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A new order on embedded coalitions: Properties and applications

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Abstract: Given a finite set of agents, an embedded coalition consists of a coalition and a partition of the rest of agents. We study a partial order on the set of embedded coalitions of a finite set of agents. An embedded coalition precedes another one if the first coalition is contained in the second and the second partition equals the first one after removing the agents in the second coalition. This poset is not a lattice. We describe the maximal lower bounds and minimal upper bounds of a finite subset, whenever they exist. It is a graded poset and we are able to count the number of elements at a given level as well as the total number of chains. The study of this structure allows us to derive results for games with externalities. In particular, we introduce a new concept of convexity and show that it is equivalent to having non-decreasing contributions to embedded coalitions of increasing size.

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Keywords: Partial order, Embedded coalition, Partition function, Convexity.

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1 Introduction

Lately, the study of cooperative games with coalitional externalities has attracted the attention of some important researchers (see Maskin, 2016). The basic ingredients of such games are coalitions of players embedded in a partition of the set of all players. Then, a game with coalitional externalities, or game in partition function form, is a real valued function on the set of all such embedded coalitions with the convention that the value attached to the empty set is zero. To date, most of the efforts have been devoted to the extensions of solution concepts like the core or the Shapley value from classic games, or games in characteristic function form, to games with externalities. In this paper we concentrate on the structure of the set of embedded coalitions endowed with a partial order. This allows us to study properties of the game itself like superadditivity and convexity. However, even if the motivation for our study and its applications are in Economics, our approach uses tools and results that belong to Discrete Mathematics.

When dealing with embedded coalitions one has to consider two types of objects, namely subsets and partitions. Even if both objects have well known ordering relations that give rise to the Boolean algebra and the lattice of partitions, respectively, it is not clear how embedded coalitions should be ordered. Indeed, Grabisch (2010) and Alonso-Meijide et al. (2017) have already studied two partial orders that give rise to two distinct lattice structures. In this paper, we consider another partial order that was first used by Bolger (1990) and, more recently, by Hu and Yang (2010) and Skibski et al. (2018), but that has not been formally defined and analyzed yet. The three partial orders agree in considering that if one embedded coalition precedes another, then the coalition of the first should be contained in the coalition of the second. The difference lies in how they deal with the partition side. According to Grabisch (2010), the first partition should be finer than the second while Alonso-Meijide et al. (2017) consider that it should be coarser.[1] In this paper, we consider that the second partition equals the first one after removing the agents in the second coalition. Then, this partial order can be considered a compromise between the other two. However, it turns out that these posets are quite

[1]The precise definitions will soon follow.

different because the new one does not have a lattice structure as the other two do. We characterize the maximal lower bounds and minimal upper bounds of an arbitrary finite set, whenever they exist. We provide examples to see that the meet and join operations are not associative, neither distributive. We show that it is a graded poset and count the number of elements at a given level. We also identify the join and meet-irreducible elements. Finally, we provide an isomorphism between the chains in our structure and the chains in the Boolean lattice. Based on this isomorphism, we count the total number of chains and describe the Möbius function.

The study of this poset allows us to derive some applications to the theory of co-operative games with externalities. To start with, we obtain an explicit expression of the coefficients of any game in the basis of unanimity games with respect to this partial order. Then, using the partial order it is very natural to define what superadditivity and convexity mean for games in partition function form. The literature on partition function form games has paid attention to these properties (see for instance, Maskin, 2003; Hafalir, 2007; Abe, 2016).

To the best of our knowledge, we are the first to consider these interesting properties of a game in partition function form based on this partial order. We see that our notion of convexity implies the one proposed by Hafalir (2007), but the reverse does not hold. Finally, we present an equivalent formulation of convexity, parallel to a well-known result for games in characteristic function form. Indeed, a game is convex if and only if the contributions of players to embedded coalitions of increasing size, with respect to our partial order, are nondecreasing.

The rest of the paper is organized as follows. In Section 2, we describe previous notions and results. Section 3 is devoted to the study of the new poset. Finally, Section 4 presents our results related to game theory.

