New Predictions for Inclusive Heavy-Quarkonium *P*-Wave Decays

Nora Brambilla,[1] Dolors Eiras,[2] Antonio Pineda,[3] Joan Soto,[2] and Antonio Vairo[4]

[1]*INFN and Dipartimento di Fisica dell'Università di Milano, via Celoria 16, 20133 Milan, Italy*

[2]*Departamento d'Estructura i Constituents de la Matèria and IFAE, U. Barcelona, Diagonal 647, E-08028 Barcelona, Catalonia, Spain*

[3]*Institut für Theoretische Teilchenphysik, U. Karlsruhe, D-76128 Karlsruhe, Germany*

[4]*Theory Division CERN, 1211 Geneva 23, Switzerland*

(Received 17 September 2001; published 17 December 2001)

We show that some nonrelativistic quantum chromodynamics color-octet matrix elements can be written in terms of (derivatives of) wave functions at the origin and of nonperturbative universal constants once the factorization between the soft and ultrasoft scales is achieved by using an effective field theory where only ultrasoft degrees of freedom are kept as dynamical entities. This allows us to derive a new set of relations between inclusive heavy-quarkonium P-wave decays into light hadrons with different principal quantum numbers and with different heavy flavors. In particular, we can estimate the ratios of the decay widths of bottomonium P-wave states from charmonium data.

DOI: 10.1103/PhysRevLett.88.012003 PACS numbers: 12.38.–t, 12.39.Hg, 13.25.Gv

Inclusive P-wave decays to light hadrons have proved to be an optimal testing ground of our understanding of heavy quarkonia. The use of nonrelativistic quantum chromodynamics (NRQCD) [1,2] allowed a description of these decays in terms of expectation values of some 4-heavy-quark operators at a quantum-field level in a systematic way. Besides the so-called color-singlet operators, for which their expectation values could be related to wave functions in an intuitive way, there were also color-octet operators. The latter were decisive in solving the infrared sensitivity of earlier calculations [3]. It has been thought so far that these color-octet expectation values could not be related to a Schrödinger-like formulation in any way.

We show in this Letter that it is not so. For certain states, the expectation values of color-octet operators can also be written in terms of wave functions and additional bound-state-independent nonperturbative parameters. We will focus on the operators relevant to P-wave decays into light hadrons, but it should become apparent that this is a general feature.

The line of developments that has led us to this result is the following. It was pointed out in Ref. [4] that NRQCD still contains dynamical scales, which are not relevant to the kinematical situation of the lower-lying states in heavy quarkonium (energy scales larger than the ultrasoft scale, mv^2, with v being the relative velocity of the heavy quark and m being its mass). Hence, further simplifications occur if we integrate them out. We call potential NRQCD (pNRQCD) the resulting effective field theory [as in [5]; note that in [6], in the situation $\Lambda_{\rm QCD} \gg mv^2$, the effective field theory (EFT) was called pNRQCD$'$]. When the typical scale of nonperturbative physics, say, $\Lambda_{\rm QCD}$, is smaller than the soft scale mv, and larger than the ultrasoft scale mv^2, the soft scale can be integrated out perturbatively. This leads to an intermediate EFT that also contains, besides the singlet, octet fields and ultrasoft gluons as dynamical degrees of freedom [4,6]. These are eventually integrated out by the (nonperturbative) matching to pNRQCD [6]. When $\Lambda_{\rm QCD}$ is of the order of the soft scale, the (nonperturbative) matching to pNRQCD has to be done in one single step. This framework has been developed in a systematic way in Ref. [5].

In this Letter we will compute the inclusive P-wave decay widths into light hadrons at leading order for $\Lambda_{\rm QCD} \gg mv^2$ by using pNRQCD. In this situation the singlet is the only dynamical field in pNRQCD (Goldstone bosons are also present, but they play a negligible role in this analysis and will be ignored), if hybrids and other degrees of freedom associated with heavy-light meson pair threshold production develop a mass gap of $\mathcal{O}(\Lambda_{\rm QCD})$, as we will assume in what follows [5,6], or if they play a minor role in the heavy-quarkonium dynamics. Therefore, the pNRQCD Lagrangian reads [5,6]

