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GT-Varieties.

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Abstract

Fixed $4 \leq d$ and a primitive dth root of unity e, we consider the ideal I_d generated by all the μ monomials of degree d invariant under the action of the diagonal matrix $M = Diag(1, e, e^2, e^3)$. We prove that I_d is a monomial Galois Togliatti system (GT-system). We study the variety F_d image of the Galois covering $\varphi_{I_d} : \mathbb{P}^3 \to \mathbb{P}^{\mu-1}$ with cyclic Galois group $\mathbb{Z}/d$ associated to I_d. We call this 3-dimensional variety GT-threefold. Finally, we demonstrate that the homogeneous ideal of GT-threefolds is a lattice ideal associated to a saturated partial character from $\mathbb{Z}^\mu$.

Keywords. *Binomial ideal, Galois covering, GT-system, lattice ideal, quotient singularities, Togliatti sytem, toric variety, weak Lefschetz property .*

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Introduction

Even though Lefschetz properties date back to the Hard Lefschetz Theorem and the Sperner Theory at the mid 20th century, the systematic study of Lefschetz properties started afterwards R. Stanley in 1980 proved, using Algebraic Topology, that any artinian monomial complete intersection has the Strong Lefschetz property. Through the following two decades another proofs of the same result followed, for example it was proved by J. Watanabe [23] using representation theory. Of major surprise it was the assortment of approaches and methods used: representation theory, commutative and linear algebra and combinatorics. And of course, it was to be expected that Lefschetz properties became a node of many areas in mathematics with an amazing number of applications on it.

Very recently, E. Mezzetti, R. M. Miró-Roig and G. Ottaviani discovered a new connection: the existence of homogeneous artinian ideals failing the weak Lefschetz property and the existence of projective varieties satisfying at least one Laplace equation. Furthermore, by means of this unexpected relation M. Michałek and R. M. Miró-Roig in [11] and later E. Mezzetti and R. M. Miró-Roig in [10] contributed remarkably on the challenging problem of classifying them. In [12], it is proved that an artinian ideal $I \subset K[x_0, \ldots, x_n]$ generated by r forms $F_1, \ldots, F_r$ of degree d fails the weak Lefschetz property in degree $d-1$ if and only if the projection of the Veronese variety of $\mathbb{P}^{\binom{n+d}{d}-1}$, parametrized by all monomials of degree d, from the linear space $\langle F_1, \ldots, F_r \rangle$ satisfies a Laplace equation of order $d-1$. They called Togliatti system to that kind of artinian ideal and smooth Togliatti system whether the apolar linear system parametrizes a smooth variety. Also explained in [12], the name is in honour to the italian mathematician E. Togliatti whose famous articles [20] and [21] together with the work of Brenner and Kaid [2] suggested them the connection mentioned above. In [20],[21] Togliatti proved that the rational surface in $\mathbb{P}^5$ parametrized by the apolar system of the ideal $I_3 = (x^3, y^3, z^3, xyz)$ satisfies a Laplace equation of order 2. While Brenner and Kaid in [2] showed that any ideal of the type $(x^3, y^3, z^3, f(x,y,z))$ with f a form of degree 3 fails the weak Lefschetz property if and only if $f \in I_3$. A complete classification of smooth Togliatti systems of cubics was achieved in [13]. In [10], focusing on minimal smooth Togliatti systems, the authors established minimal and maximal bounds for the number of generators of smooth Togliatti system and they classified Togliatti systems with minimal number of generators close to this bound. For $d \geq 4$ the problem appears intractable. On the other hand, E. Mezzetti and R. M. Miró-Roig observed that Togliatti's example I_3 has another notable property, the associated morphism $\varphi : \mathbb{P}^2 \to \mathbb{P}^5$ is a Galois covering of the surface image with cyclic Galois group $\mathbb{Z}/3$. Based on this observation, in [11] the authors constructed a new family of Togliatti systems called GT-system (Togliatti Galois system) sharing this property. Namely, a GT-system is a Togliatti system $I \subset K[x,y,z]$ generated by forms of degree $d \geq 3$ whose associated regular map is a Galois covering with cyclic Galois group $\mathbb{Z}/d$. They proved that the ideal generated by all monomials of degree $d \geq 3$ invariant under the action of the matrix $M_{a,b} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^a & 0 \\ 0 & 0 & e^b \end{pmatrix}$ where $e^d = 1$ and $gcd(a,b,d) = 1$ is a monomial GT-system. The classification of GT-systems started in [11] where the authors showed that for $d \geq 3$ prime or power prime all GT-systems are minimal. Later, this problem has been studied in [4]. Of major importance for the present work is the study in [11] of the geometry of the family of GT-systems given by the matrix $M_{1,2}$, actually the generalization of any degree $d \geq 3$ of the Togliatti's example I_3. E. Mezzetti and R. M. Miró-Roig called them generalized classical Togliatti systems and proved that the surface S_d image of the associated morphism is toric, arithmetically Cohen-Macaulay and its homogeneous ideal is generated by quadrics if d even and by quadrics and cubics if d odd.

This work started from [11] with the hope of extending the notion of GT-systems to n variables and more concrete the study of their associated variety. We stablish that the ideal I_d generated by the μ_{I_d} monomials of degree $d \geq 4$ invariant under the action of the matrix $M_{a_0,\ldots,a_n} := \begin{pmatrix} e^{a_0} & 0 & \cdots & 0 \\ 0 & e^{a_1} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & e^{a_n} \end{pmatrix}$ where $e^d = 1$ and $gcd(a_0, \ldots, a_n, d) = 1$ is a GT-system provided $\mu_{I_d} \leq \binom{n+d-1}{n-1}$. With the purpose of proving it we show that $\varphi_{I_d} : \mathbb{P}^n \to \mathbb{P}^{\mu_{I_d}-1}$ is the quotient projective variety of $\mathbb{P}^n$ by the cyclic group of automorphisms generated by $M_{a_0,\ldots,a_n}$. Among these GT-systems we restrict our attention to those extending the classical generalized Togliatti system. In [4], the authors proved that when $d = n+1$, then number of generators is bounded by $\binom{2n}{n-1}$ as required. However, in general combinatorics involved difficult considerably the task of checking the estimation for $d > n+1 \geq 5$ and we devote the main body of this work to treat the case $n = 3$ and $d \geq 4$. We show that for $n = 3$ and $d \geq 4$ the ideal generated by all monomials invariant under the action of the matrix $M_{0,1,2,3}$ is a GT-system and we call GT-threefold the associated variety. Our goal will be mimic for GT-threefolds the study of the geometric properties of Togliatti's surfaces S_d developed in [11]. We prove that GT-threefolds are toric varieties whose homogeneous ideal is a lattice ideal associated to a saturated partial character and we conjecture, substantiated on computations, a smaller system of generators of the ideal, more coherent with the results obtained in [11]. Furthermore, working with lattice ideals could shed light to the problem of determining a minimal free resolution for GT-threefolds and checking whether they are arithmetically Cohen Macaulay. The starting point is the standard free resolution of lattice ideals: the so called Hull complex. Finally, we see that GT-systems are smooth outside the image of the four fundamental points of $\mathbb{P}^3$.

Let us outline how this work is organized. In Section 1, we prepare some basic facts on Galois coverings and quotient projective varieties by finite groups acting on them in order to conclude that under suitable hypothesis quotients by finite groups of automorphisms give rise to Galois coverings. Later on, we introduce the weak and strong Lefschetz properties and we define Togliatti systems. Section 2 deals with the generalization of GT-systems to n variables and the generalized classical Togliatti system. More precisely, in Subsection 2.1 we demonstrate that the ideal I_d generated by the μ_{I_d} monomials of degree $d \geq 4$ invariant under the action of the matrix $M_{a_0,\ldots,a_n}$ is an artinian monomial ideal which fails the weak Lefschetz property from degree $d-1$ to degree d. Applying the results obtained in Section 1 we prove that the associated morphism $\varphi_{I_d} : \mathbb{P}^n \to \mathbb{P}^{\mu_{I_d}-1}$ is a Galois covering with cyclic Galois group $\mathbb{Z}/d$. In Subsection 2.2, we recreate the results obtained in [11] on the geometric properties of the generalized classical Togliatti system to end defining n-dimensional GT-varieties. Our main new results are collected in Section 3 where we study 3-dimensional GT-varieties, or abridged GT-threefold. In Subsection 3.1, we give an entire description of the monomial ideal I_d associated to the matrix $M_{0,1,2,3}$ and mainly we prove that for all $d \geq 4$ it is a GT-system. Subsection 3.2 deals with the geometry of the GT-threefolds after a strategic exposition on prime binomial ideals and saturated lattices. Using these results, we prove that the homogeneous ideal $I(F_d)$ of GT-threefolds are lattice ideals associated to saturated partial characters. Based on computations, we conjecture that $I(F_d)$ is generated by quadrics if d even, and by quadrics and cubics if d odd.

Part of the results of Subsection 2.2 are published in [4] and the result of Chapters 2 and 3 are collected in the preprint [5].

Notation: To finish, we fix the notation we will use along this work. K will denote an algebraically closed field of characteristic zero, we set $S = K[x_0, \dots, x_n]$ and $\mathbb{P}^n$ the n-dimensional projective space over K. By a variety X we always refer an irreducible affine or projective variety.

An *artinian ideal* is an ideal $I \subset S$ such that its generators do not have common non trivial zeros. Associated to any artinian ideal $I \subset S$ generated by r forms $F_1, \dots, F_r$ of degree d, there is a regular map:

$$\varphi_I : \mathbb{P}^n \to \mathbb{P}^{r-1}.$$

We note X_{n,I_d} its image. Analogously, associated to the Macaulay inverse system I^{-1} there is a rational map:

$$\varphi_{(I^{-1})_d} : \mathbb{P}^n \dashrightarrow \mathbb{P}^{\binom{n+d}{d}-r-1}.$$

We denote $X_{n,(I^{-1})_d}$ the closure of its image.

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To M.

Chapter 1

Preliminaries.

This chapter is a compilation of all background material on algebra and geometry used in the main body of this work and distributed in two sections. We start discussing the notion of Galois coverings and how it affects quotients of projective varieties by finite groups. We follow mainly [7], [9], [18] and [17]. In Section 1.2, we give the definition of the weak and strong Lefschetz properties and results needed later on. We focus on Togliatti systems, a family of ideals failing the weak Lefschetz properties and introduced by Mezzetti, Miró-Roig and Ottaviani in [12].

1.1 Galois coverings.

In this subsection we translate the topological notions of covering and Galois covering into the projective and affine context. Our goal is to see that quotients of varieties by finite groups of automorphism are Galois coverings. We start with some definitions.

Definition 1.1.1. Let X be a topological space. *A covering* of X consists of a topological space Y and a map $f : Y \to X$ such that for every point $p \in X$, there exists a neighbourhood U_p of p and a set T with the discrete topology, together with a commutative diagram

$$\begin{array}{ccc} f^{-1}(U_p) & \xrightarrow{\;\cong\;} & T \times U_p \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & U_p & \end{array}$$

where the upper map is a homeomorphism and π denotes the projection.

Given a covering $f : Y \to X$ of a topological space X, *the group of deck transformation* $G := Aut(f)$ is defined to be the group of automorphisms of Y commuting with f. We say that $f : Y \to X$ is a covering with group $Aut(f)$.

Definition 1.1.2. We say that a covering $f : Y \to X$ of a topological space X is a Galois covering if Y is connected and the group $Aut(f)$ acts transitively on a fibre $f^{-1}(x)$ for some $x \in X$, or equivalently if $Y \times_X Y := \{(z, z') \mid f(z) = f(z')\}$, then the map $\phi : G \times Y \to Y \times_X Y$ given by $\phi(g, z) = (z, gz)$ is a homeomorphism.

When a group G acts on a topological space X, there is a natural way of constructing Galois coverings. Denote X/G the quotient space of X by the equivalence relation defined by G, and endow X/G with the quotient topology given by the canonical projection $\pi : X \to X/G$.

Proposition 1.1.3. If for any orbit $G_x = \{gx \mid g \in G\}$, $x \in X$ there exists an open cover of disjoint subsets $U_{gx} \ni gx$ permuted by G, i.e $g'(U_{gx}) = U_{g'gx}$, then $\pi : X \to X/G$ is a Galois covering.

Proof. First, we start proving that $\pi : X \to X/G$ is a covering. Fixed $x \in X$, consider U_x as in the statement. We observe that if $y \in U_x$, then $G_y \cap U_x = \{y\}$, because $gy \in U_{gx}$ and the U_{gx} are disjoint pairwise, for all $g \in G$. In other words, the restriction map $\pi_{|U_x}$ is injective and a homeomorphism with the induced topology. We have $V_x = \pi(U_x) = \{\pi(y) \mid y \in U_x\}$ and $\pi^{-1}(V_x) = \{gy \mid \pi(y) \in V_x\}$, since $\pi(z) = \pi(y)$ if and only if $z = gy$ for some $g \in G$. By hypothesis $\{gy \mid \pi(y) \in V_x\} = \{gy \mid y \in U_x\} = \cup_{g\in G} U_{gx}$, and hence $\pi^{-1}(V_x)$ is an open set of X, which implies in turn that $\pi(V_x)$ is open. Clearly the map $f : \pi^{-1}(V_x) \to G \times V_x$ sending $gy \to (g, \pi(y))$ is continuous and surjective, endowed G with the discrete topology. Moreover, f is also injective. Indeed, if $gy, g'z \in \pi^{-1}(V_x)$ are such that $(g, \pi(y)) = (g', \pi(z))$, then $g = g'$ and there exists $\tilde{g} \in G$ satisfying $z = \tilde{g}y$. As consequence $gz = g\tilde{g}y \in U_{g\tilde{g}x}$, but $z \in U_x$ implies that $gz \in U_{gx}$, since the U_{gx} are disjoint pairwise we must have $g\tilde{g} = g$ and hence $gy = gz$. Observe that f is an open map. If $W \subset \pi^{-1}(V_x)$ is open, then W is a disjoint union of open subsets W_{gx} of each U_{gx}, so may we assume that W is an open subset of U_x. We have $f(W) = \{id\} \times \pi(W)$ and hence $f^{-1}(f(W)) = \pi^{-1}(W)$, which is open since $\pi_{|U_x}$ is open. In particular, the fact that f is open implies that if $V \subset U_x$ is open, $g(V) \subset U_{gx}$ is also open. So, if $W \subset X$ is open, then any point $x \in W$ has an open neighbourhood $W_x \subset U_x$ in W permuted by G such that $W = \cup_x W_x$.