2 Preliminaries

Let $(\mathcal{A}, \leq)$ be a partially ordered set (in short, a poset). Let $A \subseteq \mathcal{A}$ and $x \in \mathcal{A}$. We say that x is a *lower bound* of A if and only if $x \leq y$, for every $y \in A$.[2] We say that x is an *upper bound* of A if and only if $y \leq x$, for every $y \in A$. We say that x is a *minimal* (*maximal*) element of A if there is no $y \in A \setminus \{x\}$ such that $y \leq x$ ($x \leq y$). We say that x is the *supremum* of A, *sup*(A), if $x \leq y$ for every upper bound y of A. We say that x is the *infimum* of A, $inf(A)$, if $y \leq x$ for every lower bound y of A. If there is an element $\hat{1} \in \mathcal{A}$ such that $y \leq \hat{1}$ for every $y \in \mathcal{A}$, we say that $\hat{1}$ is the *top* element of $\mathcal{A}$. Similarly, the *bottom* element $\hat{0}$ is an element of $\mathcal{A}$ such that $\hat{0} \leq y$ for every $y \in \mathcal{A}$. We say that x is *covered* by $y \in \mathcal{A} \setminus \{x\}$ or y *covers* x if $x \leq y$ and there is no $z \in \mathcal{A} \setminus \{x, y\}$ such that $x \leq z \leq y$. An *atom* is any $x \in \mathcal{A}$ that covers $\hat{0}$. A *coatom* is any $x \in \mathcal{A}$ that is covered by $\hat{1}$. An element $x \in \mathcal{A} \setminus \{\hat{1}\}$ is *join-irreducible* if for each $A \subset \mathcal{A}$ such that *sup*(A) exists and $x =$ *sup*(A) implies $x \in A$. An element $x \in \mathcal{A} \setminus \{\hat{0}\}$ is *meet-irreducible* if for each $A \subset \mathcal{A}$ such that $inf(A)$ exists and $x = inf(A)$ implies $x \in A$. A *(irreducible) chain* $\mathcal{C}$ is a totally ordered subset of $\mathcal{A}$, $\mathcal{C} = \{x_0, x_1, \ldots, x_k\}$ such that x_{l+1} covers x_l, for every $l = 0, \ldots, k-1$.

Let $(\mathcal{A}, \leq)$ be a poset.

- If $x, y \in \mathcal{A}$ and $x \leq y$, we denote by $[x, y]_{\mathcal{A}}$ the set of elements $z \in \mathcal{A}$ such that $x \leq z \leq y$. If no confusion arises, we may simply write $[x, y]$.
- $(\mathcal{A}, \leq)$ satisfies the *Jordan-Dedekind condition* if all chains between two elements have the same length. This common length is called the *rank*.

Let $(\mathcal{A}_1, \leq_1), (\mathcal{A}_2, \leq_2)$ be two posets. An *isomorphism* ϕ is a bijective map $\phi : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ such that $\phi(x) \leq_2 \phi(y)$ if and only if $x \leq_1 y$, for every $x, y \in \mathcal{A}_1$.

A *finite lattice* is a finite poset $(\mathcal{A}, \leq)$ such that *sup*$(A) \in \mathcal{A}$ and $inf(A) \in \mathcal{A}$, for every $A \subseteq \mathcal{A}$. Apart from the Boolean lattice of a finite set, denoted by $(\mathcal{B}(N), \subseteq)$, we need to recall some notions related to the partition lattice. Let N be a finite set, $|N| = n$, and $\Pi(N)$ be the family of partitions of the set N. Let $S \subseteq N$ and $P \in \Pi(N)$.

[2] We denote: $x = y$ if $x \leq y$ and $y \leq x$; $x < y$ if $x \leq y$, but $x \neq y$.

We denote by P_{-S} the partition of $N \setminus S$ given by $P_{-S} = \{T \setminus S \ : \ T \in P\}$ and by $P \setminus R = P \setminus \{R\}$, for every $R \in P$. Let $P \in \Pi(N \setminus S)$. We denote by $P \cup \lfloor S \rfloor$ the partition given by $\{\{T : T \in P\}, \{\{i\} \ : i \in S\}\}$. Let $1 \leq k \leq n$. The total number of partitions of N with k elements is the *Stirling number of second kind*, $S_{n,k}$. The *Bell number* of n is the total number of partitions of a finite set N with $|N| = n$, i.e., $B_n = \sum_{k=1}^{n} S_{n,k}$. A well-known partial order on $\Pi(N)$ is the following. Let $P, Q \in \Pi(N)$.

$$P \preceq Q \text{ if and only if for every } S \in P \text{ there is some } T \in Q \text{ such that } S \subseteq T.$$

We denote this poset by $(\Pi(N), \preceq)$. It is well-known that $(\Pi(N), \preceq)$ is a lattice. If $P, Q \in \Pi(N)$, we denote by $P \bigwedge Q$ the infimum of P and Q and by $P \bigvee Q$ the supremum of P and Q, according to the partial order $\preceq$.

