a)$ iff $a \in I^{(k)}$.*
2. *$\mathfrak{B} \models \psi_k(a)$ iff $a^{(k)}$ is atomic in $\mathfrak{B}^{(k)}$.*
3. *$\mathfrak{B} \models \rho_k(a)$ iff $a^{(k)}$ is atomless in $\mathfrak{B}^{(k)}$.*
4. *$\mathfrak{B} \models \eta_{k,l}(a)$ iff $a^{(k)}$ majorizes at least l atoms in $\mathfrak{B}^{(k)}$.*
5. *$\mathfrak{B} \models \sigma_{k,l}(a)$ iff $a^{(k)}$ majorizes at most l atoms in $\mathfrak{B}^{(k)}$.*

Proof: We will show this by induction:
If $k = 0$,

1. $\varphi_0(x)$ is $x = 0$.
2. $\psi_0(x)$ must say that x is atomic in $\mathfrak{B}$. Then $\psi_0(x)$ is
$$\forall y(y \leqslant x \ \&\ 0 < y \to \exists z(z \leqslant y \ \&\ At(z))).$$
3. $\rho_0(x)$ must say that x is atomless in $\mathfrak{B}$. Then $\rho_0(x)$ is
$$\forall y(y \leqslant x \to \neg At(y)).$$
4. $\eta_{0,l}(x)$ must say that x majorizes at least l atoms in $\mathfrak{B}$. Then $\eta_{0,l}(x)$ is
$$\exists y_1, ..., y_l(\&_{1 \leqslant i \leqslant l}\ (y_i \leqslant x)\ \&\quad \&_{1 \leqslant i < j \leqslant l}\ (\neg y_i = y_j)\ \&_{1 \leqslant i \leqslant l}\ At(y_i)).$$
5. $\sigma_{0,l}(x)$ must say that x majorizes at most l atoms in $\mathfrak{B}$. Then $\sigma_{0,l}(x)$ is
$$\neg \eta_{0,l+1}(x).$$

Now assume we have found the formulas for k:
Notice that $\varphi_j(x)$ means $x^{(j)} = 0$. We know for every x, y, $x \leqslant y \Rightarrow x \wedge y^{\mathrm{c}} = 0$. Therefore, we will write $\varphi_j(x \wedge y^{\mathrm{c}})$ for $x^{(j)} \leqslant y^{(j)}$.

1. $x \in I^{(k+1)}$ iff $x^{(k+1)} = 0 \Leftrightarrow \exists y, z \in \mathfrak{B}^{(k)}$ such that y is atomic in $\mathfrak{B}^{(k)}$, z is atomless in $\mathfrak{B}^{(k)}$ and $x^{(k)} = y \vee z \Leftrightarrow \exists y, z \in B$ such that $y^{(k)}$ is atomic in $\mathfrak{B}^{(k)}$, $z^{(k)}$ is atomless in $\mathfrak{B}^{(k)}$ and $x^{(k)} = y^{(k)} \vee z^{(k)}$. Then the formula $\varphi_{k+1}(x)$ is
$$\exists y, z(x = y \vee z \ \&\ \psi_k(y) \ \&\ \rho_k(y)).$$
2. $x^{(k+1)}$ is atomic in $\mathfrak{B}^{(k+1)}$ iff $\forall y \in \mathfrak{B}^{(k+1)} | x^{(k+1)} (y > 0 \to \exists z \in \mathfrak{B}^{(k+1)}(0 < z \leqslant y \wedge \neg \exists u \in \mathfrak{B}^{(k+1)}(0 < u < z)))$. Then the formula $\psi_{k+1}(x)$ is
$$\forall y(\varphi_{k+1}(y \wedge x^{\mathrm{c}}) \ \&\ \neg\varphi_{k+1}(y) \to \exists z(\neg\varphi_{k+1}(z) \ \&\ \varphi_{k+1}(z \wedge y^{\mathrm{c}}) \ \&$$
$$\forall u(\varphi_{k+1}(u \wedge z^{\mathrm{c}}) \ \&\ \neg\varphi_{k+1}(z \wedge u^{\mathrm{c}}) \to \varphi_{k+1}(u)))).$$

3. $x^{(k+1)}$ is atomless in $\mathfrak{B}^{(k+1)}$ iff there is no atom in $\mathfrak{B}^{(k+1)}|x^{(k+1)}$. Then the formula $\rho_{k+1}(x)$ is

$$\forall y(\varphi_{k+1}(y \wedge x^{c}) \rightarrow \varphi_{k+1}(y) \sqcup \exists z(\neg\varphi_{k+1}(z) \,\&\, \neg\varphi_{k+1}(y \wedge z^{c}) \,\&\, \neg\varphi_{k+1}(z \wedge y^{c}))).$$

4. $\eta_{k+1,l}(x)$ is

$$\exists y_1, ..., y_l(\&_{1\leqslant i\leqslant l} \ (\varphi_{k+1}(y_i \wedge x^{c}) \ \& \ \neg\varphi_{k+1}(y_i) \ \& \ \&_{1\leqslant j\leqslant l, j\neq i} \ (\neg\varphi_{k+1}(y_i \wedge y_j^{c})$$
$$\sqcup \neg\varphi_{k+1}(y_j \wedge y_i^{c})) \ \& \ \forall z(\neg\varphi_{k+1}(z) \ \& \ \varphi_{k+1}(z \wedge y_i^{c}) \rightarrow \varphi_{k+1}(y_i \wedge z^{c}))).$$

5. $\sigma_{k+1,l}(x)$ is

$$\neg\eta_{k+1,l+1}(x).$$

□

Consider the sequence $\mathfrak{B}^{(0)}, \mathfrak{B}^{(1)}, ..., \mathfrak{B}^{(k)}$. It is obvious that if one Boolean algebra in the sequence is trivial, all the following ones will be trivial too. Nevertheless, there is always the option that $\mathfrak{B}^{(k)}$ never becomes trivial.
Assume k is the first integer such that $\mathfrak{B}^{(k+1)}$ is trivial. Then $\mathfrak{B}^{(k)}$ is not, and every element in $\mathfrak{B}^{(k)}$ is in $I(\mathfrak{B}^{(k)})$. But what can be said about the atoms in $\mathfrak{B}^{(k)}$ (if there are any)? The *invariants* will help us to classify the theories of Boolean algebras by giving this kind of information.