Now, we verify that the covering $\pi : X \to X/G$ is Galois. Set $D := X \times_{X/G} X := \{(x, x') \in X \times X \mid \pi(x) = \pi(x')\}$. Since $\pi(x) = \pi(y)$ if and only $y = g(x)$ for some $g \in G$, we can write $D = \{(x, gx) \in X \times X \mid g \in G\} = \cup_{g\in G}\{(x, gx) \mid x \in X\}$. Consider the injective map $\phi : G \times X \to D$ to be $\phi(g, x) = (x, gx)$. Since any open set $W \subset X$ admits an open cover $\cup_{x\in W} W_x$ with W_x as in the statement, to prove the continuity of ϕ it is enough to consider opens of the form $(W_x \times W_y) \cap G$. We have $(W_x \times W_y) \cap G = \cup_{g\in G}\{(z, gz) \mid z \in W_x,\ gz \in W_y\} = \cup_{g\in G}\{(z, gz) \mid z \in W_x \cap W_{g^{-1}y}\}$. Therefore, $\phi^{-1}((W_x \times W_y) \cap G) = \cup_{g\in G}\{g\} \times W_x \cap W_{g^{-1}y}$, which is open. That the covering $\pi : X \to X/G$ is Galois follows from the fact that ϕ is an open map. Indeed, for any open set $\{g\} \times W_x$ of $G \times X$, $\phi(\{g\} \times W_x\}) = W_x \times_{X/G} W_{gx} = (W_x \times W_{gx}) \cap G$. □

Remark 1.1.4. Assumptions in the Proposition 1.1.3 can be substantially reduced when G is a group of homeomorphisms of X. In fact, for $\pi : X \to X/G$ be a covering, it is enough to assume that for any point $x \in X$, there exists a neighbourhood U_x such that $g(U_x) \cap U_x = \emptyset$, for all $g \in G$.

In the context of algebraic geometry, the notion of quotients of a variety X by a group G acting on it becomes a delicate issue. Even though the quotient X/G is well defined as a topological space, it need not be in general a variety.

Definition 1.1.5. Let G be a group acting on a variety X. The quotient of X by G is defined to be a variety Y and a regular surjective map $p : X \to Y$ such that any regular map $\rho : X \to Z$ to another variety X factors through p if and only if $\rho(x) = \rho(g(x))$, for all $x \in X$ and $g \in G$.

Remark 1.1.6. The quotient variety is unique up to isomorphism. In particular, the regular map $p : X \to X/G$ verifies that if $x, y \in X$, then $p(x) = p(y)$ if and only if $g(x) = y$, for some $g \in G$.

One circumstance in which quotients always exist is the case of a finite group acting on an affine variety.

Proposition 1.1.7. Let G be a finite group acting on an affine variety X. Denote by X/G the quotient space of X by the equivalence relation defined by G, endowed with the quotient topology induced by the canonical projection $\pi : X \to X/G$. Then, X/G is the affine variety whose coordinate ring $A(X/G)$ is identified with $A(X)^G$ and $\pi : X \to X/G$ is the quotient of X by G.
Proof. See [17] Section 12, Proposition 18. □

Proposition 1.1.8. Let G be a finite group acting on a projective variety X and X/G its quotient space. If the orbit of any point $x \in X$ is contained in an affine open of X, then X/G is a projective variety and $\pi : X \to X/G$ is the quotient of X by G.
Proof. See [17] Section 12, Proposition 19. □

Quotients varieties by finite groups of automorphisms works particularity well with respect to Galois coverings.

Proposition 1.1.9. Let X be a projective variety and $G \subset Aut(X)$ be a finite group. If the quotient variety X/G exists, then $\pi : X \to X/G$ is a Galois covering.
Proof. Denote $G = \{g_1, \dots, g_{n-1}, g_n = id\}$ and let $x \in X$, and $G_x = \{g_i x \mid g_i \in G\}$ its orbit. Since X is a T_1-space, there exists a neighbourhood $\overline{U}_x$ of x such that $\overline{U}_x \cap G_x = \{x\}$. Then, $U_x = \overline{U}_x \cap V(g_i - g_j | g_i \neq g_j \in G)^c$ is still a neighbourhood of x such that $U_x \cap g_i(U_x) = \emptyset$ for all $g_i \in G$ different from id. By Remark 1.1.4, $\pi : X \to X/G$ is a covering of X/G.

The group of deck transformations $Aut(\pi)$ consists of all automorphism of X commuting with π. If $f : X \to X$ verifies that $\pi \circ f = \pi$, then for all point of X we have $\pi(f(x)) = \pi(x)$. But then given $x \in X$ there exists $g_i \in G$ such that $f(x) = g_i(x)$, and hence $X = V(f - g_1) \cup \cdots \cup V(f - g_n)$. The irreducibility of X allows us to conclude that $f = g_i$ for some $g_i \in G$. Summarizing, $Aut(\pi) = G$ and it is clear that given $\pi(x) \in X/G$, the fibre $\pi^{-1}(\pi(x)) = G_x$, so $Aut(\pi) = G$ acts transitively on the fibre $\pi^{-1}(\pi(x))$. □

1.2 Lefschetz properties.

Artinian ideals having the weak or strong Lefschetz property and its repercussion in Algebraic Geometry have been largely studied. In this section, we center in on the work of Mezzetti, Miró-Roig and Ottaviani [12], where they relate the existence of artinians ideals failing the weak Lefschetz properties to the existence of varieties satisfying at least one Laplace equation.

Definition 1.2.1. Let $I \subset S$ be a homogeneous artinian ideal.

1. We say that S/I has the *weak Lefschetz property*, or abridged WLP, if there exists a linear form $L \in (S/I)_1$ such that the multiplication map $\times L : (S/I)_j \to (S/I)_{j+1}$ has maximal rank, for all integer j, i.e it is either injective or surjective.
2. We say that S/I has the *strong Lefschetz property*, or shorted SLP, if there exists a linear form $L \in (S/I)_1$ such that the multiplication map $\times L^k : (S/I)_j \to (S/I)_{k+j}$ has maximal rank, for all integer k and j.

Let us see some examples of artinian ideals failing the WLP and the SLP.

Examples 1.2.2. 1. $I = (x^3, y^3, z^3, xyz)$ fails the WLP for $j = 2$.

2. $I = (x^4, y^4, z^4, t^4, xyzt)$ fails the WLP for $j = 5$.
3. $I = (x^5, y^5, z^5, t^5, w^5, xyztw)$ fails the WLP for $j = 8$ and $j = 9$.
4. In general, $I = (x_0^{n+1}, \dots, x_n^{n+1}, x_0 \dots x_n)$ fail the WLP for $j = \binom{n+1}{2} - 1$.
5. $I = (x_0^2, x_1^3, x_2^5, x_0x_1,\ x_0x_2^2, x_1x_2^3, x_1^2x_2^2)$ fails the SLP for $k = 2$ and $j = 1$, but has the WLP
6. $I = (x_0^3, x_1^3, x_2^3,\ (x_0 + x_1 + x_2)^3)$ fails the SLP for $k = 3$ and $j = 1$, but also has the WLP.

Having the WLP puts great constraints on the Hilbert function and determining if an artinian ideal I has the WLP or SLP is, in general, a hard problem. For instance, even though Stanley in [19] or Watanabe in [23] proved that a general artinian complete intersection has the WLP, it is an open problem to determine whether *every* artinian complete intersection has the WLP in codimension greater or equal to four. However, dealing with monomial artinian ideals can reduce this difficulty. For instance we only need to check the WLP or the SLP for the linear form $L = x_0 + \cdots + x_n$, instead of a general linear form of $(S/I)_1$. Properly, we have:

Proposition 1.2.3. Let $I \subset K[x_0, \dots, x_n]$ be an artinian monomial ideal. Then, S/I has the weak Lefschetz property (respectively SLP) if and only if the multiplication map $\times(x_0 + \cdots + x_n) : (S/I)_j \to (S/I)_{j+1}$ (respectively $\times(x_0 + \cdots x_n)^k : (S/I)_j \to (S/I)_{k+j}$) has maximal rank, for all integer j (respectively for all integer k and j).
Proof. See [14], Proposition 2.2 □

In [12], E. Mezzeti, R.M Miró-Roig and G.Ottaviani introduced the so called Togliatti systems, a particular family of artinian ideals failing the WLP. Let us recall its definition.

Definition 1.2.4. Let $I \subset S$ be an artinian ideal generated by r forms $F_1, \dots, F_r$ of degree d, with $r \leq \binom{n+d-1}{n-1}$. We say that I is a *Togliatti system* if S/I fails the weak Lefschetz property in degree $d-1$. In addition, if I can be generated by monomial, we say that I is a *monomial Togliatti system*.

In [12], using Macaulay-Matlis duality and inverse systems the authors established a connection between Togliatti systems and varieties satisfying Laplace equations (see for instance [12], Theorem 3.2). The name is in honour of the italian mathematician Eugenio

Togliatti. He studied the geometric properties of the surface X_{2,I_3} associated to the artinian ideal $I = (x^3, y^3, z^3, xyz) \subset K[x,y,z]$ and its apolar variety $X_{2,I_3^{-1}}$. He proved that I was the only Togliatti system in $K[x,y,z]$ of cubics whose apolar variety is smooth and satisfies a Laplace equation of order two.

Classifying Togliatti systems with smooth apolar variety is a challenging problem and only partial solutions have been achieved. In [12] the classification was accomplished for Togliatti systems of cubics for $n = 4$ variables and was stated a conjecture for arbitrary n, afterwards it was solved in [13]. Nevertheless, the classification is still open in general and it seems out of reach. Taking a step forward, in [11] the authors proposed the study of a particular family of Togliatti systems, the so called GT-systems, which we develop in next chapter.

Chapter 2

GT-systems.

This chapter is devoted to the study of the so called GT-systems, a new family of Togliatti systems introduced by E. Mezzetti and R.M Miró-Roig in [11]. In Section 2.1 we define GT-systems and generalize some results in [11] to n variables. By means of this approach we relate a variety associated to a GT-system to a quotient variety of $\mathbb{P}^n$ by a finite group of automorphisms. In Section 2.1, we study the geometry of *the generalized classical Togliatti systems* introduced in [11] and collect the main statements developed in [11], Section 7. We end this chapter defining GT-varieties whose geometry we will develop in next chapters.

2.1 Definition and examples.

To start, let us to focus on Togliatti's example $I = (x^3, y^3, z^3, xyz)$ for a moment. Fixed a third root of unity e, the monomials x^3, y^3, z^3 and xyz correspond to all monomials of degree 3 invariant under the action of the matrix

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e^2 \end{pmatrix},$$

and they generate all the forms of degree 3 invariant under the action of M. The morphism $\varphi_I : \mathbb{P}^2 \to \mathbb{P}^3$ associated to I is a Galois covering of degree 3 of the image $X_{2,3} = im(\varphi_I)$ with cyclic Galois group $\mathbb{Z}/3$. Furthermore, Togliatti's example admits a family of generalizations in all degree sharing this property, they are called *generalized classical Togliatti system*:

Definition 2.1.1. Fix a positive integer $d = 2k + \varepsilon$ with $0 \leq \varepsilon \leq 1$ and a dth root of unity e. The monomials of degree d invariant under the action of the matrix M gives rise to an artinian monomial ideal $I_d = (x^d, y^d, z^d, xy^{d-2}z, x^2y^{d-4}z^2, \ldots, x^k y^\varepsilon z^k) \subset K[x, y, z]$ which defines a monomial Toglitti system called *generalized classical Togliatti system.* Any such ideal I_d defines a Galois covering of degree d

$$\varphi_{I_d} : \mathbb{P}^2 \to \mathbb{P}^{k+2}$$

of the surface $S_d := im(\varphi_{I_d})$ with cyclic Galois group $\mathbb{Z}/d$ represented by M.

These examples motivated the definition of a new family of Togliatti systems:

Definition 2.1.2. Let $I \subset S$ be an artinian ideal generated by r forms $F_1, \dots, F_r$ of degree d with $r \leq \binom{n+d-1}{n-1}$. We say that I is a *GT-system* if it is a Togliatti system whose associated regular map $\varphi_I : \mathbb{P}^n \to \mathbb{P}^{r-1}$ is a Galois covering of degree d with cyclic Galois group $\mathbb{Z}/d$.

In particular, in [11], section 3, the authors proved the following:

Theorem 2.1.3. Fix an integer $d \geq 3$ and let $I \subset K[x,y,z]$ be the ideal generated by all monomials of degree d invariant under the action of $M_{a,b} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^a & 0 \\ 0 & 0 & e^b \end{pmatrix}$, with $1 \leq a < b \leq d-1$ and $gcd(a,b,d) = 1$. Then I is a GT-system.

The smart way of defining such ideals $I \subset K[x,y,z]$ simplifies significantly the work of checking the conditions of being a GT-system, even if we consider its natural generalization to $K[x_0, \dots, x_n]$. As we will see next, it is enough to check that the number of generators of I does not exceed the bound in Definition 2.1.2.

Proposition 2.1.4. Fix $n+1 \leq d \in \mathbb{Z}$, a dth primitive root of unity e and $a_0, \dots, a_n$ positive integers such that $a_i < d$, $i = 0, \dots, n$. The ideal I_d generated by all forms of degree d invariant under the action of the matrix

$$M_{a_0,\dots,a_n} := \begin{pmatrix} e^{a_0} & 0 & \cdots & 0 \\ 0 & e^{a_1} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & e^{a_n} \end{pmatrix}.$$

is an artinian monomial ideal.