An *embedded coalition* of N is a pair $(S; P)$ with $\emptyset \neq S \subseteq N$ and P a partition of $N \setminus S$, i.e., $P \in \Pi(N \setminus S)$. If we have the embedded coalition $(T; Q)$ with $T = N$ then, $Q = \{\emptyset\}$ and we take $|Q| = 0$. For simplicity we denote by $(S; N \setminus S)$ the embedded coalition $(S; \{N \setminus S\})$, for every $S \subseteq N$. The set of embedded coalitions is denoted by EC^N. Several partial orders are considered on the family of embedded coalitions of a finite set N, EC^N. One of them has been studied in Grabisch (2010). He defined a partial order as follows: for every $(S; P), (T; Q) \in EC^N$,

$$(S; P) \sqsubseteq_0 (T; Q) \text{ if and only if } S \subseteq T \text{ and } P \cup \{S\} \preceq Q \cup \{T\},$$

Alonso-Meijide et al. (2017) study a different partial order on EC^N defined as

$$(S; P) \sqsubseteq_1 (T; Q) \text{ if and only if } S \subseteq T \text{ and } Q \preceq P_{-T} \tag{1}$$

for every $(S; P), (T; Q) \in EC^N$. Both partial orders consider a fictitious bottom element $\hat{0}$. Instead of that, here we consider empty embedded coalitions given by the family $\mathcal{F}_0(N) = \{(\emptyset; P) : \ P \in \Pi(N)\}$. We denote by $\mathcal{F}_N = EC^N \cup \mathcal{F}_0(N)$. In the next section we define and study a partial order on $\mathcal{F}_N$.

3 A new poset on $\mathcal{F}_N$

In this section we formulate and study the partial order suggested by Bolger (1990), Hu and Yang (2010), and Skibski et al. (2018), among others. First, we formalize the partial order defined on $\mathcal{F}_N$.

Definition 3.1. *Let N be a finite set. We define the* inclusion *in $\mathcal{F}_N$ as follows:*

$$(S;P) \sqsubseteq (T;Q) \text{ if and only if } S \subseteq T \text{ and } Q = P_{-T} \tag{2}$$

for every $(S;P),(T;Q) \in \mathcal{F}_N$.[3]

Equation (2) implies a closed relationship between P and $Q \cup \lfloor T \setminus S \rfloor$. Since $(S;P) \sqsubseteq (T;Q)$, then $S \subseteq T$ and $P_{-T} = Q$. Thus, $Q \cup \lfloor T \setminus S \rfloor \preceq P$ and $P = P \bigvee (Q \cup \lfloor T \setminus S \rfloor)$. The reverse implication is not true as we can see by taking $(S;P) = (\{1,2\};\{3,4,5,6\})$ and $(T;Q) = (\{1,2,3\};\{\{4\},\{5,6\}\})$. Notice that $S \subseteq T$, $P = P \bigvee (Q \cup \{3\})$ but $P_{-T} \neq Q$. Then, $(S;P) \not\sqsubseteq (T;Q)$.

This binary relation defines a partial order on $\mathcal{F}_N$. The next example illustrates the differences among the three partial orders defined above, $\sqsubseteq_0$, $\sqsubseteq_1$, and $\sqsubseteq$.

Example 3.1. *Let us take $N = \{1,2,3\}$ and its set of embedded coalitions EC^N. Figure 1 depicts the Hasse diagram corresponding to $(\mathcal{F}_N, \sqsubseteq)$. Notice that $(\{1\};\{2,3\})$ and $(\{1\};\lfloor 2,3 \rfloor)$ are not comparable according to $\sqsubseteq$. Nevertheless, $(\{1\};\{2,3\}) \sqsubseteq_1 (\{1\};\lfloor 2,3 \rfloor)$ and $(\{1\};\lfloor 2,3 \rfloor) \sqsubseteq_0 (\{1\};\{2,3\})$. The figure also illustrates the fact that there is no bottom element in $(\mathcal{F}_N, \sqsubseteq)$.*

Let N be a finite set. The set of lower bounds of a pair of elements in $(\mathcal{F}_N, \sqsubseteq)$ can be empty as the next example shows.

Example 3.2. *Let $N = \{1,2,3,4\}$ and $(S;P),(T;Q) \in \mathcal{F}_N$ defined as $(S;P) = (\{1\};\{\{2\},\{3,4\}\})$, $(T;Q) = (\{1,3\};\{2,4\})$. There is no $(L;M) \in \mathcal{F}_N$ such that $(L;M) \sqsubseteq (S;P)$ and $(L;M) \sqsubseteq (T;Q)$ as we see next. We proceed by contradiction. Let*

[3] The *strict inclusion* in $\mathcal{F}$ is given by $(S;P) \sqsubset (T;Q)$ if and only if $S \subset T$ and $Q = P_{-T}$, for every $(S;P),(T;Q) \in \mathcal{F}_N$.