$$\mathcal{L}_{\rm pNRQCD} = {\rm Tr}\{S^\dagger(i\partial_0 - h)S\}, \tag{1}$$

where h is the pNRQCD Hamiltonian, to be determined by matching the EFT to NRQCD. The total decay width of the singlet heavy-quarkonium state is then given by

$$\Gamma = -2\,{\rm Im}\langle n, L, S, J|h|n, L, S, J\rangle, \tag{2}$$

where $|n, L, S, J\rangle$ are the eigenstates of the Hamiltonian h. The imaginary parts are inherited from the 4-heavy-fermion NRQCD Wilson coefficients and, for P-wave decays, first appear as local (deltalike) $\mathcal{O}(1/m^4)$ potentials in the pNRQCD Lagrangian. The relevant structure reads [we shall concentrate on potentials, which inherit imaginary parts from the NRQCD operators and which contribute to P-wave states at first order in quantum-mechanical perturbation theory (QMPT)]

$$-2\,{\rm Im}h|_{P\ \rm wave} = F_{SJ}\mathcal{T}^{ij}_{SJ}\frac{\nabla^i_{\bf r}\delta^{(3)}({\bf r})\nabla^j_{\bf r}}{m^4}, \tag{3}$$

where $\mathcal{T}^{ij}_{SJ}$ corresponds to the spin and total angular momentum wave-function projectors. What is now left is to compute F_{SJ}, i.e., to perform the matching between

NRQCD and pNRQCD. For the situation (A) where $mv \gg \Lambda_{\rm QCD} \gg mv^2$, by taking the results of [6], and for the more general situation (B) where $\Lambda_{\rm QCD} \lesssim mv$, by using the formalism of Ref. [5], we obtain

$$F_{SJ} = -2N_c \, {\rm Im} f_1(^{2S+1}P_J) - \frac{4T_F}{9N_c} \mathcal{E} \, {\rm Im} f_8(^{2S+1}S_S), \tag{4}$$

where $f_1(^{2S+1}L_J)$ and $f_8(^{2S+1}L_J)$ are the short-distance Wilson coefficients of NRQCD, as defined in Ref. [2], and

$$\mathcal{E} = T_F \int_0^\infty d\tau \, \tau^3 \langle g\mathbf{E}^a(\tau,\mathbf{0}) \Phi_{ab}(\tau,0;\mathbf{0}) g\mathbf{E}^b(\tau,\mathbf{0})\rangle. \tag{5}$$

(A) P-wave potentials for $mv \gg \Lambda_{\rm QCD} \gg mv^2$.—In this case the matching from NRQCD to pNRQCD at the scale $\Lambda_{\rm QCD}$ can be done in two steps. In the first step, which can be done perturbatively, we integrate out the scale mv and end up with an EFT, which contains singlet (S) and octet (O) fields as dynamical degrees of freedom. At the next-to-leading order in the multipole expansion the Lagrangian reads [4,6]

$$\begin{aligned} \mathcal{L} &= {\rm Tr}\{S^\dagger(i\partial_0 - h_s)S + O^\dagger(iD_0 - h_o)O\} \\ &\quad + {\rm Tr}\left\{O^\dagger \mathbf{r}\cdot g\mathbf{E} S + {\rm H.c.} + \frac{O^\dagger \mathbf{r}\cdot g\mathbf{E} O}{2} \right. \\ &\quad \left. + \frac{O^\dagger O \mathbf{r}\cdot g\mathbf{E}}{2}\right\} - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a}, \end{aligned} \tag{6}$$

where h_s and h_o have to be determined by matching to NRQCD. They read as follows:

$$\begin{aligned} h_s &= -\frac{\nabla_{\mathbf{r}}^2}{m} + V_s(r) + \ldots \\ &\quad + N_c f_1(^{2S+1}P_J) \mathcal{T}^{ij}_{SJ} \frac{\nabla^i_{\mathbf{r}} \delta^{(3)}(\mathbf{r}) \nabla^j_{\mathbf{r}}}{m^4} + \ldots, \\ h_o &= -\frac{\nabla_{\mathbf{r}}^2}{m} + V_o(r) + \ldots \\ &\quad + T_F f_8(^{2S+1}S_S) \mathcal{T}_S \frac{\delta^{(3)}(\mathbf{r})}{m^2} + \ldots, \end{aligned} \tag{7}$$