Proof. Set $N = \binom{n+d}{d}$ and let $B_d := \{m_i := x_0^{\alpha_{0,i}} \cdots x_n^{\alpha_{n,i}}, 1 \leq i \leq N\}$ be the set of all monomials of degree d in $K[x_0, \dots, x_n]$. We denote by $m_{i_1}, \dots, m_{i_\mu}$ the μ monomials invariant under the action of $M_{a_0,\dots,a_n}$. Let $F \in I_d$ and write $F = \beta_1 m_{j_1} + \cdots + \beta_t m_{j_t}$, with the $\beta_i \in K^*$ and $m_{j_i} \in B_d$. For convenience we note $M_{a_0,\dots,a_n}$ just by M. Since F is invariant under the action of M, we have $MF = F$ and so

$$\beta_1 e^{a_0\alpha_{0,j_1} + \cdots a_n\alpha_{n,j_1}} m_{j_1} + \cdots + \beta_t e^{a_0\alpha_{0,j_t} + \cdots a_n\alpha_{n,j_t}} m_{j_t} = \beta_1 m_{j_1} + \cdots + \beta_t m_{j_t},$$

which clearly implies that $\beta_i - \beta_i e^{a_0\alpha_{0,j_i} + \cdots a_n\alpha_{n,j_i}} = 0$, for all $1 \leq i \leq t$, or in other words $e^{a_0\alpha_{0,j_i} + \cdots a_n\alpha_{n,j_i}} = 1$ for all $1 \leq i \leq t$. Since $x_0^d, \dots, x_n^d$ are invariant under the action of M, I_d is artinian and the statement is completely proved. □

We are now ready to state the main result of this section.

Theorem 2.1.5. Fix $n+1 \leq d \in \mathbb{Z}$, a dth primitive root of unity e and $a_0, \dots, a_n$ positive integers such that $gcd(a_0, \dots, a_n, d) = 1$. Let I_d be the ideal generated by the μ monomials of degree d invariant under the action of $M_{a_0,\dots,a_n}$. If $\mu \leq \binom{n+d-1}{n-1}$, then I_d is a monomial GT-system.

Proof. We have to verify that I_d is Togliatti, i.e I_d fails the weak Lefschetz property from degree $d-1$ to degree d, and second that its associated morphism $\varphi_{I_d} : \mathbb{P}^n \to \mathbb{P}^{\mu_{I_d}-1}$ is a Galois covering of degree d with cyclic Galois group Z/d.

We start with the Togliatti's condition. Since I_d is monomial and artinian (see Proposition 2.1.4), we can apply Proposition 1.2.3 and we only have to check that for $L := x_0 + \cdots + x_n$, the multiplication map $\times L : (S/I_d)_{d-1} \to (S/I_d)_d$ is not injective. Since $(S/I_d)_k \cong S_k$ for $k < d$, we may assume that $\times L : S_{d-1} \to (S/I_d)_d$ and it suffices to find a form C_{d-1} of degree $d-1$ such that $L \cdot C_{d-1} \in I_d$. We consider then $C_{d-1} = \prod_{i=1}^{d-1} e^{ia_0}x_0 + \cdots + e^{ia_n}x_n \in S_{d-1}$. Clearly $L \cdot C_{d-1} = \prod_{i=0}^{d-1} e^{ia_0}x_0 + \cdots + e^{ia_n}x_n$ is invariant under the action of $M_{a_0,\ldots,a_n}$, i.e $L \cdot C_{d-1} \in I_d$ from which we conclude that I_d is a monomial Togliatti system.

We proceed now to prove that $\varphi_{I_d} : \mathbb{P}^n \to \mathbb{P}^{\mu_{I_d}-1}$ is a Galois covering of degree d with cyclic Galois group Z/d. We observe that all monomials $x_0^{\alpha_0} \cdots x_n^{\alpha_n}$ in $K[x_0, \ldots, x_n]$ invariant under the action of $M_{a_0,\ldots,a_n}$ are characterized by the solutions of the system:

$$(*) = \begin{cases} \alpha_0 + \cdots + \alpha_n = d \\ a_0\alpha_0 + \cdots + a_n\alpha_n = td \end{cases}, \quad \text{with } t = 0, \ldots, max_{0 \le i \le n}\{a_i\}.$$

The condition $gcd(a_0, \ldots, a_n, d) = 1$ makes $G = \{M_{a_0,\ldots,a_n}, M^2_{a_0,\ldots,a_n}, \ldots, M^{d-1}_{a_0,\ldots,a_n}, id\}$ a cyclic group of $GL_n(K)$ of order d. Actually, G is a cyclic group of automorphisms of $\mathbb{P}^n$ of order d, acting on $\mathbb{P}^n$ in the following way:

$$G \times \mathbb{P}^n \to \mathbb{P}^n, \quad (M^m_{a_0,\ldots,a_n}, [x_0, \ldots, x_n]) \to [e^{ma_0}x_0, \ldots, e^{ma_n}x_n],\ 1 \le m \le d.$$

From Proposition 1.1.9 it is enough to see that $\varphi_{I_d} : \mathbb{P}^n \to X_{n,I_d}$ is the quotient variety of $\mathbb{P}^n$ by G. If $f : \mathbb{P}^n \to Z$ is a regular map of projective varieties, which factors through φ_{I_d}, i.e there exists a regular map $g : X_{n,I_d} \to Z$ such that $f = g \circ \varphi_{I_d}$, then clearly for all $z = [z_0, \ldots, z_n]$, $f(M_{a_0,\ldots,a_n}z) = g(\varphi_{I_d}(M_{a_0,\ldots,a_n}z)) = g(\varphi_{I_d}(z)) = f(z)$, since $\varphi_{I_d}((M_{a_0,\ldots,a_n}z) = \varphi_{I_d}(z)$ by the definition of I_d. Conversely, suppose that $f : \mathbb{P}^n \to Z$ is a regular map of projective varieties such that $f(M_{a_0,\ldots,a_n}z) = f(z)$. In order to prove that f factors through φ_{I_d}, it suffices to verify that for any pair $z, y \in \mathbb{P}^n$ such that $\varphi_{I_d}(z) = \varphi_{I_d}(y)$, then y belongs to the orbit G_z of z. In which case the map $g(\varphi_{I_d}(z)) = f(z)$ is well defined and it is regular. We write $z = [z_0, \ldots, z_n]$ and $y = [y_0, \ldots, y_n]$ points of $\mathbb{P}^n$, if $\varphi_{I_d}(z) = \varphi_{I_d}(z)$, then there exists $\lambda \in K - \{0\}$ such that for any monomial $x_0^{\alpha_0} \cdots x_n^{\alpha_n}$ of degree d invariant under the action of $M_{a_0,\ldots,a_n}$ it satisfies $y_0^{\alpha_0} \cdots y_n^{\alpha_n} = \lambda z_0^{\alpha_0} \cdots z_n^{\alpha_n}$. In particular $y_i^d = \lambda z_i^d$, $i = 0, \ldots, n$, so we can write $y_i = \beta e^{b_i} z_i$ such that $(\beta e^{b_i})^d = \lambda$, $i = 0, \ldots, n$ and $(b_0, \ldots, b_n, d) = 1$. But then we must have $e^{\alpha_0 b_0 + \cdots + \alpha_n b_n} = 1$ for all solutions of $(*)$, or equivalently any solution of $(*)$ is solution of the system

$$(**) \begin{cases} \alpha_0 + \cdots + \alpha_n = d \\ b_0\alpha_0 + \cdots + b_n\alpha_n = td \end{cases}, \quad \text{with } t = 0, \ldots, max_{0 \le i \le n}\{b_i\}.$$

□

2.2 The geometry of the generalized classical Togliatti system.

Fix an integer $3 \leq d = 2k + \varepsilon$, with $0 \leq \varepsilon \leq 1$ and a primitive dth root of unity e. The ideal $I_d \subset K[x, y, z]$ generated by all monomials of degree d invariant under the action of the matrix $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^2 & 0 \\ 0 & 0 & e \end{pmatrix}$ is the GT-system corresponding to the generalized classical Togliatti system $I_d = (x^d, y^d, z^d, x^k y^k z^\varepsilon, \dots, x^2 y^2 z^{d-4}, xyz^{d-2})$ (see Definition 2.1.1 or [11], Definition 2.4). The associated morphism $\varphi_{I_d} : \mathbb{P}^2 \to \mathbb{P}^{k+2}$ is a Galois covering of degree d with cyclic Galois group $\mathbb{Z}/d$ of the surface image of φ_{I_d} (see Proposition 1.1.3). In [11], section 7, Mezzetti and Miró-Roig provided a detailed study of the geometry of the surface $im(\varphi_{I_d})$, they denoted it S_d. This work was proposed as an attempt to wide the generalized classical Togliatti system to $K[x_0, \dots, x_n]$ and investigate the geometry of the associated variety. We present our results for $n = 3$ in next chapters. We include a review on the main statements developed in [11], section 7 which leads our approach.

We note $I(S_d)$ the homogeneous ideal of the surface S_d.

Theorem 2.2.1. Let $I_d \subset K[x, y, z]$ be a generalized Togliatti system and set $R = K[x_0, \dots, x_{k+2}]$. Then, the following holds:

(1) If $d = 2k + 1$ is odd, then $I(S_d)$ is the ideal generated by the matrix minors of the matrix:

$$\mathcal{A} := \begin{pmatrix} x_3 & x_4 & \cdots & x_{k+1} & x_{k+2} & x_0 x_1 \\ x_4 & x_5 & \cdots & x_{k+2} & x_2 & x_3^2 \end{pmatrix}.$$

In particular, $I(S_d)$ a determinantal ideal generated by $\binom{k}{2}$ quadrics and k cubics. Its minimal free resolution is given by the Eagon-Northcott complex:

$$0 \to R(-k-2)^k \to \cdots \to R(-4)^{3\binom{k}{4}} \oplus R(-5)^{3\binom{k}{3}} \to$$

$$R(-3)^{2\binom{k}{3}} \oplus R(-4)^{2\binom{k}{2}} \to R(-2)^{\binom{k}{2}} \oplus R(-3)^k \to R \to R/I(S_d) \to 0.$$

(2) If $d = 2k$ is even, set $I_2(\mathcal{B})$ the ideal generated by the maximal minors of the matrix:

$$\mathcal{B} := \begin{pmatrix} x_3 & x_4 & \cdots & x_{k+1} & x_{k+2} \\ x_4 & x_5 & \cdots & x_{k+2} & x_2 \end{pmatrix}.$$

Then, $I(S_d) = I_2(\mathcal{B}) + (x_0 x_1 - x_3^2)$. In particular, $I(S_d)$ is a Cohen Macaulay ideal generated by $1 + \binom{k}{2}$ quadrics and its minimal free resolution is the following:

$$0 \to R(-k-2)^k \to \cdots \to R(-k-1)^{(k-1)\binom{k}{k-1}} \oplus R(-k)^k \to \cdots \to$$

$$R(-5)^{2\binom{k}{3}} \oplus R(-4)^{3\binom{k}{4}} \to R(-4)^{\binom{k}{2}} \oplus R(-3)^{2\binom{k}{3}} \to R(-2)^{1+\binom{k}{2}} \to R \to R/I(S_d) \to 0.$$

Proof. See [11], Theorem 7.2. and Theorem 7.3 □

Classical Togliatti systems I_d admits a natural generalization to $K[x_0,\dots,x_n]$. In [4], the authors prove that the μ_{I_d} monomials of degree $d = n+1$ invariant under the action of the matrix

$$M = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & e & 0 & \cdots & 0 \\ 0 & 0 & e^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e^n \end{pmatrix}$$

does not exceed $\binom{2n}{n-1}$, and therefore by Theorem 2.1.5, the ideal I_{n+1} generated by these monomials gives rise to a GT-system. So we are led to pose the following definition:

Definition 2.2.2. Fix $n+1 \leq d \in \mathbb{Z}$ and a dth primitive root of unity e. Let $I_d \subset K[x_0,\dots,x_n]$ be the ideal generated by the μ_{I_d} monomials of degree d invariant under the action of the matrix

$$M = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & e & 0 & \cdots & 0 \\ 0 & 0 & e^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e^n \end{pmatrix}.$$

If $\mu_{I_d} \leq \binom{n+d-1}{n-1}$, then by Theorem 2.1.5 I_d is a GT-system . We call *GT-variety* the n-dimensional variety $im(\varphi_{I_d}) \subset \mathbb{P}^{\mu_{I_d}-1}$ where $\varphi_{I_d} : \mathbb{P}^n \to \mathbb{P}^{\mu_{I_d}-1}$ is the Galois covering of degree d with cyclic Galois group $\mathbb{Z}/d$ associated to the GT-system I_d.

The 2-dimensional GT-varieties correspond to the Togliatti's surfaces S_d studied in [11], Section 7. Since the combinatorics needed to study GT-varieties of dimension $n \geq 3$ becomes quickly very involved, we will restrict our attention to the case $n = 3$ and we will entirely devote Chapter 3 to study GT-varieties of dimension 3.

Chapter 3

GT-threefolds.

This chapter is entirely devoted to study GT-variety of dimension 3, or abridged GT-threefold. Through this chapter we fix an integer $d \geq 4$, a dth-root of unity e and we write $d = 2k + \varepsilon = 3k' + \rho$ with $\varepsilon \in \{0,1\}$ and $\rho \in \{0,1,2\}$. We note $I_d \subset R$ the ideal generated by the μ_{I_d} monomials of degree d invariant under the action of the matrix $M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e & 0 & 0 \\ 0 & 0 & e^2 & 0 \\ 0 & 0 & 0 & e^3 \end{pmatrix}$. In Section 3.1, we describe the ideal I_d and we prove that I_d is a GT-system for all $d \geq 4$. In section 3.2, we present our main contribution on GT-varieties, we prove that the homogeneous ideal of GT-threefolds is a lattice ideal. We apply Eisenbud results on binomial ideals (see [6]) to deduce that the homogeneous ideal of GT-threefolds are prime binomials ideals defined by saturated partial characters of $\mathbb{Z}^{\mu_{I_d}}$. Finally, in Section 3.3, we relate GT-threefolds to quotient cyclic singularities in order to investigate their singular locus.