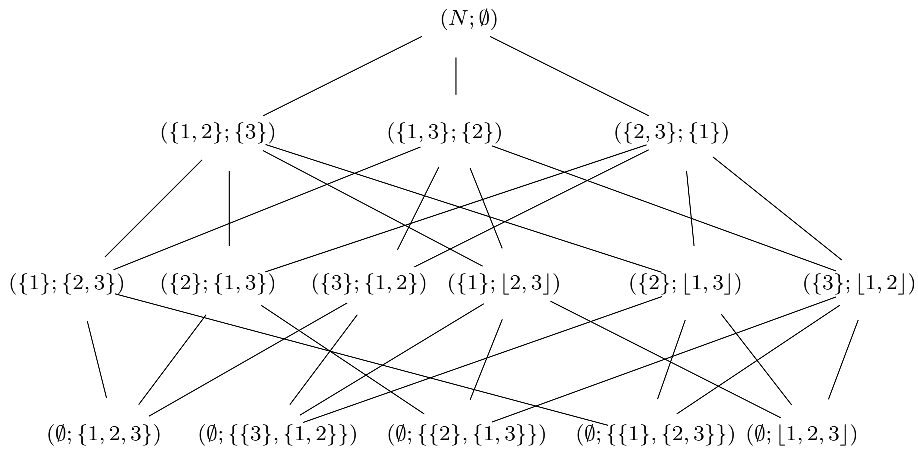

Figure 1: The partial order $\sqsubseteq$ on $\mathcal{F}_N$ with $|N| = 3$.

us assume that there is $(L;M) \in \mathcal{F}_N$ such that $(L;M) \sqsubseteq (S;P)$ and $(L;M) \sqsubseteq (T;Q)$. Then, $L \subseteq \{1\}$ and we distinguish two cases.

1. *$L = \{1\}$. Since $(L;M) \sqsubseteq (S;P)$, we have $P = M_{-S} = M$, because $L = S = \{1\}$. Then, $(L;M) = (S;P)$, but $(S;P)$ and $(T;Q)$ are not comparable.*

2. *$L = \emptyset$. We have $P_{-L} = P = M_{-S}$ because $(L;M) \sqsubseteq (S;P)$. Additionally, $(L;M) \sqsubseteq (T;Q)$ implies $M_{-T} = Q = Q_{-L}$. On one hand, we obtain that 2 and 4 belong to different elements in M due to $P = M_{-S}$ and, on the other hand, 2 and 4 must belong to the same element in M because $M_{-T} = Q$. Then, we get a contradiction.*

Moreover, if the set of maximal lower bounds is non-empty, it may have more than one element, as we see next.

Example 3.3. *Let $N = \{1,2,3\}$ and $(S;P), (T;Q) \in \mathcal{F}_N$ given by $(S;P) = (\{1\};\{2,3\})$, $(T;Q) = (\{2,3\};\{1\})$. Then, from Figure 1 it is easy to see that*

$$(\emptyset;\{1,2,3\}) \sqsubseteq (S;P), \quad (\emptyset;\{1,2,3\}) \sqsubseteq (T;Q), \quad \textit{and}$$
$$(\emptyset;\{\{1\},\{2,3\}\}) \sqsubseteq (S;P), \quad (\emptyset;\{\{1\},\{2,3\}\}) \sqsubseteq (T;Q).$$

Additionally, for the embedded coalition $(\emptyset;\{1,2,3\})$*, there is no* $(L;M)\in\mathcal{F}$ *such that* $(\emptyset;\{1,2,3\})\sqsubset(L;M)\sqsubset(S;P)$ *and* $(\emptyset;\{1,2,3\})\sqsubset(L;M)\sqsubset(T;Q)$*. Analogously, for the embedded coalition* $(\emptyset;\{\{1\},\{2,3\}\})$*, there is no* $(L;M)\in\mathcal{F}$ *such that* $(\emptyset;\{\{1\},\{2,3\}\})\sqsubset(L;M)\sqsubset(S;P)$ *and* $(\emptyset;\{\{1\},\{2,3\}\})\sqsubset(L;M)\sqsubset(T;Q)$*.*

As a consequence of all this, the set of maximal lower bounds of $(S;P)$ *and* $(T;Q)$ *is*

$$\{(\emptyset;\{1,2,3\}),(\emptyset;\{\{1\},\{2,3\}\})\}.$$

We present an auxiliary result for partitions.