Definition 6.2.3 *We assign a pair of invariants ($m(\mathfrak{B})$,$n(\mathfrak{B})$) to any non-trivial Boolean algebra $\mathfrak{B}$ as follows:*

$$m(\mathfrak{B}) = \begin{cases} \text{the least } k < \omega \text{ such that } \mathfrak{B}^{(k+1)} \text{ is trivial,} \\ \infty, \ \text{otherwise.} \end{cases}$$

$$n_0(\mathfrak{B}) = \left\{ \begin{array}{ll} \infty, & \text{if } m(\mathfrak{B}) = k \text{ and } \mathfrak{B}^{(k+1)} \text{ has infinitely many atoms,} \\ l, & \text{if } m(\mathfrak{B}) = k \text{ and } \mathfrak{B}^{(k)} \text{ has } l \text{ atoms.} \end{array} \right\}$$

$$n(\mathfrak{B}) = \left\{ \begin{array}{ll} 0, & \text{if } m(\mathfrak{B}) = \infty, \\ n_0(\mathfrak{B}), & \text{if } m(\mathfrak{B}) = k < \omega \text{ and } \mathfrak{B}^{(k)} \text{ is atomic,} \\ -n_0(\mathfrak{B}), & \text{if } m(\mathfrak{B}) = k < \omega \text{ and } \mathfrak{B}^{(k)} \text{ is not atomic.} \end{array} \right\}$$

Notice that $m(\mathfrak{B})$ indicates when does the sequence become trivial. The sign of $n(\mathfrak{B})$ indicates whether $\mathfrak{B}^{(m(\mathfrak{B}))}$ is atomic or not, and its absolute value is the number of atoms in the algebra.

Observation 6.2.4 *If $m(\mathfrak{B}) = k < \omega$ and $n(\mathfrak{B}) = 0$ then $\mathfrak{B}^{(k)}$ is atomless.*

Proposition 6.2.5 *The following properties can be expressed by a sentence for any $k, l < \omega$:*

1. $m(\mathfrak{B}) = k$
2. $m(\mathfrak{B}) = k, n(\mathfrak{B}) = l$
3. $m(\mathfrak{B}) = k, n(\mathfrak{B}) = -l$
4. $m(\mathfrak{B}) = \infty$
5. $m(\mathfrak{B}) = k, n(\mathfrak{B}) = \infty$
6. $m(\mathfrak{B}) = k, n(\mathfrak{B}) = -\infty$

Proof: There is a formula that expresses $\mathfrak{B}^{(k)}$ is trivial: $\forall x(\varphi_k(x))$. This, added to what we have proven in proposition 6.2.2, gives this result. □

Proposition 6.2.6 $\mathfrak{B}^{(k)} \cong (\mathfrak{B}|a)^{(k)} \times (\mathfrak{B}|a^{\mathrm{c}})^{(k)}$

Proof: It suffices to apply the result in observation 6.1.5 to $\mathfrak{B}^{(k)}$ and to prove $(\mathfrak{B}|a)^{(k)} = \mathfrak{B}^{(k)}|a^{(k)}$. See [5], lemma 18.3. □

6.3 Classification

Proposition 6.3.1 *Take $a \in B$. Then:*

1. $m(\mathfrak{B}) = max(m(\mathfrak{B}|a), m(\mathfrak{B}|a^{\mathrm{c}}))$
2. $m(\mathfrak{B}|a) < m(\mathfrak{B}|a^{\mathrm{c}}) \Rightarrow m(\mathfrak{B}) = m(\mathfrak{B}|a^{\mathrm{c}})$ *and* $n(\mathfrak{B}) = n(\mathfrak{B}|a^{\mathrm{c}})$
3. *Assume* $m(\mathfrak{B}|a) = m(\mathfrak{B}|a^{\mathrm{c}}) < \infty$. *Then* $n(\mathfrak{B})$ *is characterized as follows:*

   (a) $n(\mathfrak{B}) = 0$ *iff* $n(\mathfrak{B}|a) = n(\mathfrak{B}|a^{\mathrm{c}}) = 0$
   (b) $n(\mathfrak{B}) > 0$ *iff* $n(\mathfrak{B}|a) > 0$ *and* $n(\mathfrak{B}|a^{\mathrm{c}}) > 0$
   (c) $n(\mathfrak{B}) < 0$ *iff* $n(\mathfrak{B}|a) \leqslant 0$ *or* $n(\mathfrak{B}|a^{\mathrm{c}}) \leqslant 0$ *but not both 0 at the same time*
4. $m(\mathfrak{A}|a) = m(\mathfrak{B}|a^{\mathrm{c}}) < \infty \Rightarrow n_0(\mathfrak{B}) = n_0(\mathfrak{B}|a) + n_0(\mathfrak{B}|a^{\mathrm{c}})$

Proof:

1. This follows from proposition 6.2.6: $\mathfrak{B}^{(k)} \times \{0\}$ is not trivial if $\mathfrak{B}^{(k)}$ is not trivial.
2. $m(\mathfrak{B}) = m(\mathfrak{B}|a^{\mathrm{c}})$ because of 1. As a consequence, $n(\mathfrak{B}) = n(\mathfrak{B}|a^{\mathrm{c}})$, because the atoms in $\{0\} \times (\mathfrak{B}|a^{\mathrm{c}})^{(k)}$ are $(0, x)$, where x is an atom in $(\mathfrak{B}|a^{\mathrm{c}})^{(k)}$.

3. All characterizations follow from propositions 6.2.6 and 6.1.7 but the last one, which is just a consequence from the previous two.