3.1 A monomial Togliatti system.

Let $x^\alpha y^\beta z^\delta t^\gamma \in K[x,y,z,t]$ be a monomial of degree d. Asking it to be invariant under the action of M is equivalent to impose that $\beta + 2\delta + 3\gamma = \dot{d}$, where $\dot{d}$ denotes a multiple of d. In fact, since the monomial has degree d, this multiple can not exceed $3d$. So, we can determine all monomials by solving the system:

$$(*) \quad \left.\begin{array}{ccccccccc} \alpha & + & \beta & + & \delta & + & \gamma & = & d \\ & & \beta & + & 2\delta & + & 3\gamma & = & rd \end{array}\right\}, \quad r = 0,1,2,3.$$

The solutions of $(*)$ in terms of γ and r are the following:

$$\begin{array}{rcl} \alpha & = & \delta + 2\gamma + (1-r)d, \\ \beta & = & rd - 2\delta - 3\gamma, \\ \gamma & \in & \{0, \dots, rk' + \lfloor \frac{r\rho}{3} \rfloor\}, \\ \delta & \in & \{max\{0,(r-1)d - 2\gamma\}, \dots, \lfloor \frac{rd-3\gamma}{2} \rfloor\}. \end{array}$$

Given $d \geq 4$, we define

$$\mathcal{W}_d := \{(r,\gamma,\delta) \in \mathbb{Z}^3 \ \mid \ 0 \leq r \leq 3, 0 \leq \gamma \leq rk' + \lfloor \frac{r\rho}{3} \rfloor, max\{0, d-2\gamma\} \leq \delta \leq \lfloor \frac{rd-3\gamma}{2} \rfloor\}.$$

All solutions of $(*)$ are uniquely determined by a triple $(r, \gamma, \delta) \in \mathcal{W}_d$. The following tables collect the ranges of the γ's and δ's in a clearer format.

For $r = 1$,

γ	δ										
0	0	1	$\dots$	$k-6$	$k-5$	$k-4$	$k-3$	$k-2$	$k-1$	k	
1	0	1	$\dots$	$k-6$	$k-5$	$k-4$	$k-3$	$k-\lceil\frac{3-\epsilon}{2}\rceil$			
$\vdots$											
i	0	1	$\dots$	$\lfloor\frac{d-3i}{2}\rfloor$							
$\vdots$											
$k'-1$	0	1	$\dots$	$\lfloor\frac{\rho+3}{2}\rfloor$							
k'	0		$\dots$	$\lfloor\frac{\rho}{2}\rfloor$							

It will be useful to know the value for δ when $k'-1 \leq \gamma \leq k'$ for each $\rho = 0, 1, 2$:

<table>
<tr><th></th><th>$k'-1$</th><th>k'</th></tr>
<tr><td>$\rho = 0$</td><td>$0 \leq \delta \leq 1$</td><td rowspan="2">$\delta = 0$</td></tr>
<tr><td>$\rho = 1$</td><td rowspan="2">$0 \leq \delta \leq 2$</td></tr>
<tr><td>$\rho = 2$</td><td>$0 \leq \delta \leq 1$</td></tr>
</table>

For $r = 2$,

γ	δ		
0	d		
1	$d-2$		
$\vdots$			
$k+\epsilon$	0	$\dots$	$\lfloor\frac{2d-3(k+\epsilon)}{2}\rfloor$
$\vdots$			
i	0	$\dots$	$\lfloor\frac{2d-3i}{2}\rfloor$
$\vdots$			
$2k'+\lfloor\frac{\rho}{2}\rfloor-1$	0	$\dots$	$\lceil\frac{\rho}{2}\rceil+1$
$2k'+\lfloor\frac{\rho}{2}\rfloor$	0	$\dots$	$\lceil\frac{\rho}{2}\rceil-\lfloor\frac{\rho}{2}\rfloor$

As before it will be useful to specify how are the two last rows for each $\rho = 0, 1, 2$:

$\rho = 0$		$\rho = 1$		$\rho = 2$	
γ	δ	γ	δ	γ	δ
$2k'-1$	$0 \quad 1$	$2k'-1$	$0 \quad 1 \quad 2$	$2k'$	$0 \quad 1 \quad 2$
$2k'$	0	$2k'$	$0 \quad 1$	$2k'+1$	0

Remark 3.1.1. Notice that x^d, y^d, z^d and t^d are invariant under the action of M. So, the ideal I_d generated by all monomials invariant under the action of M is artinian.

Proposition 3.1.2. Fix $d \geq 4$. We define $I_d \subset K[x,y,z,t]$ to be the artinian ideal generated by $\{x^{\delta+2\gamma+(1-r)d}y^{rd-2\delta-3\gamma}z^{\delta}t^{\gamma} \,|(r,\gamma,\delta) \in \mathcal{W}_d\}$. Then, I_d is a $GT-$system.

Proof. By Theorem 2.1.5, we only have to check that $\mu(I_d) \leq \binom{2+d}{2}$. From the definition of I_d, it follows that

$$\mu(I_d) = 2 + \sum_{r=1,2} \sum_{\gamma=0}^{rk'+\lfloor \frac{r\rho}{3} \rfloor} (\lfloor \frac{rd-3\gamma}{2} \rfloor - max\{0, (r-1)d - 2\gamma\} + 1).$$

We want to prove that $\mu(I_d) \leq \binom{d+2}{2} = \frac{(d+2)(d+1)}{2}$. We sum separately for $r = 1$ and $r = 2$, we have

$$\begin{aligned} \textstyle\sum_{\gamma=0}^{k'}(k - \lceil \frac{3\gamma-\varepsilon}{2} \rceil + 1) &= (k'+1)(k+1) - \textstyle\sum_{\gamma=1}^{k'} \lceil \frac{3\gamma-\varepsilon}{2} \rceil, \text{ and} \\ \textstyle\sum_{\gamma=0}^{2k'+\lfloor \frac{2\rho}{3} \rfloor}(d - \lceil \frac{3\gamma}{2} \rceil + 1) - \textstyle\sum_{\gamma=0}^{k}(d-2\gamma) &= (d+1)(2k' + \lfloor \frac{2\rho}{3} \rfloor + 1) + k(k+1) - \\ &\quad -d(k+1) - \textstyle\sum_{\gamma=0}^{2k'+\lfloor \frac{2\rho}{3} \rfloor} \lceil \frac{3\gamma}{2} \rceil. \end{aligned}$$

We only have to focus on the sum of the series of the type $\sum_{\gamma=1}^{N} \lceil \frac{3\gamma-\varepsilon}{2} \rceil$ with $\varepsilon \in \{0,1\}$. We can rewrite the series as follows: if $N = 2j$, $\sum_{\gamma=1}^{N} \lceil \frac{3\gamma-\varepsilon}{2} \rceil = \sum_{i=1}^{j} 3i + \sum_{i=1}^{j}(3i - 1 - \varepsilon) = 3j(j+1) - j - j\varepsilon = j(3j + 2 - \varepsilon)$. Otherwise $N = 2j+1$, $\sum_{i=1}^{j} 3j + \sum_{i=1}^{j+1} 3j - 1 - \varepsilon = 3j(j+1) + 3(j+1) - (j+1) - (j+1)\varepsilon = (j+1)(3j + 2 - \varepsilon)$. In any case,

$$\sum_{\gamma=1}^{N} \lceil \frac{3\gamma - \varepsilon}{2} \rceil = \lceil \frac{N}{2} \rceil (3\lfloor \frac{N}{2} \rfloor + 2 - \varepsilon).$$

From this, we conclude

$$\begin{aligned} \mu(I_d) &= 2 + (k'+1)(k+1) + (d+1)(2k' + \lfloor \tfrac{2\rho}{3} \rfloor + 1) + k(k+1) - \\ &\quad -d(k+1) - \lceil \tfrac{k'}{2} \rceil (3\lfloor \tfrac{k'}{2} \rfloor + 2 - \varepsilon) - \lceil \tfrac{2k'+\lfloor \frac{2\rho}{3} \rfloor}{2} \rceil (3\lfloor \tfrac{2k'+\lfloor \frac{2\rho}{3} \rfloor}{2} \rfloor + 2). \end{aligned}$$

Substituting $d = 3k' + \rho$ by $k = \frac{3k'+\rho-\varepsilon}{2}$ we verify that $\mu(I_d) \leq 2 + (k'+1)(\frac{3k'+\rho}{2} + 1) + (3k' + \rho + 1)(2k' + 2) + \frac{3k'+\rho}{2}(\frac{3k'+\rho}{2} + 1) - (3k' + \rho)(\frac{3k'+\rho}{2} + 1) - \frac{k'}{2}(\frac{3(k'-1)}{2} + 1) - k'(3k' + 2) = \frac{1}{4}(20 + 6(k')^2 + 8\rho - \rho^2 + k'(29 + 4\rho))$. It holds that $\frac{1}{4}(20 + 6(k')^2 + 8\rho - \rho^2 + k'(29 + 4\rho)) < \frac{1}{2}(3k' + \rho + 1)(3k' + \rho + 2) = \frac{1}{2}(d+2)(d+1) \Leftrightarrow 1/4(16 - 12(k')^2 + k'(11 - 8\rho) + 2\rho - 3\rho^2) \leq 0$, which vanishes for all $k' \geq 2$ and $\forall \rho \in \{0,1,2\}$. Under the relation between k' and ρ, we get that the inequality holds for all $d \geq 4$. $\square$

From now on, we call I_d the 4-*generalized GT-system* and the 3-dimensional GT-variety *GT-threefold*. We will often note it by F_d.

For sake of completeness, let us formalize the shape of I_d and exhibit it explicitly for $d = 4, 5, 6, 7, 8$ and 9. For these values of d we cover all possibilities of ε and ρ.

In general we have:

$$
\begin{aligned}
I_d \;=\; (&x^d,\\
&y^d, xy^{d-2\delta}z, \cdots, x^k y^{\varepsilon} z^k, x^2y^{d-3}t, \cdots, x^{k+\varepsilon}y^{1-\varepsilon}z^{k-2+\varepsilon}t, \cdots, x^{\delta+2\gamma}y^{d-2\delta-3\gamma}z^{\delta}t^{\gamma}, \cdots,\\
&z^d, yz^{d-2}t, y^2z^{d-4}t^2, xz^{d-3}t^2, \cdots, x^{\delta+2\gamma-d}y^{2d-2\delta-3\gamma}z^{\delta}t^{\gamma}, \cdots,\\
&t^d).
\end{aligned}
$$

Examples 3.1.3. $I_4 = (x^4, y^4, xy^2z, x^2z^2, x^2yt, z^4, yz^2t, y^2t^2, xzt^2, t^4)$, $\mu_{I_4} = 10$.

$I_5 = (x^5, y^5, xy^3z, x^2yz^2, tx^2y^2, tx^3z, z^5, tyz^3, t^2y^2z, t^2xz^2, t^3xy, t^5)$, $\mu_{I_5} = 12$.

$I_6 = (x^6, y^6, xy^4z, x^2y^2z^2, x^3z^3, tx^2y^3, tx^3yz, t^2x^4, z^6, tyz^4, t^2y^2z^2, t^2xz^3, t^3y^3, t^3xyz, t^4x^2, t^6)$, $\mu_{I_6} = 16$.

$I_7 = (x^7, y^7, xy^5z, x^2y^3z^2, x^3yz^3, tx^2y^4, tx^3y^2z, tx^4z^2, t^2x^4y, z^7, tyz^5, t^2y^2z^3, t^2xz^4, t^3y^3z, t^3xyz^2, t^4xy^2, t^4x^2z, t^7)$, $\mu_{I_7} = 18$.

$I_8 = (x^8, y^8, xy^6z, x^2y^4z^2, x^3y^2z^3, x^4z^4, tx^2y^5, tx^3y^3z, tx^4yz^2, t^2x^4y^2, t^2x^5z, z^8, tyz^6, t^2y^2z^4, t^2xz^5, t^3y^3z^2, t^3xyz^3, t^4y^4, t^4xy^2z, t^4x^2z^2, t^5x^2y, t^8)$, $\mu_{I_8} = 22$.

$I_9 = (x^9, y^9, xy^7z, x^2y^5z^2, x^3y^3z^3, x^4yz^4, tx^2y^6, tx^3y^4z, tx^4y^2z^2, tx^5z^3, t^2x^4y^3, t^2x^5yz, t^3x^6, z^9, tyz^7, t^2y^2z^5, t^2xz^6, t^3y^3z^3, t^3xyz^4, t^4y^4z, t^4xy^2z^2, t^4x^2z^3, t^5xy^3, t^5x^2yz, t^6x^3, t^9)$, $\mu_{I_9} = 26$.

3.2 Geometric properties of GT-threefolds.

The associated morphism $\varphi_{I_d} : \mathbb{P}^3 \to \mathbb{P}^{\mu(I_d)-1}$ of the 4-generalized GT-system I_d is a Galois covering of degree d with cyclic Galois group $\mathbb{Z}/d$ (see Proposition 3.1.2). In particular, this means that a general fibre of φ_{I_d} consists on d points, and hence the image of φ_{I_d} is a 3-dimensional projective variety, we call GT-threefold (see Definition 2.2.2). The rest of this work is devoted to study the geometry of F_d.

For instance, we will prove that the homogeneous ideal of F_d is a lattice ideal associated to a saturated partial character and generated by binomials. As one can expect, there is a stronger relation between GT-threefold and the Togliatti's surface surface S_d, associated to the generalized classical Togliatti system. In fact, both are toric varieties whose homogeneous ideas are binomial prime ideals.

3.2.1 Laurent binomial ideals and binomial primes.

As it requires, we start with the theory on binomial ideals that we will apply later. The reader can see [6] for more details.

Let $K[x^{\pm}]$ be the ring of Laurent polynomials. We can think $K[x^{\pm}]$ as the quotient of the polynomial ring $K[y_1, \ldots, y_n, z_1, \ldots, z_n]$ by the binomial ideal $(y_iz_i - 1 \mid i = 1, \ldots, n)$. In that sense, the Laurent polynomial ring $K[x^{\pm}]$ is often denoted by $K[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]$. We call *Laurent binomial* to any Laurent polynomial of the form $ax^{\alpha} + bx^{\beta}$ where $a, b \in K$, $\alpha, \beta \in \mathbb{Z}^n$, and x^{α} just denotes the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, with $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$. A *Laurent binomial ideal* is an ideal of $K[x^{\pm}]$ generated by Laurent binomials.

The Laurent polynomial ring can be regarded as the coordinate ring of the group of multiplicative characters on $\mathbb{Z}^n$ to the multiplicative group K^*. By a *multiplicative character* or just a *character on* $\mathbb{Z}^n$ *to* k^* we mean a homomorphism of groups $\eta : \mathbb{Z}^n \to K^*$. From this point of view it follows a nice characterization of binomial ideals in $K[x^{\pm}]$.

Definition 3.2.1. A *partial character* on $\mathbb{Z}^n$ is a pair consisting of a lattice $L_\eta \subseteq \mathbb{Z}^n$ and a homomorphism $\eta : L_\eta \to K^*$.