Lemma 3.1. *Let* N *be a finite set,* $S\subset N$*,* $P,Q\in\Pi(N)$ *such that* $\lfloor S\rfloor\in Q$*. Then,* $(P\bigvee Q)_{-S}=P_{-S}\bigvee Q_{-S}$*.*

Proof. Let us take $L\in P\bigvee Q$. If $L\cap S=\emptyset$, then $L\in P_{-S}\bigvee Q_{-S}$. Let us assume that $L\cap S\neq\emptyset$. By the choice of L, there are $L_1,\ldots,L_k\in P$ such that $L=\cup_{j=1}^{k}L_j$ with $L_j\cap S\neq\emptyset$ for some $j\in\{1,\ldots,k\}$. Besides, there are $L'_1,\ldots L'_r\in Q$ with $L'_j\cap S=\emptyset$, for every $j=1,\ldots,r$, such that $L=(\cup_{j=1}^{r}L'_j)\cup(\lfloor L\cap S\rfloor)$. Then,

$$L\setminus S=\cup_{j=1}^{r}L'_j=\cup_{j=1}^{k}(L_j\setminus S)$$

and $L\setminus S\in P_{-S}\bigvee Q_{-S}$.

Now take $L\in P_{-S}\bigvee Q_{-S}$. There are $L_1,\ldots,L_k\in P_{-S}$ and $L'_1,\ldots L'_r\in Q_{-S}$ such that $L=\cup_{j=1}^{k}L_j=\cup_{j=1}^{r}L'_j$. Take $R_1,\ldots,R_k\in P$ such that $L_j\subseteq R_j$, for every $j=1,\ldots,k$ and define $R=\cup_{j=1}^{k}R_j$. Thus,

$$R=\cup_{j=1}^{k}R_j=L\cup(R\cap S)=(\cup_{j=1}^{r}L'_j)\cup(R\cap S),$$

$R_{-S}=L$, and $R\in P\bigvee Q$. □

Next, we characterize the set of maximal lower bounds of a finite subset of $\mathcal{F}_N$.

Proposition 3.1. *Let* N *be a finite set and* $(S;P)$*,* $(T;Q)\in\mathcal{F}_N$ *with* $(S;P)\neq(T;Q)$*.*

1. *If* $Q_{-S}\neq P_{-T}$*, a lower bound of* $(S;P)$ *and* $(T;Q)$ *does not exist.*

2. *If* $Q \cup \{T\} = P \cup \{S\}$*, then the set of maximal lower bounds of* $(S; P)$ *and* $(T; Q)$ *is the set* $\{(\emptyset; P \cup \{S\}), (\emptyset; P_{-T} \cup \{T \cup S\})\}$*.*

3. *If* $P_{-T} = Q_{-S}$ *but* $Q \cup \{T\} \neq P \cup \{S\}$*, then the set of maximal lower bounds of* $(S; P)$ *and* $(T; Q)$ *is given by*

$$\{(S \cap T; M)\} \cup \{(S \cap T; M_{-(R \cup R')} \cup \{R \cup R'\}) \ : R \subseteq T, R' \subseteq S, R, R' \in M\}$$

being $M = (P \cup \lfloor S \setminus T \rfloor) \bigvee (Q \cup \lfloor T \setminus S \rfloor)$*.*

Proof. Let N be a finite set and $(S; P)$, $(T; Q) \in \mathcal{F}_N$ with $(S; P) \neq (T; Q)$.

1. Take $(S; P)$, $(T; Q) \in \mathcal{F}_N$ such that $Q_{-S} \neq P_{-T}$. We proceed by contradiction. Let us assume that $(L; M) \in \mathcal{F}_N$ is a lower bound of $\{(S; P), (T; Q)\}$. Then, $(L; M) \sqsubseteq (S; P)$ and $(L; M) \sqsubseteq (T; Q)$. This implies $L \subseteq S \cap T$, $M_{-S} = P$ and $M_{-T} = Q$. Notice that

$$M_{-(S \setminus L)} = M_{-S} = P = P_{-L}, \quad M_{-(T \setminus L)} = M_{-T} = Q = Q_{-L}.$$

Thus,

$$P_{-T} = P_{-(T \setminus L)} = M_{-((S \cup T) \setminus L)} = Q_{-(S \setminus L)} = Q_{-S},$$

reaching a contradiction.