4. The atoms in $\mathfrak{B}^{(k)}$ are isomorphic to the elements $(b, 0)$ and $(0, c)$, where b is an atom in $(\mathfrak{B}|a)^{(k)}$ and c is an atom in $(\mathfrak{B}|a^{\mathrm{c}})^{(k)}$. Therefore, $n_0(\mathfrak{B}) = n_0(\mathfrak{B}|a) + n_0(\mathfrak{B}|a^{\mathrm{c}})$

□

Lemma 6.3.2 *Let $\mathfrak{A}, \mathfrak{B}$ be two non-trivial Boolean algebras such that $(m(\mathfrak{A}), n(\mathfrak{A})) = (m(\mathfrak{B}), n(\mathfrak{B}))$. Assume $\mathfrak{B}$ is ω-saturated. If we take $a \in A, 0 \neq a \neq 1$, then there is $b \in B$ such that $0 \neq b \neq 1$ and*

$$(m(\mathfrak{A}|a), n(\mathfrak{A}|a)) = (m(\mathfrak{B}|b), n(\mathfrak{B}|b))$$
$$(m(\mathfrak{A}|a^{\mathrm{c}}), n(\mathfrak{A}|a^{\mathrm{c}})) = (m(\mathfrak{B}|b^{\mathrm{c}}), n(\mathfrak{B}|b^{\mathrm{c}}))$$

Proof: First of all, we need to distinguish several cases:

- Assume $m(\mathfrak{A}|a) = m(\mathfrak{A}|a^{\mathrm{c}}) = \infty$.
Then $\infty = m(\mathfrak{A}) = m(\mathfrak{B})$. As a consequence, for every $k < \omega$ there is $c \in B$ such that $c^{(k)} \neq 0, 1$ in $\mathfrak{B}^{(k)}$. This is because, even though the condition of not being trivial only gives us that there is $c \in B$ such that $c^{(k)} \neq 0$, if $\mathfrak{B}^{(k)}$ was $\{0, 1\}$ for some k, then $\mathfrak{B}^{(k+1)}$ would be trivial. What we need to find is a certain $b \in B$ such that $m(\mathfrak{B}|b) = m(\mathfrak{B}|b^{\mathrm{c}}) = \infty$ and, in order to do this, it suffices to find $b \in B$ such that $b^{(k)} \neq 0, 1$ in $\mathfrak{B}^{(k)}$ for every k. This is equivalent to see that the set of formulas $\{\neg\varphi_k(x), \neg\varphi_k(x^{\mathrm{c}})\}_{k<\omega}$, where φ_k is the one described in proposition 6.2.2, is a type. Let's define $\Sigma = \{\neg\varphi_k(x), \neg\varphi_k(x^{\mathrm{c}})\}_{k<\omega}$. To prove its consistency, it suffices to take any finite subset and see it is consistent: consider $\{\neg\varphi_k(x), \neg\varphi_k(x^{\mathrm{c}})\}_{k\in I}$, where $I \subseteq \omega$ is finite, and let l be the maximum of I. Since we know that there is $x \in \mathfrak{B}^{(l)}$ such that $x^{(l)} \neq 0, 1$, we get that there is $x \in \mathfrak{B}$ such that $x^{(k)} \neq 0, 1$ for every $k \leqslant l$ and, therefore, the consistency of this set is granted. As a result, Σ must be consistent too. Now that we know that Σ is a type over $\emptyset$, we get that it is realized in $\mathfrak{B}$, since $\mathfrak{B}$ is ω-saturated.
Therefore, we can conclude that there is $b \in B$ such that $b^{(k)} \neq 0$ for every $k < \omega$, thus $m(\mathfrak{B}|b) = m(\mathfrak{B}|b^{\mathrm{c}}) = \infty$.

- Assume $m(\mathfrak{A}|a) < m(\mathfrak{A}|a^{\mathrm{c}})$.
Claim: to prove the lemma it suffices to find $b \in B$ such that $m(\mathfrak{B}|b) = m(\mathfrak{A}|a)$ and $n(\mathfrak{B}|b) = n(\mathfrak{A}|a)$.
This is because, in this case, by lemma 6.3.1
$m(\mathfrak{A}|a^{\mathrm{c}}) = m(\mathfrak{A}) = m(\mathfrak{B}) = max(m(\mathfrak{B}|b), m(\mathfrak{B}|b^{\mathrm{c}})) = m(\mathfrak{B}|b^{\mathrm{c}})$
$n(\mathfrak{A}|a^{\mathrm{c}}) = n(\mathfrak{A}) = n(\mathfrak{B}) = n(\mathfrak{B}|b^{\mathrm{c}})$
And the result will be proven.
Let $k = m(\mathfrak{A}|a)$. It is important to keep in mind during this part of the proof that $\mathfrak{B}^{(k)}$ is not atomic and it has infinitely many atoms, because otherwise

we would get that $m(\mathfrak{B}) = k$ due to proposition 6.1.10, and we know $k < m(\mathfrak{A}|a^{\mathrm{c}}) = m(\mathfrak{B})$. We can now distinguish several cases:

1. If $n(\mathfrak{A}|a) = +\infty$, then we need to find $x \in B$ such that $x \neq 0, x \neq 1, x^{(k+1)} = 0, x^{(k)} \neq 0$ and $(\mathfrak{B}|x)^{(k)}$ is atomic and has infinitely many atoms. It will be done if we can prove the consistency of the set of formulas: $\{\varphi_{k+1}(x)\} \cup \{\neg\varphi_k(x)\} \cup \{\psi_k(x)\} \cup \{\eta_{k,l}(x)\}_{l<\omega}$, because of the ω-saturation of the Boolean algebra $\mathfrak{B}$. Now, an infinite set of formulas is consistent iff any of its finite subsets is also consistent. For this reason, we consider $\{\varphi_{k+1}(x)\} \cup \{\neg\varphi_k(x)\} \cup \{\psi_k(x)\} \cup \{\eta_{k,0}(x), ..., \eta_{k,l}(x)\}$ for some $l < \omega$, and claim that any b defined as $b = b_1 \vee ... \vee b_l$, where $b_i^{(k)}$ are different atoms in $\mathfrak{B}^{(k)}$ for all $i \leqslant l$ satisfies the set of formulas. First notice that the atoms $b_i^{(k)}$ actually exist because $\mathfrak{B}^{(k)}$ has infinitely many atoms. Let's see that b satisfies all the conditions required. It is obvious that $b^{(k)} \neq 0$ because $b^{(k)} = b_1^{(k)} \vee ... \vee b_l^{(k)}$, and $b_i^{(k)} \neq 0$ for every i, thus $b \neq 0$. $(\mathfrak{B}|(b_1 \vee ... \vee b_l))^{(k)}$ has l atoms, $b_1^{(k)}, ..., b_l^{(k)}$, and it is atomic, because all elements in $\mathfrak{B}^{(k)}|(b_1 \vee ... \vee b_l)^{(k)}$ are of the form $\bigvee_{i\in I} b_i^{(k)}$ for some $I \subseteq \{1, ..., l\}$, so they majorize the atoms $b_i^{(k)}, i \in I$. Notice that $b \neq 1$ because otherwise $b^{(k)} = b_1^{(k)} \vee ... \vee b_l^{(k)} = 1$, and this would mean that $\mathfrak{B}^{(k)}$ only has finitely many atoms, which is a contradiction with the fact that $m(\mathfrak{B}) > k$. In conclusion, we get that b satisfies every formula in the subset, so $\{\varphi_{k+1}(x)\} \cup \{\neg\varphi_k(x)\} \cup \{\psi_k(x)\} \cup \{\eta_{k,l}(x)\}_{l<\omega}$ is consistent, as required.