Next result shows us the strong relation between Laurent binomial ideals and the lattice domain associated to it. For our purpose, it will be capital the characterization of Laurent prime binomial ideals in terms of its lattice.

Theorem 3.2.2. Let $K[x^{\pm}]$ be a Laurent polynomial ring.

(i) Any proper Laurent binomial ideal $I \subseteq K[x^{\pm}]$ is uniquely determined by a partial character (L_η, η) on $\mathbb{Z}^n$, $I = I(\eta) := (x^m - \eta(m) \mid m \in L_\eta)$.

(ii) If $m_1, \dots, m_r$ is a basis of the lattice L_η, then $I(\eta) = (x^{m_i} - \eta(m_i) \mid i = 1, \dots, r)$ and the Laurent binomials $x^{m_i} - \eta(m_i)$ form a regular sequence in $K[x^{\pm}]$. In particular, $codim(I(\eta)) = rank(L_\eta)$.

(iii) $I(\eta)$ is prime if and only if L_η is saturated, i.e $Sat(L_\eta) := \{m \in \mathbb{Z}^d \mid dm \in L_\eta \text{ for some } d \in \mathbb{Z}\} = L_\eta$. Or equivalent, if L_η is a direct summand of $\mathbb{Z}^n$.

Proof. See [6], Theorem 2.1. □

For instance, we think $K[x^{\pm}]$ as the quotient $T := K[y_1, \dots, y_n, z_1, \dots, z_n]/(y_i z_i - 1 \mid i = 1, \dots, n)$. Given a Laurent binomial ideal $I(\eta)$, we denote by $I'(\eta)$ the pre-image of $I(\eta)$ in T. Therefore, $I'(\eta)$ is generated by the set $\{y^a z^b - \eta(a - b - c + d) y^c z^d \mid a, b, c, d \in \mathbb{N}^n,\ a - b \equiv c - d (mod L_\eta)\}$. And hence, the above result can be translated to certain binomial ideals in the polynomial ring $K[x_1, \dots, x_n]$.

Definition 3.2.3. Given a partial character η on $\mathbb{Z}^n$, we define the ideal $I_+(\eta) := (x^{m_+} - \eta(m) x^{m_-} \mid m \in L_\eta) \subseteq K[x_1, \dots, x_n]$, where m_+ and m_- denote the positive and negative part of $m \in \mathbb{Z}^n$, respectively.

Corollary 3.2.4. Let I be a binomial ideal in $K[x_1, \dots, x_n]$ not containing any monomial.

(i) There is a unique partial character η on $\mathbb{Z}^n$ such that $I_+(\eta) = I \cdot K[x^{\pm}] \cap K[x_1, \dots, x_n]$ $= I'(\eta) \cap K[x_1, \dots, x_n]$.

(ii) The generators of $I_+(\eta)$ form a Gröbner basis for any monomial order in $K[x_1, \dots, x_n]$.

(iii) $I_+(\eta)$ is radical and all its associated primes are minimal and have the same codimension $rank(L_\eta)$.

Proof. See [6], Corollary 2.5. □

Corollary 3.2.5. Let P a binomial ideal in $K[x_1, \dots, x_n]$, $\{y_1, \dots, y_s\} := \{x_1, \dots, x_n\} \cap P$ and let $\{z_1, \dots, z_t\} := \{x_1, \dots, x_n\} - P$. The ideal P is prime if and only if $P = (y_1, \dots, y_s) + I_+(\eta)$ for a saturated partial character η on $\mathbb{Z}^t$.
Proof. See [6], Corollary 2.6. □

3.2.2 The homogeneous ideal.

We want to determine the homogeneous ideal $I(F_d)$ of the GT-threefold $F_d \subset \mathbb{P}^{\mu_{I_d}-1}$ defined by the 4-generalized GT-system I_d. Since φ_{I_d} is a regular map and $\mathbb{P}^3$ is irreducible, $I(F_d)$ is necessarily a prime ideal of co-dimension $\mu(I_d) - 4$. We will see that $I(F_d)$ is a prime binomial ideal of co-dimension $\mu(I_d) - 4$, which implies, in particular, that GT-threefolds are toric varieties.

The ideal I_d is generated by the set $W := \{x^{\delta+2\gamma+(1-r)d}y^{rd-2\delta-3\gamma}z^{\delta}t^{\gamma} \mid (r,\gamma,\delta) \in \mathcal{W}_d\} \subset K[x,y,z,t]$ (see Section 3.1). All monomials in W are uniquely determined by a triple $(r,\gamma,\delta) \in \mathcal{W}_d$, in that sense we will denote $x^{\delta+2\gamma+(1-r)d}y^{rd-2\delta-3\gamma}z^{\delta}t^{\gamma}$ by $w_{(r,\gamma,\delta)}$. Moreover, the $w_{(r,\gamma,\delta)}$'s define a set of homogeneous coordinates in $\mathbb{P}^{\mu(I_d)-1}$, ordered lexicographically.

Definition 3.2.6. We define the binomial ideal $I = (w_{(r_1,\gamma_1,\delta_1)}w_{(r_2,\gamma_2,\delta_2)} - w_{(r_3,\gamma_3,\delta_3)}w_{(r_4,\gamma_4,\delta_4)} \mid r_1+r_2 = r_3+r_4,\ \gamma_1+\gamma_2 = \gamma_3+\gamma_4,\ \delta_1+\delta_2 = \delta_3+\delta_4) \subset K[w_{(r,\gamma,\delta)}]_{(r,\gamma,\delta)\in W_d}$.

And now for illustrating it we exhibit the ideal I for $d = 4$, $(k = 2, k' = 1, \varepsilon = 0, \rho = 1)$.

Examples 3.2.7. In this case we have $I_4 = (x^4, y^4, xy^2z, x^2z^2, x^2yt, z^4, yz^2t, y^2t^2, xzt^2, t^4)$ and $\mathcal{W}_4 = \{(0,0,0),(1,0,0),(1,0,1),(1,0,2),(1,1,0),(2,0,4),(2,1,2),(2,2,0),(2,2,1),(3,4,0)\}$ (Example 3.1.3). Solving the equation $(r_1,\gamma_1,\delta_1)+(r_2,\gamma_2,\delta_2) = (r_3,\gamma_3,\delta_3)+(r_4,\gamma_4,\delta_4)$ in $\mathcal{W}_4$ we obtain twelve generators for I:

$$\begin{array}{lcl}
w_{(0,0,0)}w_{(2,0,4)} & - & w^2_{(1,0,2)} \\
w_{(0,0,0)}w_{(2,1,2)} & - & w_{(1,0,2)}w_{(1,1,0)} \\
w_{(0,0,0)}w_{(2,2,0)} & - & w^2_{(1,1,0)} \\
w_{(1,0,0)}w_{(1,0,2)} & - & w^2_{(1,0,1)} \\
w_{(1,0,0)}w_{(2,2,1)} & - & w_{(1,0,1)}w_{(2,2,0)} \\
w_{(1,0,0)}w_{(3,4,0)} & - & w^2_{(2,2,0)} \\
w_{(1,0,1)}w_{(2,2,1)} & - & w_{(1,0,2)}w_{(2,2,0)} \\
w_{(1,0,1)}w_{(3,4,0)} & - & w_{(2,2,0)}w_{(2,2,1)} \\
w_{(1,0,2)}w_{(2,1,2)} & - & w_{(1,1,0)}w_{(2,0,4)} \\
w_{(1,0,2)}w_{(2,2,0)} & - & w_{(1,1,0)}w_{(2,1,2)} \\
w_{(1,0,2)}w_{(3,4,0)} & - & w^2_{(2,2,1)} \\
w_{(2,0,4)}w_{(2,2,0)} & - & w^2_{(2,1,2)}.
\end{array}$$

By construction it follows that I vanishes on F_d, and hence $I \subseteq I(F_d)$. Let $K[w^{\pm}_{(r,\gamma,\delta)}]$ be the ring of Laurent polynomials over K. To each binomial in I we associate a Laurent binomial $w_\alpha := w_{(r_1,\gamma_1,\delta_1)}w_{(r_2,\gamma_2,\delta_2)}w^{-1}_{(r_3,\gamma_3,\delta_3)}w^{-1}_{(r_4,\gamma_4,\delta_4)} - 1$. They generate a Laurent binomial ideal whose associated partial character is the trivial one $\eta : L_\eta \to K^*$, sending $\eta(m) = 1$ for all $m \in L_\eta$, where $L_\eta = \langle \alpha \mid w^{\alpha_+} - w^{\alpha_-} \in I \rangle$.

Assuming that L_η is saturated and $rank(L_\eta) = 4$, then $I_+(\eta) = (w^{\alpha_+} - w^{\alpha_-} \mid \alpha \in L_\eta)$ is a prime ideal of codimension 4. We will see that $I_+(\eta)$ vanishes on F_d, as consequence $I_+(\eta)$ is the homogeneous ideal of a 3-dimensional variety contained in F_d, and hence $I(F_d)$ must be $I_+(\eta)$. Furthermore, we trivially have that $I \subseteq I_+(\eta)$. We will end this section discussing when the equality holds.

To achieve our goal, we will prove that L_η is saturated showing that it is isomorphism to $\mathbb{Z}^{\mu(I_d)-4}$. We will proceed in the usual way, by showing a basis of the lattice L_η formed by $\mu(I_d) - 4$ elements. For convenience, we will denote the natural basis of $\mathbb{Z}^{\mu(I_d)}$ by $\{(r,\gamma,\delta) \in \mathcal{W}_d\}$.

Definition 3.2.8. Fix $d \geq 4$ and write $d = 2k + \varepsilon = 3k' + \rho$, with $\varepsilon \in \{0,1\}$ and $\rho \in \{0,1,2\}$. We define $\mathcal{D}_d$ to be the following list of elements in $\mathbb{Z}^{\mu(I_d)}$.

- $D_{(0,0,0)} := (0,0,0) + (2,2k',0) - (1,k',0) - (1,k',0)$.
- For $0 \leq \gamma \leq k' - (\lceil \frac{\rho}{2} \rceil - \lfloor \frac{\rho}{2} \rfloor)$,
  - If $\varepsilon = 1$, let $D_{(1,0,0)} := (1,0,0) + (2, 2k' + \lfloor \frac{\rho}{2} \rfloor, \lceil \frac{\rho}{2} \rceil - \lfloor \frac{\rho}{2} \rfloor) - (1, 2k' + \lfloor \frac{\rho}{2} \rfloor - k - \varepsilon', \lceil \frac{\rho}{2} \rceil - \lfloor \frac{\rho}{2} \rfloor) - (2, k+\varepsilon, 0)$.
  - $D_{(1,\gamma,\delta)} := (1,\gamma,\delta) + (3,d,0) - (2, \lfloor \frac{d+\gamma}{2} \rfloor, \delta) - (2, \lceil \frac{d+\gamma}{2} \rceil, 0)$, for all $max\{0, d - 2\lfloor \frac{d+\gamma}{2} \rfloor\} \leq \delta \leq \lfloor \frac{2d-3\lfloor \frac{d+\gamma}{2} \rfloor}{2} \rfloor$.
  - $D_{(1,\gamma,\delta)} := (1,\gamma,\delta) + (3,d,0) - (2, \lfloor \frac{d+\gamma}{2} \rfloor, \lfloor \frac{2d-3\lfloor \frac{d+\gamma}{2} \rfloor}{2} \rfloor) - (2, \lceil \frac{d+\gamma}{2} \rceil, \delta - \lfloor \frac{2d-3\lfloor \frac{d+\gamma}{2} \rfloor}{2} \rfloor)$ for all $\lfloor \frac{2d-3\lfloor \frac{d+\gamma}{2} \rfloor}{2} \rfloor < \delta < \lfloor \frac{d-3\gamma}{2} \rfloor$, and $\delta = \lfloor \frac{d-3\gamma}{2} \rfloor$ with $\lfloor \frac{d+\gamma}{2} \rfloor$ and $\lceil \frac{d+\gamma}{2} \rceil$ both not odd.
  - $D_{(1,\gamma,\lfloor \frac{d-3\gamma}{2} \rfloor)} := (1,\gamma,\lfloor \frac{d-3\gamma}{2} \rfloor) + (3,d,0) - (2, \lfloor \frac{d+\gamma}{2} \rfloor - 1, \lfloor \frac{2d-3\lfloor \frac{d+\gamma}{2} \rfloor + 3}{2} \rfloor) - (2, \lceil \frac{d+\gamma}{2} \rceil + 1, \lfloor \frac{2d-3\lceil \frac{d+\gamma}{2} \rceil - 3}{2} \rfloor)$, for $\lfloor \frac{d+\gamma}{2} \rfloor$and $\lceil \frac{d+\gamma}{2} \rceil$ both odd, with $\lceil \frac{d+\gamma}{2} \rceil + 1 \leq 2k' + \lfloor \frac{\rho}{2} \rfloor$.
- For $0 \leq \gamma \leq 2k' + \lfloor \frac{\rho}{2} \rfloor - 2$,
  - $D_{(2,\gamma,\delta)} := (2,\gamma,\delta) + (2, 2k' + \lfloor \frac{\rho}{2} \rfloor, \lceil \frac{\rho}{2} \rceil - \lfloor \frac{\rho}{2} \rfloor) - (2, k' + \lfloor \frac{\gamma + \lfloor \frac{\rho}{2} \rfloor}{2} \rfloor, \delta + \lceil \frac{\rho}{2} \rceil - \lfloor \frac{\rho}{2} \rfloor - max\{0, d - 2(k' + \lceil \frac{\gamma + \lfloor \frac{\rho}{2} \rfloor}{2} \rceil)\}) - (2, k' + \lceil \frac{\gamma + \lfloor \frac{\rho}{2} \rfloor}{2} \rceil, max\{0, d - 2(k' + \lceil \frac{\gamma + \lfloor \frac{\rho}{2} \rfloor}{2} \rceil)\})$, for all $max\{0, d - 2(k' + \lceil \frac{\gamma + \lfloor \frac{\rho}{2} \rfloor}{2} \rceil)\} \leq \delta + \lceil \frac{\rho}{2} \rceil - \lfloor \frac{\rho}{2} \rfloor \leq \lfloor \frac{2d - 3(k' + \lfloor \frac{\gamma + \lfloor \frac{\rho}{2} \rfloor}{2} \rfloor}{2} \rfloor$.
  - $D_{(2,\gamma,\delta)} := (2,\gamma,\delta) + (2, 2k' + \lfloor \frac{\rho}{2} \rfloor, \lceil \frac{\rho}{2} \rceil - \lfloor \frac{\rho}{2} \rfloor) - (2, k' + \lfloor \frac{\gamma + \lfloor \frac{\rho}{2} \rfloor}{2} \rfloor, \lfloor \frac{2d - 3k' - \lfloor \frac{\gamma + \lfloor \frac{\rho}{2} \rfloor}{2} \rfloor}{2} \rfloor) - (2, k' + \lceil \frac{\gamma + \lfloor \frac{\rho}{2} \rfloor}{2} \rceil, \delta + \lceil \frac{\rho}{2} \rceil - \lfloor \frac{\rho}{2} \rfloor - \lfloor \frac{2d - 3k' - \lfloor \frac{\gamma + \lfloor \frac{\rho}{2} \rfloor}{2} \rfloor}{2} \rfloor)\}$, for all $\lfloor \frac{2d - 3(k' + \lfloor \frac{\gamma + \lfloor \frac{\rho}{2} \rfloor}{2} \rfloor}{2} \rfloor < \delta + \lceil \frac{\rho}{2} \rceil - \lfloor \frac{\rho}{2} \rfloor < \lfloor \frac{2d-3\gamma}{2} \rfloor + \lceil \frac{\rho}{2} \rceil - \lfloor \frac{\rho}{2} \rfloor$, and $\delta = \lfloor \frac{2d-3\gamma}{2} \rfloor$ with $k' + \lfloor \frac{\gamma + \lfloor \frac{\rho}{2} \rfloor}{2} \rfloor$ and $k' + \lceil \frac{\gamma + \lfloor \frac{\rho}{2} \rfloor}{2} \rceil$ both not odd.
  - $D_{(2,\gamma,\lfloor \frac{2d-3\gamma}{2} \rfloor)} := (2,\gamma,\delta) + (2, 2k' + \lfloor \frac{\rho}{2} \rfloor, \lceil \frac{\rho}{2} \rceil - \lfloor \frac{\rho}{2} \rfloor) - (2, k' + \lfloor \frac{\gamma + \lfloor \frac{\rho}{2} \rfloor}{2} \rfloor - 1, \lfloor \frac{2d - 3k' - \lfloor \frac{\gamma + \lfloor \frac{\rho}{2} \rfloor}{2} + 3 \rfloor}{2} \rfloor) - (2, k' + \lceil \frac{\gamma + \lfloor \frac{\rho}{2} \rfloor}{2} \rceil + 1, \lfloor \frac{2d - 3k' - \lceil \frac{\gamma + \lfloor \frac{\rho}{2} \rfloor}{2} \rceil - 3}{2} \rfloor)\}$, for $k' + \lfloor \frac{\gamma + \lfloor \frac{\rho}{2} \rfloor}{2} \rfloor$ and $k' + \lceil \frac{\gamma + \lfloor \frac{\rho}{2} \rfloor}{2} \rceil$ both odd, and $\gamma \leq 2k' + \lfloor \frac{\rho}{2} \rfloor - 3$.
- If $\rho = 1$,
  - $D_{(2,2k'-2,4)} := (2, 2k'-2, 4) + (2, 2k', 0) - 2(2, 2k'-1, 2)$.
  - $D_{(2,2k'-1,0)} := (2, 2k'-1, 0) + (2, 2k', 1) - (2, 2k'-1, 1) - (2, 2k', 0)$.