neglecting center-of-mass recoil terms; $\mathcal{T}_S$ corresponds to the total spin projector. Beyond $\mathcal{O}(1/m)^0$ we have displayed only the terms that are relevant to our calculation. In the second step we integrate out (nonperturbatively) the gluons and the octet field, ending up with the pNRQCD Lagrangian (1). The Hamiltonian h has to be determined by matching the two effective field theories. It reads $h = h_s + \delta h_s$, with (at leading nonvanishing order in the multipole expansion)

$$\begin{aligned} \delta h_s &= -i \frac{T_F}{N_c} \int_0^\infty d\tau \, e^{ih_s(\tau/2)} \langle \mathbf{r}\cdot g\mathbf{E}^a(\tau,\mathbf{0}) \\ &\quad \times e^{-ih_o\tau} \Phi_{ab}(\tau,0;\mathbf{0}) g\mathbf{E}^b(0,\mathbf{0}) \cdot \mathbf{r}\rangle e^{ih_s(\tau/2)}, \end{aligned} \tag{8}$$

where consistency with $\Lambda_{\rm QCD} \gg mv^2$ requires an expansion of the exponentials of h_o and h_s. By taking into account the fact that we are interested in P-wave states, only the perturbation that puts one h_o to each side of the $\mathcal{O}(1/m^2)$ S-wave potential survives at leading order. The final result reads

$$ {\rm Im}\delta h_s|_{P\ \rm wave} = \frac{2T_F}{9N_c} \mathcal{E} \frac{\nabla_{\mathbf{r}} \delta^{(3)}(\mathbf{r}) \nabla_{\mathbf{r}}}{m^4} \mathcal{T}_S \, {\rm Im} f_8(^{2S+1}S_S) \tag{9}$$

which plugged into Eq. (3), gives Eq. (4). This shows how a color-octet operator in NRQCD becomes a color-octet potential in the EFT of Eq. (6) and, eventually, contributes to a color-singlet potential in pNRQCD, which is one of our main points.

(B) P-wave potentials for $\Lambda_{\rm QCD} \lesssim mv$.—In the case $\Lambda_{\rm QCD} \lesssim mv$ the matching from NRQCD to pNRQCD at the scale $\Lambda_{\rm QCD}$ has to be done directly, since no other relevant scales are supposed to lie between m and mv. The only dynamical degree of freedom of pNRQCD is the heavy-quarkonium singlet field S. The Lagrangian has been written in (1). The Hamiltonian h is obtained by matching (nonperturbatively) to NRQCD, order by order in $1/m$, within a Hamiltonian formalism [5]. In this Letter we sketch only the main steps of the derivation. In short, we can formally expand the NRQCD Hamiltonian in $1/m$:

$$H_{\rm NRQCD} = H^{(0)}_{\rm NRQCD} + \frac{1}{m} H^{(1)}_{\rm NRQCD} + \ldots. \tag{10}$$

The eigenstates of the heavy quark-antiquark sector can be labeled as

$$|\underline{g}; \mathbf{x}_1, \mathbf{x}_2\rangle = |\underline{g}; \mathbf{x}_1, \mathbf{x}_2\rangle^{(0)} + \frac{1}{m} |\underline{g}; \mathbf{x}_1, \mathbf{x}_2\rangle^{(1)} + \ldots,$$

where $\underline{g}$ labels the color-related degrees of freedom (we do not explicitly display spin labels for simplicity). Assuming a mass gap of $\mathcal{O}(\Lambda_{\rm QCD})$ much larger than mv^2, all the excitations ($\underline{g} \neq 0$) decouple and the ground state ($\underline{g} = \underline{0}$) corresponds to the singlet state. Therefore, the matching condition reads

$$\begin{aligned} &\langle \underline{0}; \mathbf{x}_1, \mathbf{x}_2 | H | \underline{0}; \mathbf{x}'_1, \mathbf{x}'_2\rangle \\ &\quad = h(\mathbf{x}_1, \mathbf{x}_2, \nabla_{\mathbf{x}_1}, \nabla_{\mathbf{x}_2}) \delta^{(3)}(\mathbf{x}_1 - \mathbf{x}'_1) \delta^{(3)}(\mathbf{x}_2 - \mathbf{x}'_2). \end{aligned} \tag{11}$$