2. If $0 < n(\mathfrak{A}|a) < \infty$. We denote $l = n(\mathfrak{A}|a)$. Then we need to find $x \in B$ such that $x^{(k+1)} = 0, x^{(k)} \neq 0, x \neq 0, x \neq 1$, and $(\mathfrak{B}|x)^{(k)}$ is atomic and has l atoms. It suffices to take $x = b_1 \vee ... \vee b_l$ where $b_i^{(k)}$ is an atom in $\mathfrak{B}^{(k)}$, for the reasons explained in the previous case. Remember that it is obvious that there are at least l atoms in $\mathfrak{B}^{(k)}$ because otherwise $\mathfrak{B}^{(k+1)}$ would be trivial.

3. If $-\infty < n(\mathfrak{A}|a) < 0$. We denote $l = n(\mathfrak{A}|a)$. Notice that, in order to prove the lemma, we need to find $x \in \mathfrak{B}^{(k)}$ such that $x \neq 0, x \neq 1, x^{(k+1)} = 0, x^{(k)} \neq 0, x^{(k)} \neq 1$ and $(\mathfrak{B}|x)^{(k)}$ has l atoms but it is not atomic. The situation is similar to the previous cases but now we need to use the fact that $\mathfrak{B}^{(k)}$ is not atomic: we pick an element $c \in \mathfrak{B}^{(k)}$ such that c does not majorize any atom. Then it suffices to take $x = b_1 \vee ... \vee b_l \vee c$ where $b_i^{(k)}$ are different atoms in $\mathfrak{B}^{(k)}$ for every $i \leqslant l$, because $\mathfrak{B}|(b_1^{(k)} \vee ... \vee b_l^{(k)} \vee c)$ has l atoms but it is not atomic because c does not majorize any of them. The rest of the conditions $(b \neq 0, 1, b^{(k+1)} = 0$ but $b^{(k)} \neq 0)$ can be verified exactly as in the previous cases.

4. If $n(\mathfrak{A}|a) = -\infty$. Then we need to find $x \in B$ such that $x^{(k+1)} = 0, x^{(k)} \neq 0, x \neq 0, 1$ and $(\mathfrak{B}|x)^{(k)}$ has infinitely many atoms but is not atomic. It will be done if we can prove the consistency of the set of formulas: $\{x \neq 0 \& x \neq 1\} \cup \{\varphi_{k+1}(x)\} \cup \{\neg\varphi_k(x)\} \cup \{\neg\psi_k(x)\} \cup \{\eta_{k,l}(x)\}_{l<\omega}$, because of the ω-saturation. This is an infinite set of formulas, so it is consistent iff any of its finite subsets is consistent. Now, any subset has the form $\{x \neq 0 \ \& \ x \neq 1\} \cup \{\varphi_{k+1}(x)\} \cup \{\neg\varphi_k(x)\} \cup \{\neg\psi_k(x)\} \cup \{\eta_{k,1}(x), ..., \eta_{k,l}(x)\}$. Now, this is consistent because in case (3) we found an element that satistfied every formula in the set.

5. If $n(\mathfrak{A}|a) = 0$. Since $\mathfrak{B}^{(k)}$ is not atomic or atomless, there must be $b \in B$ such that $b^{(k)} \neq 0, 1$ and it does not majorize any atom. This means $(\mathfrak{B}|b)^{(k)}$ is atomless, so $n(\mathfrak{B}|b) = 0$. This is the element b we are looking for (we know $b \neq 0, b \neq 1$).

- The proof is analogous if $m(\mathfrak{A}|a^c) < m(\mathfrak{A}|a) = m(\mathfrak{A})$
- Assume $m(\mathfrak{A}|a) = m(\mathfrak{A}|a^c) < \infty$.
  We note $k = m(\mathfrak{A}) = m(\mathfrak{A}|a) = m(\mathfrak{A}|a^c) = m(\mathfrak{B})$. We can distinguish several cases:

1. If $n(\mathfrak{A}) = 0$, then $n(\mathfrak{B}) = 0$ and, for every $x \in B, n(\mathfrak{B}|x) = n(\mathfrak{B}|x^c) = 0$. As a consequence, to prove the result it suffices to take $b \in B$ such that $b^{(k)} \neq 0, 1$ in $\mathfrak{B}^{(k)}$. We know that this element actually exists because otherwise $\mathfrak{B}^{(k)}$ would be atomic ($\{0, 1\}$ is an atomic algebra).