- $D_{(2,2k'-1,1)} := (2,2k'-1,1) + (2,2k',1) - (2,2k'-1,2) - (2,2k',0)$.

- If $\rho = 2$, let $D_{(2,2k',0)} := (2,2k',0) + (2,2k',2) - 2(2,2k',1)$.

We call the elements of $\mathcal{D}_d$ *distinguished elements*. We define one extra element D^ρ of $\mathbb{Z}^{\mu(I_d)}$, we will call *ghost distinguish element*,

- $D^0 = (2,2k'-2,3) + (2,2k'-1,0) + (2,2k',0) - 3(2,2k'-1,1)$;
- $D^1 = (1,k',0) + (2,2k'-1,2) + (3,d,0) - (2,2k',0) - 2(2,2k',1)$; and
- $D^2 = (1,k',1) + (2,2k',1) + (3,d,0) - (2,2k',2) - 2(2,2k'+1,0)$.

Remark 3.2.9. For $\rho \in \{0,1,2\}$, it holds that $\lfloor \frac{2\rho}{3} \rfloor = \lfloor \frac{\rho}{2} \rfloor$. We will use both expression indifferently.

Examples 3.2.10. Continuing with Example 3.2.7, we list the distinguished elements and the ghost distinguished element for $d = 4$:

- $D_{(0,0,0)} = (0,0,0) + (2,2,0) - 2(1,1,0)$.
- $D_{(1,0,0)} = (1,0,0) + (3,4,0) - 2(2,2,0)$.
- $D_{(1,0,1)} = (1,0,1) + (3,4,0) - (2,2,0) - (2,2,1)$.
- $D_{(1,0,2)} = (1,0,2) + (3,4,0) - 2(2,2,1)$.
- $D_{(2,0,4)} = (2,0,4) + (2,2,0) - 2(2,1,2)$.
- $D^1 = (1,1,0) + (2,1,2) + (3,4,0) - (2,2,0) - 2(2,2,1)$.

Now, we can formulate the main result of this section.

Theorem 3.2.11. Fix $d \geq 4$ and write $d = 2k + \varepsilon = 3k' + \rho$, with $\varepsilon \in \{0,1\}$ and $\rho \in \{0,1,2\}$. Then,

(i) $\mathcal{D}_{d,\rho} := \mathcal{D}_d \cup \{D^\rho\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^{\mu(I_d)-4}$,

(ii) $L_\eta = \langle \mathcal{D}_{d,\rho} \rangle$.

Or equivalently, $L_\eta \cong \mathbb{Z}^{\mu(I_d)}$.

The key point for proving the above theorem is the following lemma.

Lemma 3.2.12. $\mathcal{D}_d$ is parametrized by all the elements $(r_1, \gamma_1, \delta_1) \in \mathcal{W}_d$ admitting a non trivial generator $w_{(r_1,\gamma_1,\delta_1)} w_{(r_2,\gamma_2,\delta_2)} - w_{(r_3,\gamma_3,\delta_3)} w_{(r_4,\gamma_4,\delta_4)} \in I$ such that $(r_1, \gamma_1, \delta_1) < (r_i, \gamma_i, \delta_i)$, $i = 2,3,4$. In particular, $\mathcal{D}_d \subseteq L_\eta$.

Proof. Let $\mathcal{W}'_d$ be the set of all elements $(r, \gamma, \delta) \in \mathcal{W}_d$ admitting a non trivial generator as in the statement. We want to prove that $\mathcal{D}_d$ is parametrized by $\mathcal{W}'_d$. To this end we will see that the binomial associated to $D_{(r,\gamma,\delta)}$ is in fact a generator of I satisfying $(r_1, \gamma_1, \delta_1) < (r_i, \gamma_i, \delta_i)$, $i = 2,3,4$; and conversely, if an element (r, γ, δ) does not have a distinguished element in $\mathcal{D}_d$, it does not admit an equation as we require.

Let us first prove that each $D_{(r,\gamma,\delta)} \in \mathcal{D}_d$ defines a binomial of the form $w_{(r,\gamma,\delta)}w_{(r_2,\gamma_2,\delta_2)} - w_{(r_3,\gamma_3,\delta_3)}w_{(r_4,\gamma_4,\delta_4)}$ such that $r + r_2 = r_3 + r_4$, $\gamma + \gamma_2 = \gamma_3 + \gamma_4$ and $\delta + \delta_2 = \delta_3 + \delta_4$. For $D_{(0,0,0)}$, $D_{(2,2k'-2,4)}$, $D_{(2,2k'-1,0)}$, $D_{(2,2k'-1,1)}$ (if $\rho = 1$), and $D_{(2,2k',0)}$ (if $\rho = 2$) it is clear. We treat the elements $D_{(1,\gamma,\delta)}$. Observe that $\lceil \frac{d+\gamma}{2} \rceil \leq 2k' + \lfloor \frac{\rho}{2} \rfloor \Leftrightarrow \lfloor \frac{4k'+2\lfloor \frac{\rho}{2} \rfloor - d - \gamma}{2} \rfloor \geq 0 \Leftrightarrow \lfloor \frac{k'-\rho+2\lfloor \frac{\rho}{2} \rfloor - \gamma}{2} \rfloor \geq 0 \Leftrightarrow \gamma \leq k' - (\lceil \frac{\rho}{2} \rceil - \lfloor \frac{\rho}{2} \rfloor)$. Moreover, if $\gamma = k' - (\lceil \frac{\rho}{2} \rceil - \lfloor \frac{\rho}{2} \rfloor)$, then $\lceil \frac{d+\gamma}{2} \rceil = \lceil \frac{d+k'-(\lceil \frac{\rho}{2} \rceil - \lfloor \frac{\rho}{2} \rfloor)}{2} \rceil = 2k' + \lfloor \frac{\rho}{2} \rfloor = \lfloor \frac{d+\gamma}{2} \rfloor$. So, the ranges of $\gamma_3 \leq \gamma_4$ are well-defined. Since $\lfloor \frac{d+\gamma}{2} \rfloor = k + \lfloor \frac{\gamma+\varepsilon}{2} \rfloor$, for these γ's it holds that $max\{0, d-2\gamma\} = 0$, with the exception of $\gamma = 0$ only when $\varepsilon = 1$, in which case $max\{0, d-2\gamma\} = 1$. Formally, we have

$$0 \leq \delta \leq \lfloor \frac{d-3\gamma}{2} \rfloor,\ max\{0, d-2\gamma\} \leq \delta_3 \leq \lfloor \frac{2d - 3\lfloor \frac{d+\gamma}{2} \rfloor}{2} \rfloor,\ 0 \leq \delta_4 \leq \lfloor \frac{2d - 3\lceil \frac{d+\gamma}{2} \rceil}{2} \rceil.$$

Using the properties of the floor function, it verifies that $\lfloor \frac{2d-3\lfloor \frac{d+\gamma}{2} \rfloor}{2} \rfloor + \lfloor \frac{2d-3\lceil \frac{d+\gamma}{2} \rceil}{2} \rceil$ equals to

$$\begin{cases} \lfloor \frac{4d-3d-3\gamma}{2} \rfloor = \lfloor \frac{d-3\gamma}{2} \rfloor, & if\ \lfloor \frac{d+\gamma}{2} \rfloor, \lceil \frac{d+\gamma}{2} \rceil \text{ are not both odd,} \\ \lfloor \frac{4d-3d-3\gamma}{2} \rfloor - 1 = \lfloor \frac{d-3\gamma}{2} \rfloor - 1, & if\ \lfloor \frac{d+\gamma}{2} \rfloor, \lceil \frac{d+\gamma}{2} \rceil \text{ are both odd.} \end{cases}$$

As we have just argued, when $\gamma = k' - (\lceil \frac{\rho}{2} \rceil - \lfloor \frac{\rho}{2} \rfloor)$, $\frac{d+\gamma}{2} = 2k' + \lfloor \frac{\rho}{2} \rfloor$, which is odd if and only if $\rho = 2$. So, we conclude that for all $\gamma \in \{0, \dots, k' - \lfloor \frac{\rho}{2} \rfloor\}$ and $0 \leq \delta \leq \lfloor \frac{d-3\gamma}{2} \rfloor$, all the elements in $D_{(1,\gamma,\delta)}$ belong to $\mathcal{W}_d$, and hence its associated binomial is a generator of I. Moreover, it trivially holds that $(1,\gamma,\delta) < (r_i, \gamma_i, \delta_i)$, $i = 2,3,4$. On the other hand, the elements of the type $(1,\gamma,\delta)$ which not appear in the list are just:

1. for $\rho = 1$, $(1, k', 0)$, and
2. for $\rho = 2$, $(1, k', 1)$,

since $\lfloor \frac{d-3k'}{2} \rfloor = \lfloor \frac{\rho}{2} \rfloor$. But these are the last elements of the type $(1,\gamma,\delta)$ and they do not have any associated binomial of this kind.

Now, we analyze the distinguished elements $D_{(2,\gamma,\delta)}$. Observe that a binomial associated to $(2,\gamma,\delta)$ as we require only can involve variables $w_{(2,\gamma_i,\delta_i)}$. We need to impose $\gamma \leq \gamma_3 \leq \gamma_4 \leq \gamma_2$ in order to avoid trivial cancellations. For instance, we have $\gamma < \gamma_3 = k' + \lfloor \frac{\gamma+\lfloor \frac{\rho}{2} \rfloor}{2} \rfloor \Leftrightarrow \gamma \leq 2k' + \lfloor \frac{\rho}{2} \rfloor - 2$ and $\gamma_4 = k' + \lceil \frac{\gamma+\lfloor \frac{\rho}{2} \rfloor}{2} \rceil \leq k' + \lceil \frac{2k'+\lfloor \frac{\rho}{2} \rfloor + \lfloor \frac{\rho}{2} \rfloor}{2} \rceil = 2k' + \lfloor \frac{\rho}{2} \rfloor$. For $0 \leq \gamma \leq 2k' + \lfloor \frac{\rho}{2} \rfloor - 2$, we formally have $max\{0, d-2\gamma\} \leq \delta \leq \lfloor \frac{2d-3\gamma}{2} \rfloor$,

$$max\{0, d - 2(k' + 2(k' + \lfloor \frac{\gamma + \lfloor \frac{\rho}{2} \rfloor}{2} \rfloor)\} \leq \delta_3 \leq \lfloor \frac{2d - 3(k' + \lfloor \frac{\gamma+\lfloor \frac{\rho}{2} \rfloor}{2} \rfloor)}{2} \rfloor,$$

and

$$max\{0, d - 2(k' + 2(k' + \lceil \frac{\gamma + \lfloor \frac{\rho}{2} \rfloor}{2} \rceil)\} \leq \delta_4 \leq \lfloor \frac{2d - 3(k' + \lceil \frac{\gamma+\lfloor \frac{\rho}{2} \rfloor}{2} \rfloor)}{2} \rceil.$$

Firstly we study the lower bounds of δ_3 and δ_4 respect to δ. Consider the following observation.