Up to $\mathcal{O}(1/m^4)$ the imaginary contributions are carried only by the Wilson coefficients of the dimension-6 and dimension-8 4-heavy-fermion operators in NRQCD. Since we are interested only in Eq. (3), a huge simplification occurs and only two contributions survive. From the dimension-8 operators we obtain

$$\begin{aligned}
&\mathrm{Im}\,\delta h\,\delta^{(3)}(\mathbf{x}_1 - \mathbf{x}_1')\delta^{(3)}(\mathbf{x}_2 - \mathbf{x}_2') \\
&\quad = \frac{1}{m^4}\,\mathrm{Im}^{(0)}\langle \underline{0}; \mathbf{x}_1, \mathbf{x}_2 | H^{(4)}_{\mathrm{NRQCD}} | \underline{0}; \mathbf{x}_1', \mathbf{x}_2'\rangle^{(0)}|_{P \text{ wave}} \\
&\quad = N_c \mathcal{T}^{ij}_{SJ}\,\mathrm{Im} f_1(^{2S+1}P_J)\,\frac{\nabla^i_{\mathbf{r}}\delta^{(3)}(\mathbf{r})\nabla^j_{\mathbf{r}}}{m^4} \\
&\qquad \times \delta^{(3)}(\mathbf{x}_1 - \mathbf{x}_1')\delta^{(3)}(\mathbf{x}_2 - \mathbf{x}_2'). \qquad (12)
\end{aligned}$$

On the other hand, we also have contributions from the iteration of lower-order $1/m$ corrections to the NRQCD Hamiltonian with the dimension-6 4-heavy-fermion operators. The only term that contributes to Eq. (3) is

$$\begin{aligned}
&\mathrm{Im}\,\delta h\,\delta^{(3)}(\mathbf{x}_1 - \mathbf{x}_1')\delta^{(3)}(\mathbf{x}_2 - \mathbf{x}_2') \\
&\quad = \frac{1}{m^4}\,\mathrm{Im}^{(1)}\langle \underline{0}; \mathbf{x}_1, \mathbf{x}_2 | H^{(2)}_{\mathrm{NRQCD}} | \underline{0}; \mathbf{x}_1', \mathbf{x}_2'\rangle^{(1)}|_{P \text{ wave}}\,. \qquad (13)
\end{aligned}$$

The explicit computation of the right-hand side of Eq. (13) gives (as far as the P-wave contribution is concerned) Eq. (9). Therefore, the sum of the contributions from Eqs. (12) and (13) coincides with Eq. (3), after the replacements (4) and (5).

We can now obtain the decay widths by using Eq. (3). At first order in QMPT, we obtain

$$\Gamma(\chi^S_{QJ}(nP) \to LH) = \left[\frac{3N_c}{\pi}\,\mathrm{Im} f_1(^{2S+1}P_J) + \frac{2T_F}{3\pi N_c} \times \mathrm{Im} f_8(^{2S+1}S_S)\mathcal{E}\right]\frac{|R'_{Qn1}(0)|^2}{m^4}\,, \qquad (14)$$

where $\chi^1_{QJ}(nP) := \chi_{QJ}(nP)$, $\chi^0_{QJ}(nP) := h_Q(nP)$ ($Q = b, c$), n is the principal quantum number, and $R_{Qn1}(r)$ is the radial wave function at leading order. Comparing with the standard NRQCD formula, where spin symmetry has already been used, we have

$$\langle h_Q(nP)|O_8(^1S_0)\,|h_Q(nP)\rangle(\mu) = \frac{|R'_{Qn1}(0)|^2}{3\pi N_c m^2}\,T_F\mathcal{E}(\mu)\,. \qquad (15)$$

The information gained with this formula is that all nonperturbative flavor and principal quantum number dependence are encoded in the wave function, as in the color-singlet operators. The additional nonperturbative parameter $\mathcal{E}(\mu)$ is universal: it depends only on the light degrees of freedom of QCD. This implies that the following relation between decay widths is also universal:

$$\mathcal{E}(\mu) = -\frac{9N_c^2}{2T_F} \times \frac{\mathrm{Im} f_1(^{2S+1}P_J) - \mathrm{Im} f_1(^{2S'+1}P_{J'})\,\frac{\Gamma[\chi^S_{QJ}(nP)\to LH]}{\Gamma[\chi^{S'}_{QJ'}(nP)\to LH]}}{\mathrm{Im} f_8(^{2S+1}S_S) - \mathrm{Im} f_8(^{2S'+1}S_{S'})\,\frac{\Gamma[\chi^S_{QJ}(nP)\to LH]}{\Gamma[\chi^{S'}_{QJ'}(nP)\to LH]}}\,. \qquad (16)$$