2. If $n(\mathfrak{A}) > 0$, then $n(\mathfrak{A}|a) > 0, n(\mathfrak{A}|a^c) > 0$ and $n(\mathfrak{B}) = n(\mathfrak{A}) = n(\mathfrak{A}|a) + n(\mathfrak{A}|a^c)$. In order to prove the lemma we need to find $b \in B$ such that $b^{(k)}$ is atomic and has $n(\mathfrak{A}|a)$ atoms in $\mathfrak{B}^{(k)}$ and $(b^c)^{(k)}$ is atomic and has $n(\mathfrak{A}|a^c)$ atoms in $\mathfrak{B}^{(k)}$. It is obvious that, for any b we pick to prove the lemma, $b^{(k+1)} = 0$, because $k = m(\mathfrak{B})$. Now, several situations ought to be distinguished here:
   (a) If $n(\mathfrak{A}) = l$ and $n(\mathfrak{A}|a) = l_1, n(\mathfrak{A}|a^c) = l_2$, where l_1, l_2 are positive integers. Then $l = n(\mathfrak{A}|a) + n(\mathfrak{A}|a^c) = l_1 + l_2$ and we know that $\mathfrak{B}^{(k)}$ has l atoms. Let $b_1, ..., b_{l_1}, ..., b_l \in B$ be the elements such that $b_i^{(k)}$ are all the different atoms in $\mathfrak{B}^{(k)}$ for all $i \leqslant l$. It suffices to take $b = b_1 \vee ... \vee b_{l_1}$ because, in this algebra, $1^{(k)} = b_1^{(k)} \vee ... \vee b_l^{(k)}$, so $(b^c)^{(k)} = b_{l_1+1}^{(k)} \vee ... \vee b_l^{(k)}$. This means that $(\mathfrak{B}|b)^{(k)}$ has l_1 atoms $(b_1^{(k)}, ..., b_n^{(k)})$, and $(\mathfrak{B}|b^c)^{(k)}$ has $l - l_1 = l_2$ atoms. Also, we know that $b \neq 0, 1, b^{(k)} \neq 0$.
   (b) If $n(\mathfrak{A}) = \infty$ and $n(\mathfrak{A}|a) = n(\mathfrak{A}|a^c) = \infty$. We need to find x such that $x \neq 0, 1$, $x^{(k)} \neq 0$, $x^{(k+1)} = 0$ and $(\mathfrak{B}|x)^{(k)}, (\mathfrak{B}|x^c)^{(k)}$ are both atomic and have infinitely many atoms. Such element must verify

every formula in the set $\Sigma = \{x \neq 0, x \neq 1\} \cup \{\varphi_{k+1}(x), \neg\varphi_k(x)\} \cup \{\eta_{k,l}(x), \eta_{k,l}(x^{\mathrm{c}})\}_{l<\omega}$. If we consider a finite subset of Σ and see it is consistent, we will get that such b exists because of the ω-saturation of $\mathfrak{B}$. An arbitrary finite subset of Σ can be reduced to $\{x \neq 0, x \neq 1\} \cup \{\varphi_{k+1}(x), \neg\varphi_k(x)\} \cup \{\eta_{k,l}(x), \eta_{k,l}(x^{\mathrm{c}})\}$ for some l, and we have already shown this is consistent in the previous case.

(c) If $n(\mathfrak{A}) = \infty$, $n(\mathfrak{A}|a) = l$ but $n(\mathfrak{A}|a^{\mathrm{c}}) = \infty$. We need to find x such that $x \neq 0, 1$, $x^{(k)} \neq 0$, $x^{(k+1)} = 0$ and $(\mathfrak{B}|x)^{(k)}$, $(\mathfrak{B}|x^{\mathrm{c}})^{(k)}$ are both atomic but the first one has l atoms and the second one has infinitely many. Such element must verify every formula in the set $\Sigma = \{x \neq 0, x \neq 1\} \cup \{\varphi_{k+1}(x), \neg\varphi_k(x)\} \cup \{\eta_{k,l}(x), \eta_{k,r}(x^{\mathrm{c}})\}_{r<\omega} \cup \{\sigma_{k,l}(x)\}$. Therefore, such b exists iff Σ is consistent, and this is iff every finite subset of Σ is consistent. A finite arbitrary subset of Σ is of the form $\{x \neq 0, x \neq 1\} \cup \{\varphi_{k+1}(x), \neg\varphi_k(x)\} \cup \{\psi_k(x), \neg\psi_k(x^{\mathrm{c}})\} \cup \{\eta_{k,l}(x), \eta_{k,l}(x^{\mathrm{c}})\} \cup \{\sigma_{k,l}(x^{\mathrm{c}})\}$, and the fact that it is consistent results from the previous cases.

(d) If $n(\mathfrak{A}) = \infty$, $n(\mathfrak{A}|a^{\mathrm{c}}) = l$ but $n(\mathfrak{A}|a) = \infty$. This case is handled in an analogous way.

3. If $n(\mathfrak{A}) < 0$, then $n(\mathfrak{B}) < 0$ and $n(\mathfrak{B}) = n(\mathfrak{A})$. Given this situation, the only possible cases are:

   (a) $n(\mathfrak{A}|a) > 0$ and $n(\mathfrak{A}|a^{\mathrm{c}}) = 0$, or $n(\mathfrak{A}|a^{\mathrm{c}}) > 0$ and $n(\mathfrak{A}|a) = 0$,

   (b) $n(\mathfrak{A}|a) < 0$ and $n(\mathfrak{A}|a^{\mathrm{c}}) = 0$ or $n(\mathfrak{A}|a^{\mathrm{c}}) < 0$ and $n(\mathfrak{A}|a) = 0$,

   (c) $n(\mathfrak{A}|a) > 0$ and $n(\mathfrak{A}|a^{\mathrm{c}}) < 0$ or $n(\mathfrak{A}|a^{\mathrm{c}}) > 0$ and $n(\mathfrak{A}|a) < 0$,

   (d) $n(\mathfrak{A}|a) < 0$ and $n(\mathfrak{A}|a^{\mathrm{c}}) < 0$.

   We will only prove the first situation of each case, because the other one is done similarly. Also, we will not prove conditions like $b \neq 0, 1, b^{(k)} \neq 0$ or $b^{(k+1)} = 0$ because the proof is exactly the same as in previous cases. Of course, in any of this cases we also need to distinguish if $n(\mathfrak{A})$ is finite or infinite.

   (a) We need to find $b \in B$ such that $b \neq 0, 1, b^{(k)} \neq 0, b^{(k+1)} = 0$ and $(\mathfrak{B}|b)^{(k)}$ is atomic has $l = n(\mathfrak{A})$ atoms and $(\mathfrak{B}|b^{\mathrm{c}})^{(k)}$ is atomless. Let $b_1, ..., b_l \in B$ be the elements such that $b_i^{(k)}$ are all the different atoms in $\mathfrak{B}^{(k)}$ for every $i \leqslant l$. Then it suffices to take $b = b_1 \vee ... \vee b_l$. It is easy to see that $(\mathfrak{B}|b)^{(k)}$ is atomic and has l atoms, and $(\mathfrak{B}|b^{\mathrm{c}})^{(k)}$ is atomless because there are no atoms left in $\mathfrak{B}^{(k)}$: all of them are in $(\mathfrak{B}|b)^{(k)}$.