$$d - 2k' - 2\lfloor \frac{\gamma + \lfloor \frac{\rho}{2} \rfloor}{2} \rfloor \leq 0 \Leftrightarrow k' + \lfloor \frac{\gamma + \lfloor \frac{\rho}{2} \rfloor}{2} \rfloor \geq k + \varepsilon$$

$$\Leftrightarrow \lfloor \frac{2k'-2k-2\varepsilon+\gamma+\lfloor\frac{\rho}{2}\rfloor}{2}\rfloor \geq 0 \Leftrightarrow \gamma \geq d+\varepsilon-2k'-\lfloor\frac{\rho}{2}\rfloor = k'+\varepsilon+\lceil\frac{\rho}{2}\rceil.$$

Therefore, if $max\{0, d-2\gamma\} = 0$, or equivalently $\gamma \geq k+\varepsilon$, then $max\{0, d-2k'-2\lfloor\frac{\gamma+\lfloor\frac{\rho}{2}\rfloor}{2}\rfloor\}$ $= max\{0, d-2k'-2\lfloor\frac{\gamma+\lceil\frac{\rho}{2}\rceil}{2}\rceil\} = 0$. Otherwise, we always have

$$d-2k'-2\lfloor\frac{\gamma+\lfloor\frac{\rho}{2}\rfloor}{2}\rfloor + d-2k'-2\lceil\frac{\gamma+\lfloor\frac{\rho}{2}\rfloor}{2}\rceil = 2d-4k'-2\gamma-2\lfloor\frac{\rho}{2}\rfloor$$
$$= d-2\gamma+d-4k'-2\lfloor\frac{\rho}{2}\rfloor = d-2\gamma-k'+\rho-2\lfloor\frac{\rho}{2}\rfloor$$
$$= d-2\gamma+\lceil\frac{\rho}{2}\rceil-\lfloor\frac{\rho}{2}\rfloor-k' \leq d-2\gamma.$$

Now we argue similarly as in $(1,\gamma,\delta)$. In this case, $\lfloor\frac{2d-3(k'+\lfloor\frac{\gamma+\lfloor\frac{\rho}{2}\rfloor}{2}\rfloor}{2}\rfloor + \lfloor\frac{2d-3(k'+\lceil\frac{\gamma+\lfloor\frac{\rho}{2}\rfloor}{2}\rceil}{2}\rfloor$ equals to

$$\begin{cases} \lfloor\frac{2d-3\gamma}{2}\rfloor+\lceil\frac{\rho}{2}\rceil-\lfloor\frac{\rho}{2}\rfloor \; if\; k'+\lfloor\frac{\gamma+\lfloor\frac{\rho}{2}\rfloor}{2}\rfloor, k'+\lceil\frac{\gamma+\lfloor\frac{\rho}{2}\rfloor}{2}\rceil \text{ are not both odd.} \\ \lfloor\frac{2d-3\gamma}{2}\rfloor+\lceil\frac{\rho}{2}\rceil-\lfloor\frac{\rho}{2}\rfloor-1, \; if\; k'+\lfloor\frac{\gamma+\lfloor\frac{\rho}{2}\rfloor}{2}\rfloor, k'+\lceil\frac{\gamma+\lfloor\frac{\rho}{2}\rfloor}{2}\rceil \text{ are both odd.} \end{cases}.$$

If $\gamma = 2k'+\lfloor\frac{\rho}{2}\rfloor-2$, then $k'+\frac{\gamma+\lfloor\frac{\rho}{2}\rfloor}{2} = 2k'+\lfloor\frac{\rho}{2}\rfloor-1$, which is odd if and only if $\rho \neq 2$. In these cases, $\gamma_4 = 2k'+\lfloor\frac{\rho}{2}\rfloor$, so $\delta_4 = \lceil\frac{\rho}{2}\rceil-\lfloor\frac{\rho}{2}\rfloor$ and hence the equation is trivial. Summarizing, we have just proved that all the elements in $\mathcal{D}_d$ verifies the lemma. It remains to prove that the elements of the type $(2,\gamma,\delta)$ not appearing in $\mathcal{D}_d$ do not belong to $\mathcal{W}'_d$. They are:

1. $(2, 2k'+\lfloor\frac{\rho}{2}\rfloor, \lceil\frac{\rho}{2}\rceil-\lfloor\frac{\rho}{2}\rfloor)$,
2. for $\rho = 0$, $(2, 2k'-2, 3)$.

The first one is the biggest element of the type $(2,\gamma,\delta)$, so it is clear. Now, notice that for $\rho = 0$ we have the following configuration

- $\gamma = 2k'-1,\ 0 \leq \delta \leq 1$, and
- $\gamma = 2k',\ \delta = 0$.

There is no way to achieve $\delta_3+\delta_4 = 3$ in any case. □

Remark 3.2.13. The previous lemma allows us to describe $\mathcal{W}'_d$ explicitly,

- $\rho = 0$, $\mathcal{W}'_d = \mathcal{W}_d - \{(2,2k'-2,3),(2,2k'-1,0),(2,2k'-1,1),(2,2k',0),(3,d,0)\}$,
- $\rho = 1$, $\mathcal{W}'_d = \mathcal{W}_d - \{(1,k',0),(2,2k'-1,2),(2,2k',0),(2,2k',1),(3,d,0)\}$, and
- $\rho = 2$, $\mathcal{W}'_d = \mathcal{W}_d - \{(1,k',1),(2,2k',0),(2,2k',1),(2,2k',2),(3,d,0)\}$.

Examples 3.2.14. For $d = 4$ we check from Example 3.2.7 that

$$\begin{aligned} \mathcal{W}_4' &= \mathcal{W}_4 - \{(1,1,0),(2,1,2),(2,2,0),(2,2,1),(3,4,0)\} \\ &= \{(0,0,0),(1,0,0),(1,0,1),(1,0,2),(2,0,4)\}. \end{aligned}$$

Corollary 3.2.15. $\mathcal{D}_{d,\rho} \subseteq L_\eta$.
Proof. We can check directly from tables in Section 3.1 that the following three binomials belong to I_d,

- if $\rho = 0$, $w_{(2,2k'-2,2)}w_{(2,2k'-1,1)} - w_{(2,2k'-2,3)}w_{(2,2k'-1,0)}$,
- if $\rho = 1$, $w_{(1,k'-1,2)}w_{(2,2k',0)} - w_{(1,k',0)}w_{(2,2k'-1,2)}$, and
- if $\rho = 2$, $w_{(1,k',0)}w_{(2,2k',2)} - w_{(1,k',1)}w_{(2,2k',1)}$.

Moreover, from the above remark $(2,2k'-2,2),(1,k'-1,2)$ and $(1,k',0)$ are in $\mathcal{W}_d'$. It is a straightforward computation to verify that

$$\begin{aligned} D^0 &= D_{(2,2k'-2,2)} - [(2,2k'-2,2)+(2,2k'-1,1)-(2,2k'-2,3)-(2,2k'-1,0)], \\ D^1 &= D_{(1,k'-1,2)} - [(1,k'-1,2)+(2,2k',0)-(1,k',0)-(2,2k'-1,2)] \text{ and} \\ D^2 &= D_{(1,k',0)} - [(1,k',0)+(2,2k',2)-(1,k',1)-(2,2k',1)]. \end{aligned}$$

Since $\mathcal{D}_d \subseteq L_\eta$, the result follows. □

Corollary 3.2.16. $\mathcal{D}_{d,\rho} \subseteq L_\eta$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^{\mu(I_d)-4}$.
Proof. It follows from Lemma 3.2.12 that $\mathcal{D}_d$ is a system of $\mu(I_d)-5$ linearly independent elements of $\mathbb{Z}^{\mu(I_d)}$. Indeed, $\mathcal{D}_d$ is parametrized by all the elements in $\mathcal{W}_d$ admitting a non-trivial generator $w_{(r_1,\gamma_1,\delta_1)}w_{(r_2,\gamma_2,\delta_2)} - w_{(r_3,\gamma_3,\delta_3)}w_{(r_4,\gamma_4,\delta_4)} \in I$ such that $(r_1,\gamma_1,\delta_1) \leq (r_i,\gamma_i,\delta_i)$, $i = 2,3,4$. Since (r,γ,δ) is the smallest element involved in $D_{(r,\gamma,\delta)}$, the matrix associated to $\mathcal{D}_d$ is an upper triangular matrix. Moreover, the coefficient of (r,γ,δ) in $D_{(r,\gamma,\delta)}$is 1. In fact Lemma 3.2.12 implies that $\mathcal{D}_d$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^{\mu(I_d)-5}$.

Now, observe that the ghost distinguished element verify that $(2,2k'-2,3),(1,k',0)$ or $(1,k',1)$ is the smallest element involved in D^ρ with coefficient 1. So, from Remark 3.2.13 the matrix associated to $\mathcal{D}_{d,\rho}$ is obtained from the matrix associated to $\mathcal{D}_d$ by adding a single row representing D^ρ, which is linearly independent to the rows of $\mathcal{D}_d$. □

Examples 3.2.17. Following with Example 3.2.7, the matrix associated to $\mathcal{D}_{4,1} \subset L_\eta$ is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 \end{pmatrix}$$

verifying as well Corollary 3.2.16.

The second part of Theorem 3.2.11 say that $\mathcal{D}_{d,\rho}$ is a system of generators of the lattice L_η. To prove it, it is enough to see that for any $\alpha \in L_\eta$ representing a generator of I, there exists a non-trivial linear combination $\sum D_{(r,\gamma,\delta)}$ in $\mathcal{D}_{d,\rho}$ such that $\alpha - \sum D_{(r,\gamma,\delta)} = 0$.

Lemma 3.2.18. Let Ω be the subset of L_η corresponding to the generators of I. For any $\alpha \in \Omega - \mathcal{D}_{d,\rho}$ there is a linear combination in $\Omega \cup \{\alpha\}$ only involving elements of the type $(2, \gamma, \delta)$.

Proof. Since Ω generates L_η, it is enough to see it for elements in Ω. Any $\alpha \in \Omega$ will be of the form $(r_1, \gamma_1, \delta_1) + (r_2, \gamma_2, \delta_2) - (r_3, \gamma_3, \delta_3) - (r_4, \gamma_4, \delta_4)$ such that $(r_i, \gamma_i, \delta_i) \in \mathcal{W}_d$, $r_1 + r_2 = r_3 + r_4, \gamma_1 + \gamma_2 = \gamma_3 + \gamma_4$ and $\delta_1 + \delta_2 = \delta_3 + \delta_4$. We can assume that $(r_1, \gamma_1, \delta_1)$ is the smallest element involved in α and from the equality $r_1 + r_2 = r_3 + r_4$ that $r_1 \leq r_3 \leq r_4 \leq r_2$ and r_i's has one of the following patrons:

r_1	r_2	r_3	r_4
0	2	1	1
0	3	1	2
1	1	1	1
1	2	1	2
1	3	2	2
2	2	2	2

Clearly the last type does not need any manipulation. We will proceed as follows. For each $(r_i, \gamma_i, \delta_i)$ with $r_i \neq 2, 3$, we consider the distinguished or eventually ghost distinguished element associated to it, $D_{(r_i,\gamma_i,\delta_i)}$. Next, we do the linear combination $\alpha_1 = \alpha + \sum_{i=3,4}^{*} D_{(r_i,\gamma_i,\delta_i)} - \sum_{i=1,2}^{*} D_{(r_i,\gamma_i,\delta_i)}$, where $*$ indicates that we are summing over $r_i \neq 2, 3$. Remember that $D_{(0,0,0)} = (0,0,0) + (2, \gamma_2, \delta_2) - (1, \gamma_3, \delta_3) - (1, \gamma_4, \delta_4)$, $D_{(1,\gamma,\delta)} = (1, \gamma, \delta) + (3, d, 0) - (2, \gamma_3, \delta_4) - (2, \gamma_4, \delta_4)$, and in the worst case $D_{(1,0,0)} = (1,0,0) + (2, \gamma_2, \delta_2) - (1, \gamma_3, \delta_3) - (2, \gamma_4, \delta_4)$. While $D^1 = (1, k', 0) + (2, 2k' - 1, 2) + (3, d, 0) - (2, 2k', 0) - 2(2, 2k', 1)$ and $D^2 = (1, k', 1) + (2, 2k', 1) + (3, d, 0) - (2, 2k', 2) - 2(2, 2k' + 1, 0)$. So, if α does not involves the element $D_{(1,0,0)}$ in the worst case, α_1 is the linear combination that we want in the following cases,

r_1	r_2	r_3	r_4
1	1	1	1
1	2	1	2
1	3	2	2

Otherwise, let $\alpha_1 = \alpha_1^+ - \alpha_1^-$ where α_1^+ compiles all the (r, γ, δ) with positive sing, and α_1^- with negative sign. Then, α_1 becomes of the type:

r^+						r^-					
1	2	2	2	3	3	2	3	2	2	2	2
1	2	2	3			2	2	2	2		
1	2	3				2	2	2			

where the unique element corresponding to $r = 1$ is different from $(1, 0, 0)$. Now, applying again the above procedure on α_1 the result follows. It remains to treat the first two types of α. Assuming that $(1, 0, 0)$ does not appear in the expression, we obtain the following scheme for α_1,

r^+					r^-				
1	1	2	3	3	2	2	2	2	2
1	1	3	3		2	2	2	2	

Since $(1,0,0)$ does not appear also in α_1, it is enough to apply the process on α_1. Otherwise, we will obtain similar expressions which do not involve $(1,0,0)$ in the linear combination α_2 obtained from α_1, and hence it will suffices apply now the process to α_2. □

Remark 3.2.19. Note by $\tilde{\alpha} = \sum_{i=1}^{l}(2,\gamma_i,\delta_i) - \sum_{j=1}^{s}(2,\gamma_j,\delta_j)$ the linear combination obtained from $\alpha \in L_\eta - \mathcal{D}_{d,\rho}$. The algorithm described in the above lemma does not modified the essential structure of α in the sense that $l = l'$, $\sum_{i=1}^{l}\gamma_i = \sum_{j=1}^{l'}\gamma_j$ and $\sum_{i=1}^{l}\delta_i = \sum_{j}^{l'}\delta_j$. In particular, $w^{\tilde{\alpha}_+} - w^{\tilde{\alpha}_-}$ is also a binomial vanishing in all points of $I(\varphi(I_d))$.

Before starting with the final part of the argument, let us consider and example of how Lemma 3.2.18 works.