It is interesting to notice that the UV behavior of $\mathcal{E}$ has the logarithmic divergence

$$\mathcal{E}(\mu) \simeq 12 N_c C_F\,\frac{\alpha_s}{\pi}\ln\mu\,, \qquad (17)$$

which matches exactly the IR log of the $\mathcal{O}(\alpha_s)$ correction of $\mathrm{Im} f_1(^{2S+1}P_J)$, and hence the cancellation originally observed in [1] is fulfilled. One could then consider the leading log (LL) renormalization group improvement of $\mathcal{E}$ by using the results of Ref. [2] for the running of the octet-matrix element. One obtains ($\beta_0 = 11C_A/3 - 4n_f T_F/3$):

$$\mathcal{E}(\mu) = \mathcal{E}(\mu') + \frac{24 N_c C_F}{\beta_0}\ln\frac{\alpha_s(\mu')}{\alpha_s(\mu)}\,. \qquad (18)$$

Let us apply the above results to actual quarkonium, under the assumption that our framework, discussed in the text preceding Eq. (1), provides a reasonable description for the P-wave states observed in nature. The numerical extraction of $\mathcal{E}$ is a delicate task, since several of the Wilson coefficients (see Ref. [7] for a full list of them) have large next-to-leading order contributions, which may spoil the convergence of the perturbative series. This problem is not specific to our formalism, but belongs to the standard formulation of NRQCD. Here, in order to give an estimate, we use only those data that provide more stable results in going from the leading to the next-to-leading order (NLO), more precisely the average of Eq. (16) for $(J, S) = (1, 1)$, $(J', S') = (0, 1)$ and $(J, S) = (1, 1)$, $(J', S') = (2, 1)$. The experimental data were taken from [8] and updated according to [9,10]. Our final value reads

$$\mathcal{E}(1\ \mathrm{GeV}) = 5.3^{+3.5}_{-2.2}(\mathrm{expt.})\,, \qquad (19)$$

where we have used the NLO results for the Wilson coefficients with a LL improvement. The errors refer only to the experimental uncertainties of the decay widths. Theoretical uncertainties mainly come from subleading operators in the power counting [$\mathcal{O}(v)$ suppressed] and subleading terms in the perturbative expansion of the Wilson coefficients [$\mathcal{O}(\alpha_s)$ suppressed], whose bad convergence may affect Eq. (19) considerably. We feel, therefore, that further studies, maybe along the lines of Ref. [11], are needed before a complete numerical analysis, including theoretical uncertainties, can be done. In any case, the above equation is compatible with the values that are usually assigned to the NRQCD octet and singlet matrix elements [e.g., from the fit of [12], one obtains $\mathcal{E}(1\ \mathrm{GeV}) = 3.6^{+3.6}_{-2.9}(\mathrm{expt.})$].

The above equation is also compatible with the charmonium (quenched) lattice data of [13], whereas, if the running (18) is taken into account, bottomonium lattice data, quenched [13] and unquenched [14], appear to give a lower value. Note that, in the language of Refs. [13,14], Eq. (15) reads $\mathcal{E}(\mu) = 81 m_{b,c}^2 \mathcal{H}_8(\mu)/\mathcal{H}_1|_{b,c}$, which implies $\mathcal{H}_8(\mu)/\mathcal{H}_1|_b \times \mathcal{H}_1/\mathcal{H}_8(\mu)|_c = m_c^2/m_b^2$. For all quarkonium states that satisfy our assumptions this equality must be fulfilled by lattice results for any number of light fermions and for any value of the heavy-quark masses.