   (b) We need to find $b \in B$ such that $b \neq 0, 1, b^{(k)} \neq 0, b^{(k+1)} = 0$ and $(\mathfrak{B}|b)^{(k)}$ is not atomic and has got $l = n_0(\mathfrak{A})$ atoms and $(\mathfrak{B}|b^{\mathrm{c}})^{(k)}$

is atomless. Since $n(\mathfrak{B}) < 0$, there has to be an element $c \in B$ such that $c^{(k)}$ does not majorize any atoms. We can split c in two disjoint elements $c = c_1 \vee c_2$, where $c_1, c_2 \neq 0$. Then it suffices to take $b = b_1 \vee ... \vee b_l \vee c_1$, where $b_i^{(k)}$ are all the different atoms in $\mathfrak{B}^{(k)}$. Since $c_1 < c$, $b^{(k)} \neq 1$. We know that $(\mathfrak{B}|b)^{(k)}$ has l atoms but it is not atomic, because $c_1^{(k)} \in (\mathfrak{B}|b)^{(k)}$ and it does not majorize any atom. Also $(\mathfrak{B}|b^c)^{(k)}$ is atomless because all the atoms in $\mathfrak{B}^{(k)}$ are in $(\mathfrak{B}|b)^{(k)}$.

(c) We need to find $b \in B$ such that $b \neq 0, 1, b^{(k)} \neq 0, b^{(k+1)} = 0$ and $(\mathfrak{B}|b)^{(k)}$ is atomic has got $l_1 = n(\mathfrak{A}|a)$ atoms and $(\mathfrak{B}|b^c)^{(k)}$ is not atomic but has $l_2 = n_0(\mathfrak{A}|a^c)$ atoms. We will note $l = l_1 + l_2$, and we know there are l atoms in $\mathfrak{B}^{(k)}$. Notice that it suffices to take $b = b_1 \vee ... \vee b_{l_1}$, where $b_i^{(k)}$ are different atoms in $\mathfrak{B}^{(k)}$, because then $b^c = b_{l_1+1} \vee ... \vee b_l \vee c$, where c is defined as $b_1 \vee ... \vee b_l \vee c = 1$. Notice that c not atomic, because $n(\mathfrak{B}) < 0$, so $(\mathfrak{B}|b^c)^{(k)}$ has $l - l_1 = l_2$ atoms, and it is not atomic, as required.

(d) We need to find $b \in B$ such that $b \neq 0, 1, b^{(k)} \neq 0, b^{(k+1)} = 0$ and $(\mathfrak{B}|b)^{(k)}$, $(\mathfrak{B}|b^c)^{(k)}$ are both not atomic and have $l_1 = n_0(\mathfrak{A}|a), l_2 = n_0(\mathfrak{A}|a^c)$ atoms, respectively. Since $\mathfrak{B}^{(k)}$ is not atomic, $b_1 \vee ... \vee b_l \vee c = 1$, where $c \neq 0$ is atomless in $\mathfrak{B}^{(k)}$. Then, we can split c into $c = c_1 \vee c_2$, where $c_1, c_2 \neq 0$ but $c_1 \wedge c_2 = 0$, and we can take $b = b_1 \vee ... \vee b_{l_1} \vee c_1$. We know that $(\mathfrak{B}|b)^{(k)}$ has l_1 atoms but it is not atomic because $c_1 \in B|b$, and $b^c = b_{l_1} \vee ... \vee b_l \vee c_2$, which means that $(\mathfrak{B}|b^c)^{(k)}$ is not atomic either, and has $l - l_1 = l_2$ atoms, as required.

□

Definition 6.3.3 *Let $\mathfrak{A}, \mathfrak{B}$ be two Boolean algebras and take $a_1, ..., a_n \in A$ and $b_1, ..., b_n \in B$. It is said that $(\mathfrak{A}, a_1...a_n)$ is* similar *to $(\mathfrak{B}, b_1...b_n)$ iff for every $\varepsilon \in 2^n, \mathfrak{A}|\varepsilon\bar{a}$ and $\mathfrak{B}|\varepsilon\bar{b}$ are both trivial or they have the same invariants. It is written $(\mathfrak{A}, a_1...a_n) \approx (\mathfrak{B}, b_1...b_n)$*

Observation 6.3.4 *The previous definition is also valid for $n = 0$: It is said $\mathfrak{A}, \mathfrak{B}$ are similar iff they are both trivial or they have the same invariants. In this case we write $\mathfrak{A} \approx \mathfrak{B}$.*

Lemma 6.3.5 *Let $\mathfrak{B}$ be an ω-saturated Boolean algebra. Then, for any $b \in B$, $\mathfrak{B}|b$ is ω-saturated.*

Proof: This follows from the fact that a type over $a_1, ..., a_n$ in $\mathfrak{B}|b$ can be translated into a type over $a_1, ..., a_n, b$ in $\mathfrak{B}$, thus it must be realized. □

Theorem 6.3.6 *Let $\mathfrak{A}, \mathfrak{B}$ be Boolean algebras. Then they are elementary equivalent iff they are both trivial or they have the same invariants.*