Examples 3.2.20. The elements of $\Omega - \mathcal{D}_{4,1}$ are reduced to

$$\begin{array}{lcllcl}
(0,0,0)+(2,0,4) & - & 2(1,0,2) & \rightarrow & (2,0,4)+(2,2,0)-2(2,1,2) \\
(0,0,0)+(2,1,2) & - & (1,0,2)-(1,1,0) & \rightarrow & 0 \\
(1,0,0)+(1,0,2) & - & 2(1,0,1) & \rightarrow & 0 \\
(1,0,0)+(2,2,1) & - & (1,0,1)-(2,2,0) & \rightarrow & 0 \\
(1,0,1)+(2,2,1) & - & (1,0,2)-(2,2,0) & \rightarrow & 0 \\
(1,0,2)+(2,1,2) & - & (1,1,0)-(2,0,4) & \rightarrow & -[(2,0,4)+(2,2,0)-2(2,1,2)] \\
(1,0,2)+(2,2,0) & - & (1,1,0)-(2,1,2) & \rightarrow & 0
\end{array}$$

Now, fix $\alpha \in \Omega - \mathcal{D}_{d,\rho}$ and consider its linear combination $\tilde{\alpha}$ obtained from Lemma 3.2.18. Our aim is to design an algorithm which returns a linear combination in $\mathcal{D}_{d,\rho} \cup \{\tilde{\alpha}\}$ which only involves elements of $\mathcal{W}_d$ not admitting a distinguished or ghost distinguished element. Precisely, a linear combination of the last three elements of $\mathcal{W}_d$. For convenience, let $\mathcal{W}_{d,\rho}^3 = \{w_0, w_1, w_2\}$ be the set of these three last elements, ordered in the natural way.

Algorithm
Input: $\tilde{\alpha} := \sum_i (2,\gamma_i,\delta_i) - \sum_j (2,\overline{\gamma}_j,\overline{\delta}_j)$.
Output: a linear combination in $\mathcal{W}_{d,\rho}^3$.

- **i.a** If all $(2,\gamma_i,\delta_i)$ in $\tilde{\alpha}$ belong to $\mathcal{W}_{d,\rho}^3$, stop the algorithm and return $\beta = \tilde{\alpha}$.
- **i.b** Otherwise, do $\tilde{\alpha}_1 = \tilde{\alpha} + \sum_j^* D_{(2,\gamma_j,\delta_j)} - D_i^*(2,\gamma_i,\delta_i)$, where $*$ indicates that the sum runs over the elements out of $\mathcal{W}_{d,\rho}^3$. Define $\tilde{\alpha} = \tilde{\alpha}_1$ and return to step **i.a.**

For instance, assume that the algorithm stops. Both algorithms, the above and the one in Lemma 3.2.18, works in the same way. As before, **Algorithm** neither changes the structure of $\tilde{\alpha}$. Let $\beta = \sum_{i=1}^{L}(2,\gamma_i,\delta_i) - \sum_{j=1}^{L'}(2,\gamma_j,\delta_j)$. It holds that $L = L'$, $\sum_{i=1}^{L}\gamma_i = \sum_{j=1}^{L'}\gamma_j$ and $\sum_{i=1}^{L}\delta_i - \sum_{j=1}^{L'}\delta_j$. Therefore,

$$\beta = Aw_0 + Bw_1 + Cw_2 - A'w_0 - B'w_1 - C'w_2,$$

where $A, B, C, D, A', B', C', D'$ are non negative integers. We translate the structure of β in terms of these coefficients. Since $\mathcal{W}_{d,\rho}^3$ depends on ρ, we specify it at each case:

(a) $\mathcal{W}^3_{d,0} = \{(2, 2k' - 1, 0), (2, 2k' - 1, 1)(2, 2k', 0)\}$,

(b) $\mathcal{W}^3_{d,1} = \{(2, 2k' - 1, 2), (2, 2k', 0), (2, 2k', 1)\}$, and

(c) $\mathcal{W}^3_{d,2} = \{(2, 2k' + 1, 0), (2, 2k', 1), (2, 2k', 2)\}$.

Clearly, we always have $A + B + C = A' + B' + C'$. In (a), from the sum of $\delta's$ it follows that $B = B'$, and hence $A + C = A' + C'$. From the sum of γ's and the equality $B = B'$, $A(2k' - 1) + B(2k' - 1) + C(2k') = A'(2k' - 1) + B'(2k' - 1) + C'(2k') \Rightarrow (A + C)(2k') - A = (A' + C')(2k') - A'$. Since $A + C = A' + C'$, $A = A'$ and the result follows. In (b), analogously $2A + C = 2A' + C'$ and $A(2k' - 1) + B(2k') + C(2k') = A'(2k' - 1) + B'(2k') + C'(2k')$. The last equality is equivalent to $2k'(A + B + C) - A = 2k'(A' + B' + C') - A'$, and hence $A = A'$. Therefore, from the first equality $C = C'$ and the result follows. Finally, in (c) we have $B + 2C = B' + 2C'$ and $A(2k' + 1) + B(2k') + C(2k') = A'(2k' + 1) + B'(2k') + C'(2k')$, clearly we can apply the same argument as in (b). As consequence $\beta = 0$. But notice that this proves Theorem 3.2.11, indeed β is a zero non-trivial linear combination in $\mathcal{D}_{d,\rho} \cup \{\alpha\}$.

It only remains to check that **Algorithm** stops. Suppose that we are at the step s of the algorithm and i.a does not take place. This means that there exists $(2, \gamma, \delta) \notin \mathcal{W}^3_{d,\rho}$ in $\tilde{\alpha}$, so the algorithm does $\tilde{\alpha} \pm D_{(2,\gamma,\delta)}$. We write $\tilde{\alpha} \pm D_{(2,\gamma,\delta)} = \tilde{\alpha}' \pm (2, \gamma, \delta) \mp D_{(2,\gamma,\delta)}$. Firstly we assume that $D_{(2,\gamma,\delta)}$ is a distinguished element and write

$$\tilde{\alpha}' \pm (2, \gamma, \delta) \mp D_{(2,\gamma,\delta)} = \tilde{\alpha}' \mp [(2, \gamma_2, \delta_2) - (2, \gamma_3, \delta_3) - (2, \gamma_4, \delta_4)].$$

Since the element $(2, \gamma_2, \delta_2)$ appearing in $D_{(2,\gamma,\delta)}$ always belongs to $\mathcal{W}^3_{d,\rho}$, we focus on the study of $(2, \gamma_3, \delta_3)$ and $(2, \gamma_4, \delta_4)$. But observe that we always have $(2, \gamma, \delta) < (2, \gamma_i, \delta_i)$, $i = 3, 4$ from the Lemma 3.2.12, which is enough to conclude that **Algorithm** stops. Indeed, $\mathcal{W}_d$ is a finite set and at any step $\tilde{\alpha}$ has a finite number of summands. Finally, if $D_{(2,\gamma,\delta)}$ corresponds to the ghost distinguished element $D^0 = (2, 2k' - 2, 3) + (2, 2k' - 1, 0) + (2, 2k', 0) - 3(2, 2k' - 1, 1)$, it holds too that $(2, 2k' - 2, 3)$ is the smallest element. Even more, all the elements involved in the expression belong to $\mathcal{W}^3_{d,0}$. □

For sake of completeness, observe that in the reduction process on Example 3.2.20 **Algorithm** is already finished, since all the elements reduce to zero or to the distinguished $D_{(2,0,4)}$.

Summarizing, we have proved that $I_+(\eta)$ is a prime ideal of codimension 4 generated by all the binomials $w^{\alpha_+} - w^{\alpha_-} \in K[w_{(r,\gamma,\delta)}]$ such that $\alpha = \alpha_+ - \alpha_- \in L_\eta$ and $I \subseteq I_+(\eta)$. Since we know now that L_η is generated by $\mathcal{D}_{d,\rho}$, we can express α as a linear combination of the distinguished elements and the ghost distinguished element. Moreover, if $\alpha \notin \mathcal{D}_{d,\rho}$, a linear combination of that kind will be of the form $\sum_{i=1}^{l} D_{(2,\gamma_i,\delta_i)}$ for some $l \leq \mu_{I_d} - 4$. Therefore, we can write $\alpha = \sum_{i=1}^{l}(2, \gamma_i, \delta) - \sum_{i=1}^{l}(2, \overline{\gamma}_i, \overline{\delta}_i)$ such that $\sum_{i=1}^{l} \gamma_i = \sum_{i=1}^{l} \overline{\gamma}_i$ and $\sum_{i=1}^{l} \delta_i = \sum_{i=1}^{l} \overline{\delta}_i$, which is a suffice condition implying that $w^{\alpha_+} - w^{\alpha_-}$ vanishes in the GT-threefold F_d. Keeping notation, we get:

Theorem 3.2.21. $I(F_d) = I_+(\eta) = I \cup (\prod_{i=1}^{l} w_{(2,\gamma_i,\delta_i)} - \prod_{i=1}^{l} w_{(2,\overline{\gamma}_i,\overline{\delta}_j)} \in K[w_{(r,\gamma,\delta)}] \mid l \leq \mu_{I_d} - 4,\ \sum_i^l \gamma_i = \sum_i^l \overline{\gamma}_i,\ \sum_{i=1}^{l} \delta_i = \sum_{i=1}^{l} \overline{\delta}_i)$. □

Provided Teorema 3.2.21, we are in better position to investigate the relation between the ideal I (Definition 3.2.6) and the homogeneous ideal of GT-threefolds. Based on computations with the program Macaulay2 up to $d = 15$, we conjecture:

Conjecture 1. Fix $d \geq 4$ and write $d = 2k + \varepsilon = 3k' + \rho$, with $\varepsilon \in \{0, 1\}$ and $\rho \in \{0, 1, 2\}$.

(i) For d even, $I = I_{+}(\rho)$. So, the homogeneous ideal of GT-threefolds is generated by quadrics.

(ii) For d odd, $I \subsetneq I_{+}(\rho)$ and $I_{+}(\rho)$ is generated by quadrics and cubics.

Remark 3.2.22. Conjecture 1 is partially supported by the analogue result in [11] on the generalized classical Togliatti system (Theorem 2.2.1 or [11], Theorem 7.2).

We end this subsection referring to an interesting consequence of Theorem 3.2.11 and Theorem 3.2.21, namely the homogeneous ideal $I(F_d)$ of a GT-threefold is a lattice ideal. Another aim we still pursue, apart from Conjecture 1, is to find a minimal free resolution of GT-threefolds and to check that, as the surfaces S_d (Theorem 2.2.1), F_d is arithmetically Cohen Macaulay. Both Theorems 3.2.11 and Theorem 3.2.21 clear a path for achieving this goal, since it is known that the so called Hull complex is a free resolution of lattice ideals. Our starting point will be the article of Bayer and Sturmfels [1], where they construct a canonical free resolution for arbitrary monomial modules and lattice ideals. Our first goal will be attempting to derive a minimal free resolution from this one. Until now, we only have been able to compute a minimal free resolution of F_d for $d = 4, 5$ and 6 using the program Macaulay2. Even though, substantiated on Theorem 2.2.1 we modestly conjecture that:

Conjecture 2. GT-threefolds are arithmetically Cohen Macaulay varieties.

3.2.3 Singularities of GT-threefolds.

In Section 2 and 3, we have established that $\varphi_{I_d} : \mathbb{P}^3 \to \mathbb{P}^{\mu_{I_d}-1}$ is the quotient variety of $\mathbb{P}^3$ by the cyclic group of automorphisms of $\mathbb{P}^3$ of order d generated by M. When we work over $K = \mathbb{C}$, singularities of quotient varieties are called *quotient singularities* and they have been largely studied. We take advantage of the description of quotient singularities in [16] or [8] to see that F_d is smooth outside the image of the four fundamental points $P_0 = [1, 0, 0, 0], P_1 = [0, 1, 0, 0], P_2 = [0, 0, 1, 0]$ and $P_3 = [0, 0, 0, 1]$ of $\mathbb{P}^3$. We follow mainly [16] and [8].

Given $G \subset GL(n, \mathbb{C})$ a finite subgroup, we say that $g \in G$ is a *quasi-reflection* if $rank(g - Id) = 1$. We denote G_q the largest subgroup of G generated by quasi-reflections, when $G_q = \{Id\}$ the subgroup G is called *small*. Keeping notations in Section 1, we denote $\mathbb{A}^n/G$ the affine quotient variety of $\mathbb{A}^n$ by G (we see it as a finite group of automorphism).

A classical result from Chevalley [3] characterizes smooth quotient varieties $\mathbb{A}^n/G$ by means of quasi-reflections. Precisely,

Theorem 3.2.23. $\mathbb{A}^n/G$ is smooth if and only if G is generated by quasi-reflections.
Proof. See [3], Theorem (B). □

Therefore, considering the small group $\tilde{G} := G/G_q$ we derive from Theorem 3.2.23 that it is indistinguishable to study quotient singularities of $\mathbb{A}^n/G$ or $\mathbb{A}^n/\tilde{G}$. In other words, when studying quotient singularities we may restrict to small groups. We present now our main tool.

Theorem 3.2.24. Let G be a small finite subgroup of $GL(n, \mathbb{C})$ and let $S = \{p \in \mathbb{A}^n \mid g(p) = p \text{ for some } g \neq Id\}$. Then, the singular locus of the quotient variety $\mathbb{A}^n/G$ is S/G.

Proof. See [8] Corollary. □

In our case, Theorem 3.2.24 is easy to apply since the set S is just given by the fixed points of the automorphism induced by M. Cover $\mathbb{P}^3$ by the usual affine cards $C_i := \mathbb{P}^n - V(x_i), i = 0, 1, 2, 3$ and consider for instance the corresponding to $i = 0$. The restricted point of $p = [1, \alpha_1, \alpha_2, \alpha_3] \in C_i$ fixed by M have to satisfy:

$$(1, \alpha_1, \alpha_2, \alpha_3) = (1, e\alpha_1, e^2\alpha_2, e^3\alpha_3)$$

which directly implies $p = P_0$. Analogous arguments allow us to conclude that $S = \{P_0, P_1, P_2, P_3\}$ and hence to state:

Proposition 3.2.25. For all $d \geq 4$, the singular points of the GT-three fold F_d are the cyclic quotient singularities $\varphi_{I_d}(P_i),\ i = 0, 1, 2, 3.$ □

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