2. Let us assume that $Q \cup \{T\} = P \cup \{S\}$. Hence, $T \in P$, $S \in Q$, $S \cap T = \emptyset$, and $P_{-T} = Q_{-S}$. It is clear that

$$(\emptyset; P \cup \{S\}), (\emptyset; P_{-T} \cup \{T \cup S\}) \sqsubseteq (S; P), \quad \text{and}$$
$$(\emptyset; P \cup \{S\}), (\emptyset; P_{-T} \cup \{T \cup S\}) \sqsubseteq (T; Q).$$

Moreover, if $(L; M) \sqsubseteq (S; P)$ and $(L; M) \sqsubseteq (T; Q)$, we have $L = \emptyset$, $M_{-S} = P$, and $M_{-T} = Q$. Since $T \in P$ and $S \in Q$, we have $T \in M_{-S}$ and $S \in M_{-T}$. Then, either $S, T \in M$ or $S \cup T \in M$. In the first case $(L; M) = (\emptyset; P \cup \{S\})$. In the second case, $(L; M) = (\emptyset; P_{-T} \cup \{S \cup T\})$.

3. Let us assume that $P_{-T} = Q_{-S}$ but $Q \cup \{T\} \neq P \cup \{S\}$. Then, either $T \notin P$ or $S \notin Q$. Take $(L; M)$ with $L = S \cap T$ and $M = (P \cup \lfloor S \setminus T \rfloor) \bigvee (Q \cup \lfloor T \setminus S \rfloor)$.

We need to prove $(L;M) \sqsubseteq (S;P)$, and $(L;M) \sqsubseteq (T;Q)$. Clearly $L \subseteq S$ and $L \subseteq T$. It remains to prove that $M_{-S} = P$ and $M_{-T} = Q$. Using Lemma 3.1 and $P_{-T} = Q_{-S}$, we obtain

$$\begin{aligned} M_{-S} &= M_{-(S\setminus T)} = [(P \cup \lfloor S \setminus T \rfloor) \bigvee (Q \cup \lfloor T \setminus S \rfloor)]_{-(S\setminus T)} \\ &= P \bigvee (Q \cup \lfloor T \setminus S \rfloor)_{-(S\setminus T)} = P \bigvee (Q_{-(S\setminus T)} \cup \lfloor T \setminus S \rfloor) = P. \end{aligned}$$

In a similar way, we can prove $M_{-T} = Q$.

Now take $R, R' \in M$ such that $R \subseteq T, R' \subseteq S$. Let us consider $M' = M_{-(R\cup R')} \cup \{R \cup R'\}$ and $(S \cap T; M') \in \mathcal{F}_N$. By the choice of R and R' and the definition of M and M', it is clear that $M'_{-S} = M_{-S} = P$ and $M'_{-T} = M_{-T} = Q$. Then, $(S \cap T; M') \sqsubseteq (S;P)$ and $(S \cap T; M') \sqsubseteq (T;Q)$.

Finally, we need to prove that these elements in $\mathcal{F}_N$ are the maximal lower bounds. First, we take $(S \cap T; M)$. If there is $(U;W) \in \mathcal{F}_N$ such that $(S \cap T; M) \sqsubset (U;W) \sqsubset (S;P)$ and $(S \cap T; M) \sqsubset (U;W) \sqsubset (T;Q)$, then $S \cap T \subset U \subset S$ and $S \cap T \subset U \subset T$, but $S \cap T$ is the maximal lower bound of S and T. Thus, we get a contradiction. In a similar way, we can proceed if we take any element of the type $(S \cap T; M_{-(R\cup R')})$ with $R \subseteq T, R' \subseteq S, R, R' \in M$.

□

We present an example to illustrate Item 3.

Example 3.4. *Let* $N = \{1,2,3\}$, $(S;P) = (\{1,2\};\{3\})$, *and* $(T;Q) = (\{3\}; \lfloor 1,2 \rfloor)$. *Clearly,* $P_{-T} = Q_{-S}$, $P \cup \{S\} \neq Q \cup \{T\}$, *and* $(P \cup \lfloor 1,2 \rfloor) \bigvee (Q \cup \{3\}) = Q \cup \{3\} = \lfloor 1,2,3 \rfloor$. *Then, the set of maximal lower bounds is given by*

$$\{(\emptyset; \lfloor 1,2,3 \rfloor),\ (\emptyset; \{\{1,3\},\{2\}\}),\ (\emptyset; \{\{2,3\},\{1\}\})\}.$$

Next, we generalize the result in Proposition 3.1.

Corollary 3.1. *Let* N *be a finite set and* $A \subseteq \mathcal{F}_N$.

1. *If there is* $(S;P), (T;Q) \in A$ *such that* $P_{-T} \neq Q_{-S}$ *then, there is no lower bound of* A.