Proof: The first implication is obvious because, if they are elementary equivalent, for any sentence φ, $\mathfrak{A} \models \varphi \Leftrightarrow \mathfrak{B} \models \varphi$. Due to proposition 6.2.5, having a determinate pair of invariants can be expressed using a sentence. As a consequence, $\mathfrak{A}$ and $\mathfrak{B}$ must have the same invariants.
Conversely, assume $\mathfrak{A}, \mathfrak{B}$ have the same invariants. We may also assume that they are ω-saturated, because otherwise they both have ω-saturated elementary extension. We will prove $\mathfrak{A}$ and $\mathfrak{B}$ are partially isomorphic, *i.e.*, there is a set I of partial isomorphisms such that $I : \mathfrak{A} \cong_p \mathfrak{B}$.
Claim: $I = \{\{(a_i, b_i) : i < n\} : (\mathfrak{A}, a_i)_{i<n} \approx (\mathfrak{B}, b_i)_{i<n}, n < \omega\}$ verifies $I : \mathfrak{A} \cong_p \mathfrak{B}$
It is obvious that $I \neq \emptyset$ because $\emptyset \in I$. Now, to prove that the elements in I are partial isomorphisms it suffices to see that, for every $\varepsilon \in 2^n, \varepsilon\overline{a} = 0$ iff $\varepsilon\overline{b} = 0$, where $\overline{a} = (a_1, ..., a_n)$ and $\overline{b} = (b_1, ..., b_n)$. If we fix $\varepsilon \in 2^n$, assuming $(\mathfrak{A}, a_1...a_n) \approx (\mathfrak{B}, b_1...b_n)$, then $\mathfrak{A}|\varepsilon\overline{a} \approx \mathfrak{B}|\varepsilon\overline{b}$. Therefore, if $\varepsilon\overline{a} = 0$, then $\mathfrak{A}|\varepsilon\overline{a}$ is the trivial algebra and $\mathfrak{B}|\varepsilon\overline{b}$ must be trivial too. For this to happen, $\varepsilon\overline{b}$ must be 0, because otherwise $\varepsilon\overline{b} \in \mathfrak{B}|\varepsilon\overline{b}$, and it would not be trivial. The argument is the same if we start assuming $\varepsilon\overline{b} = 0$.
The only thing left to prove is that the back-and-forth conditions hold. In fact, it suffices to see that one of them holds, because we get the other one by symmetry. We want to see that, for every $a_1, ..., a_{n+1} \in A$, and every $b_1, ..., b_n \in B$, if $(\mathfrak{A}, a_1...a_n) \approx (\mathfrak{B}, b_1...b_n)$ then there is $b_{n+1} \in B$ such that $(\mathfrak{A}, a_1...a_{n+1}) \approx (\mathfrak{B}, b_1...b_{n+1})$. In order to see this, first we need to prove that, assuming $(\mathfrak{A}, a_1...a_n) \approx (\mathfrak{B}, b_1...b_n)$, for every $\varepsilon \in 2^n$ there is $c \leqslant \varepsilon\overline{b}$ such that $(\mathfrak{A}|\varepsilon\overline{a} \wedge a_{n+1}) \approx (\mathfrak{B}|\varepsilon\overline{b} \wedge c)$ and $(\mathfrak{A}|\varepsilon\overline{a} \wedge a^{\mathrm{c}}_{n+1}) \approx (\mathfrak{B}|\varepsilon\overline{b} \wedge c^{\mathrm{c}})$.
Now, if $(\mathfrak{A}, a_1...a_n) \approx (\mathfrak{B}, b_1...b_n)$ then $\mathfrak{A}|\varepsilon\overline{a} \approx \mathfrak{B}|\varepsilon\overline{b}$. If they are both trivial, it suffices to take $c = 0$. Otherwise:

- If $\varepsilon\overline{a} \wedge a_{n+1} = 0$, then $c = 0$. The first condition is trivial and the second one follows from the fact that, in this case, $\varepsilon\overline{a} \wedge a^{\mathrm{c}}_{n+1} = \varepsilon\overline{a}$, so $\mathfrak{A}|\varepsilon\overline{a} \wedge a^{\mathrm{c}}_{n+1} = \mathfrak{A}|\varepsilon\overline{a} \approx \mathfrak{B}|\varepsilon\overline{b} = \mathfrak{B}|\varepsilon\overline{b} \wedge c^{\mathrm{c}}$.

- If $\varepsilon\overline{a} \wedge a_{n+1} = \varepsilon\overline{a}$, then $c = \varepsilon\overline{b}$. Similarly, the first condition is trivial and the second one follows from the fact that, in this case, $\varepsilon\overline{a} \wedge a^{\mathrm{c}}_{n+1} = 0$.

- If $0 \neq \varepsilon\overline{a} \wedge a_{n+1} \neq 0$. Then, due to lemmas 6.3.5 and 6.3.2, we know that there is $c \in \mathfrak{B}|\varepsilon\overline{b}$ such that $\mathfrak{A}|\varepsilon\overline{a} \wedge a_{n+1} \approx \mathfrak{B}|\varepsilon\overline{b} \wedge c$ and $\mathfrak{A}|\varepsilon\overline{a} \wedge a^{\mathrm{c}}_{n+1} \approx \mathfrak{B}|\varepsilon\overline{b} \wedge c^{\mathrm{c}}$.

Claim: If $\varepsilon_1, ..., \varepsilon_l$ are the only elements in 2^n such that $\varepsilon_i\overline{a} \wedge a_{n+1} \neq 0$ for every $i = 1, ..., l$, we can take $b_{n+1} = \bigvee_{1 \leqslant i \leqslant l} c_i$
We want to check if $(\mathfrak{A}, a_1...a_{n+1}) \approx (\mathfrak{B}, b_1...b_{n+1})$, *i.e.* $\mathfrak{A}|\varepsilon_i\overline{a'} \approx \mathfrak{B}|\varepsilon_i\overline{b'}$ for any $\varepsilon_i \in 2^{n+1}$, where $\overline{a}' = (a_1, ..., a_n, a_{n+1})$ and $\overline{b}' = (b_1, ..., b_n, b_{n+1})$.

- If $\varepsilon_i(n) = 1$, we want to see $\mathfrak{A}|\varepsilon_i\overline{a} \wedge a_{n+1} \approx \mathfrak{B}|\varepsilon_i\overline{b} \wedge b_{n+1}$. Now $\mathfrak{B}|\varepsilon_i\overline{b} \wedge b_{n+1} = \mathfrak{B}|\varepsilon_i\overline{b} \wedge (\bigvee c_j) = \mathfrak{B}|\bigvee_{1 \leqslant j \leqslant l}(\varepsilon_i\overline{b} \wedge c_j)$. As a consequence, if we see that

$\varepsilon_i\bar{b} \wedge c_j = 0$ for every $j \neq i$, it will be done, because then $\bigvee_{1\leqslant j\leqslant l}(\varepsilon_i\bar{b} \wedge c_j) = \bigvee_{1\leqslant j\leqslant l, j\neq i}(\varepsilon_i\bar{b} \wedge c_j) \vee \varepsilon_i\bar{b} \wedge c_i = \varepsilon_i\bar{b} \wedge c_i$, and we just proved $\mathfrak{A}|\varepsilon_i\bar{a} \wedge a_{n+1} \approx \mathfrak{B}|\varepsilon_i\bar{b} \wedge c_i$.
Let's prove $\varepsilon_i\bar{b} \wedge c_j = 0$ for every $j \neq i$. We know $c_j \leqslant \varepsilon_j\bar{b}$, so $\varepsilon_i\bar{b} \wedge c_j = \varepsilon_i\bar{b} \wedge \varepsilon_j\bar{b} \wedge c_j = 0$ for all $j \neq i$. For this reason, we can conclude the proof in this particular case.