By using the estimate (19), we can also predict the ratios of the decay widths for the $n = 1, 2$ P-wave bottomonium states. We obtain

$$\frac{\Gamma[\chi_{b1}^1(1P)]}{\Gamma[\chi_{b2}^1(1P)]} = \frac{\Gamma[\chi_{b1}^1(2P)]}{\Gamma[\chi_{b2}^1(2P)]} = 0.50^{+0.06}_{-0.04}, \tag{20}$$

where only the errors inherited from Eq. (19) have been included. For what concerns theoretical uncertainties, the comments after Eq. (19) apply also here (with a better behavior of the perturbative series). Note that the first equality holds independently from Eq. (19) and from the use of charmonium data and, hence, provides a more robust prediction. The remaining ratios of the decay widths can be obtained by using spin symmetry. Notice also that, although no model-independent predictions can be made for the decay widths (they depend on the wave function at the origin, which is flavor and state dependent), our results allow any model that gives a definite value to $R'_{Qn1}(0)$ to make definite predictions.

In conclusion, we have exploited the fact that NRQCD still contains irrelevant degrees of freedom for certain heavy quarkonium states, which can be integrated out in order to constrain the form of the matrix elements of color-octet operators. We have focused on the operators relevant to P-wave decays, which allowed us to produce concrete, new, rigorous results. However, it should be clear from the structure of the pNRQCD Lagrangian itself that similar results can be obtained for matrix elements of any color-octet operator.

N. B. thanks R. Mussa for illuminating correspondence. N. B. and A. V. thank G. Bali and C. Davies for discussions. D. E. and J. S. are supported by the AEN98-031 (Spain) and the 1998SGR 00026 (Catalonia). D. E. also acknowledges financial support from a MEC FPI fellowship (Spain). A. P. thanks the University of Barcelona and CERN for hospitality while part of this work was carried out. A. V. thanks F. Maltoni for providing him with the second part of Ref. [7]. A. V. is supported by the European Community through the Marie-Curie fellowship HPMF-CT-2000-00733.

[1] G. T. Bodwin, E. Braaten, and G. P. Lepage, Phys. Rev. D **46**, R1914 (1992).
[2] G. T. Bodwin, E. Braaten, and G. P. Lepage, Phys. Rev. D **51**, 1125 (1995); *ibid.* **55**, 5853(E) (1997).
[3] R. Barbieri, R. Gatto, and E. Remiddi, Phys. Lett. **61B**, 465 (1976); R. Barbieri, M. Caffo, and E. Remiddi, Nucl. Phys. **B162**, 220 (1980); R. Barbieri, M. Caffo, R. Gatto, and E. Remiddi, Phys. Lett. **95B**, 93 (1980); Nucl. Phys. **B192**, 61 (1981).
[4] A. Pineda and J. Soto, Nucl. Phys. (Proc. Suppl.) **B64**, 428 (1998).
[5] N. Brambilla, A. Pineda, J. Soto, and A. Vairo, Phys. Rev. D **63**, 014023 (2001); A. Pineda and A. Vairo, Phys. Rev. D **63**, 054007 (2001); *ibid.* **64**, 039902(E) (2001).
[6] N. Brambilla, A. Pineda, J. Soto, and A. Vairo, Nucl. Phys. **B566**, 275 (2000).
[7] A. Petrelli, M. Cacciari, M. Greco, F. Maltoni, and M. L. Mangano, Nucl. Phys. **B514**, 245 (1998); F. Maltoni, Ph.D. thesis, University of Pisa, 1999 (http://web.hep.uiuc.edu/home/maltoni/thesis.ps).
[8] D. E. Groom *et al.,* Eur. Phys. J. C **15**, 1 (2000).
[9] M. Ambrogiani *et al.,* Phys. Rev. D **62**, 052002 (2000).
[10] R. Mussa (private communication); see also Proceedings of the Symposium on Charmonium Spectroscopy: Past and Future, Genoa, 2001 (to be published) (http://www.ge.infn.it/charm2001/).
[11] G. T. Bodwin and Y. Chen, hep-ph/0106095; Phys. Rev. D **60**, 054008 (1999); E. Braaten and Y. Chen, Phys. Rev. D **57**, 4236 (1998).
[12] F. Maltoni, hep-ph/0007003.
[13] G. T. Bodwin, D. K. Sinclair, and S. Kim, Int. J. Mod. Phys. A **12**, 4019 (1997).
[14] G. T. Bodwin, D. K. Sinclair, and S. Kim, hep-lat/0107011.