2. *If $P_{-T} = Q_{-S}$ for every $(S;P),(T;Q) \in A$ then, the set of maximal lower bounds of A is given by the set of elements $(\cap_{(S;P)\in A} S; L) \in \mathcal{F}_N$ with*

$$L = M = \bigvee_{(S;P)\in A}(P \cup \lfloor S \setminus (\cap_{(T;Q)\in A} T)\rfloor), \text{ or}$$
$$L = M_{-(\cap_{(T';Q')\neq(T;Q)}(R_{(T;Q)}\cup R_{(T';Q')}))} \cup \{\cap_{(T';Q')\neq(T;Q)}(R_{(T;Q)} \cup R_{(T';Q')})\}$$

for some $R_{(S;P)} \subseteq S, R_{(S;P)} \in M, (S;P) \in A$.

In the following example, we illustrate Item 2 in Corollary 3.1.

Example 3.5. *Let $N = \{1,2,3,4\}$, $(S;P) = (\{1\}; \lfloor 2,3,4\rfloor)$, $(T;Q) = (\{2,3\}; \lfloor 1,4\rfloor)$, $(H;L) = (\{3,4\}; \lfloor 1,2\rfloor)$ and $A = \{(S;P),(T;Q),(H;L)\}$. Notice that $P_{-T} = Q_{-S}$, $P_{-H} = L_{-S}$, and $Q_{-H} = L_{-T}$. Moreover, $S \cap T \cap H = \emptyset$ and $M = \lfloor 1,2,3,4\rfloor$. The set of lower bounds is given by*

$$\{(\emptyset; \lfloor 1,2,3,4\rfloor), (\emptyset; \{\{1,3\}, \lfloor 2,4\rfloor\})\}.$$

Notice that the partition $\{\{1,3\}, \lfloor 2,4\rfloor\}$ is obtained by taking $R_{(S;P)} = \{1\}$, $R_{(T;Q)} = \{3\}$, and $R_{(H;L)} = \{3\}$.

In what follows, given a finite set $A \subseteq \mathcal{F}_N$, we denote by $\wedge_{(S;P)\in A}(S;P)$ the set of maximal lower bounds of A. The operation $\wedge$ does not satisfy the associative property.

Example 3.6. *Let $N = \{1,2,3,4,5,6\}$, $(S;P) = (\{1,2\}; \{\{3,4\},\{5,6\}\})$, $(T;Q) = (\{1,4\}; \{\{3\},\{2,5,6\}\})$, and $(L;M) = (\{2,3\}; \{\lfloor 1,4\rfloor, \{5,6\}\})$. Notice that $P_{-T} = Q_{-S}$, $P_{-L} = M_{-S}$, and $Q_{-L} = M_{-T}$. We use Corollary 3.1 to obtain the set of maximal lower bounds of $\{(S;P),(T;Q),(L;M)\}$. First, we obtain*

$$\{\lfloor 1,2\rfloor, \{3,4\}, \{5,6\}\} \bigvee \{\lfloor 1,3,4\rfloor, \{2,5,6\}\} \bigvee \{\lfloor 1,2,3,4\rfloor, \{5,6\}\} =$$
$$\{\{1\}, \{2,5,6\}, \{3,4\}\}.$$

The unique lower bound of $\{(S;P),(T;Q),(L;M)\}$ is $(\emptyset; \{\{1\},\{2,5,6\},\{3,4\}\})$. Nevertheless, using Proposition 3.1,

$$(T;Q) \wedge (L;M) = \{(\emptyset; H) : \; H \in \mathcal{P}\}$$

with

$$\mathcal{P} = \{\{\lfloor 1,3,4\rfloor, \{2,5,6\}\},\ \{\{1,3\},\{4\},\{2,5,6\}\},\ \{\{3,4\},\{1\},\{2,5,6\}\}\},$$

and $(S;P)\wedge[(T;Q)\wedge(L;M)]$ *does not exist as a consequence of applying Proposition 3.1 to* $(S;P)$ *and* $(\emptyset;\{\lfloor 1,3,4\rfloor,\{2,5,6\}\})$*.*

Next, we characterize the minimal upper bounds of two elements in $\mathcal{F}_N$. It is possible to have more than one minimal upper bound.

Example 3.7. *Take* $N = \{1,2,3\}$*,* $(S;P) = (\{1\};\lfloor 2,3\rfloor)$ *and* $(T;Q) = (\{1\};\{2,3\})$*. Then,* $(S;P)$*,* $(T;Q) \sqsubseteq (\{1,3\};\{2\})$ *there is no* $(L;M)$ *such that* $(S;P)$*,* $(T;Q) \sqsubset (L;M) \sqsubset (\{1,3\};\{2\})$*. The same happens if we take* $(\{1,2\};\{3\})$*. Thus,* $\{(\{1,3\};\{2\}),(\{1,2\};\{3\})\}$ *is the set of minimal upper bounds of* $(S;P)$ *and* $(T;Q)$*.*

Previously to the characterization of the set of minimal upper bounds of two elements in $\mathcal{F}_N$, we obtain an auxiliary result related to two partitions and their infimum.