- If $\varepsilon_i(n) = 0$, what we want to show is $\mathfrak{A}|\varepsilon_i\bar{a} \wedge a^{\mathrm{c}}_{n+1} \approx \mathfrak{B}|\varepsilon_i\bar{b} \wedge b^{\mathrm{c}}_{n+1}$ for every $\varepsilon_i \in 2^n$. Now, $b^{\mathrm{c}}_{n+1} = \bigvee_{1\leqslant j\leqslant l}(\varepsilon_j \wedge c^{\mathrm{c}}_j)$. As a consequence, $\varepsilon_i\bar{b} \wedge b_{n+1} = \varepsilon_i\bar{b} \wedge \bigvee_{1\leqslant j\leqslant l}(\varepsilon_j\bar{b} \wedge c^{\mathrm{c}}_j) = \bigvee_{1\leqslant j\leqslant l}(\varepsilon_i\bar{b} \wedge \varepsilon_j\bar{b}) \wedge c^{\mathrm{c}}_j = \varepsilon_i\bar{b} \wedge c^{\mathrm{c}}_i$, due to the fact that $\varepsilon_i\bar{b} \wedge \varepsilon_j\bar{b} = 0$ if $i \neq j$. This means $\mathfrak{A}|\varepsilon_i\bar{a} \wedge a^{\mathrm{c}}_{n+1} \approx \mathfrak{B}|\varepsilon_i\bar{b} \wedge c^{\mathrm{c}}_i = \mathfrak{B}|\varepsilon_i\bar{b} \wedge b^{\mathrm{c}}_{n+1}$, as we wanted to show.

Since we have proven the result for all the possible cases, we can end the proof saying that the back-and-forth conditions hold and, therefore, $\mathfrak{A}$ and $\mathfrak{B}$ are elementary equivalent. □

Proposition 6.3.7 *For every pair of invariants* (m, n)*, there is a Boolean algebra* $\mathfrak{B}$ *such that* $(m(\mathfrak{B}), n(\mathfrak{B})) = (m, n)$.

Proof: See [8]. Proposition 18.5. □

This is, that given a pair (m, n), we can construct $T_{(m,n)}$: a theory in the usual language of Boolean algebras which consists on expanding the usual one with the formulas that mean "the invariants of the models of this theory are (m, n)".

Let's take a moment to think about the consequences of this last theorem. Remember that we said in chapter 3 that a theory is complete iff any two models of the theory are elementary equivalent. If we consider $T_{(m,n)}$, for any pair (m, n), we know that any two models of this theory are elementarily equivalent, because they are Boolean algebras with the same invariants. Hence, $T_{(m,n)}$ is a complete theory, and we reach the conclusion that we only need to add the invariants to the theory of Boolean algebras in order to complete it. Moreover, all completions of the theory of Boolean algebras appear as $T_{(m,n)}$ for some m, n. The theory of the trivial algebra is formed by adding the axiom $0 = 1$ to the general theory of Boolean algebras.

In chapter 4 we studied the theories of atomless Boolean algebras and atomic and infinite Boolean algebras and, after a rather long proof, we got that they were both complete. Let's see what happens if we study their invariants:

- Because of proposition 6.1.10, we know that the invariant m of an atomless Boolean algebra is 0. Besides, the number of atoms of this kind of algebra is also 0, so the invariants of atomless Boolean algebras are $(0, 0)$.

- In the case of infinite and atomic Boolean algebras, the invariant m is 0 also because of proposition 6.1.10. Now, an atomic and infinite Boolean algebra must have infinitely many atoms, because if a Boolean algebra is atomic, the set of the atoms in the algebra generates it, and a finitely generated Boolean algebra is always finite. For this reason, the invariants of this kind of algebras are $(0, \infty)$.

Since the invariants only depend on the fact that the Boolean algebra is atomless or infinite and atomic, respectively, any two models of these Boolean algebras have the same invariants, thus they must be elementary equivalent. Notice that we reach the same conclusion as we did before: the theory of both atomless Boolean algebras and infinite atomic Boolean algebras is complete. However, we cannot obtain, using this method, any result on quantifier elimination and ω-categoricity.

One can also ask about other kind of Boolean algebras, for example, the finite ones. Every finite Boolean algebra is isomorphic to a power set algebra, thus it is atomic, and its invariant m must be 0. Moreover, its cardinality is 2^k, so the number of atoms in the Boolean algebra is k. As a result, the invariants of these Boolean algebras are $(0, k)$. This means that the theory of finite Boolean algebras with cardinal 2^k is complete. In particular, the Boolean algebra $\{0, 1\}$ has the invariants $(0, 1)$.

Bibliography

[1] BELL, J. L., AND MACHOVER, M. *A Course in Mathematical Logic.* North Holland P.C., Amsterdam, 1977.

[2] CASANOVAS, E. *Teoría de modelos.* //www.ub.edu/modeltheory/casanovas/e.html, 2000.

[3] CHANG, C. C., AND KEISLER, H. *Model theory.* North Holland P.C., Amsterdam, 1977.

[4] EBBINGHAUS, H., FLUM, J., AND THOMAS, W. *Mathematical Logic.* Springer-Verlag, 1984.

[5] ERSHOV, Y. Decidability of the elementary theory of relatively complemented lattices and of the theory of filters. *Algebra i Logika 3* (1964), 17–38.

[6] HALMOS, P., AND GIVANT, S. *Introduction to Boolean algebras.* Springer, 2000.

[7] JANÉ, I. *Álgebras de Boole y lógica.* Publicacions Universitat de Barcelona, 1989.

[8] KOPPELBERG, S. *HandBook of Boolean Algebras, Volume I.* North Holland P.C., 1989.

[9] POIZAT, B. *A Course in model theory: an introduction to contemporary mathematical logic.* Springer, New York, 2000.

[10] TARSKI, A. Arithmetical classes and types of Boolean algebras. *Bulletin of American Mathematical Society 55* (1949), 64.

Index