Lemma 3.2. *Let* N *be a finite set,* $P,\ Q \in \Pi(N)$*, and* $S \subseteq N$*. Then,* $P_{-S} \bigwedge Q_{-S} = (P \bigwedge Q)_{-S}$*.*

Proof. Let N be a finite set, $P,\ Q \in \Pi(N)$, and $S \subseteq N$. Let $R \in P$ and $\tilde{R} \in Q$. Then, $(R \setminus S) \cap (\tilde{R} \setminus S) \in P_{-S} \bigwedge Q_{-S}$ and $(R \setminus S) \cap (\tilde{R} \setminus S) = (R \cap \tilde{R}) \setminus S \in (P \bigwedge Q)_{-S}$. Let $R \in (P \bigwedge Q)_{-S}$. Then, there are $R' \in P$ and $\tilde{R} \in Q$ such that $R = (R' \cap \tilde{R}) \setminus S = (R' \setminus S) \cap (\tilde{R} \setminus S) \in P_{-S} \bigwedge Q_{-S}$. □

The next result characterizes the set of minimal upper bounds of two embedded coalitions in $\mathcal{F}_N$.

Proposition 3.2. *Let* N *be a finite set and* $(S;P)$*,* $(T;Q) \in \mathcal{F}_N$*,* $M = P_{-T} \bigwedge Q_{-S}$*, and* $L \subseteq N \setminus (S \cup T)$*.*

1. $(R;M') \in \mathcal{F}_N$ *is an upper bound of* $\{(S;P),(T;Q)\}$ *if and only if* $R = S \cup T \cup L$ *with* $L \subseteq N \setminus (S \cup T)$*, and* $M' = P_{-(T\cup L)} = Q_{-(S\cup L)}$*.*

2. $(S\cup T\cup L; P_{-(T\cup L)})$ *is a minimal upper bound of* $(S;P)$ *and* $(T;Q)$ *if and only if* $P_{-(T\cup L)} = Q_{-(S\cup L)} = M_{-L}$ *and for every* $L'\subseteq N\setminus(S\cup T)$ *with* $P_{-(T\cup L')} = Q_{-(S\cup L')} = M_{-L'}$ *it holds* $L\subseteq L'$ *or* $L\subseteq N\setminus L'$ *.*

Proof. Let N be a finite set, $(S;P),\ (T;Q)\in\mathcal{F}_N$ and $M = P_{-T}\bigwedge Q_{-S}$.

1. Let $(R;M')\in\mathcal{F}_N$ be an upper bound of $\{(S;P),(T;Q)\}$. Then, $S\cup T\subseteq R$ and $P_{-R} = M' = Q_{-R}$. There is some $L\subseteq N\setminus(S\cup T)$ such that $R = S\cup T\cup L$ and $P_{-R} = P_{-(T\cup L)} = M' = Q_{-R} = Q_{-(S\cup L)}$.

   If $R = S\cup T\cup L$ with $L\subseteq N\setminus(S\cup T)$ and $M' = P_{-(T\cup L')} = Q_{-(S\cup L')}$, clearly $(S;P),(T;Q)\sqsubseteq(R;M')$.

2. First, take $L\subseteq N\setminus(S\cup T)$ such that $(S\cup T\cup L; P_{-(T\cup L)})$ is a minimal upper bound of $(S;P)$ and $(T;Q)$. Then, $(S;P),\ (T;Q)\sqsubseteq(S\cup T\cup L; P_{-(T\cup L)})$ and we have $P_{-(T\cup L)} = Q_{-(S\cup L)} = M_{-L}$ using Lemma 3.2. Let $L'\subseteq N\setminus(S\cup T)$ with $P_{-(T\cup L')} = Q_{-(S\cup L')} = M_{-L'}$. We consider two cases. If $(S\cup T\cup L; P_{-(T\cup L)})\sqsubseteq(S\cup T\cup L'; P_{-(T\cup L')})$, then $L\subseteq L'$. It remains to study the case $(S\cup T\cup L; P_{-(T\cup L)})$ and $(S\cup T\cup L'; P_{-(T\cup L')})$ are not comparable. Since both are upper bounds of $(S;P)$ and $(T;Q)$, we have $P_{-(T\cup L')} = Q_{-(S\cup L')}$, $S\cup T\subseteq S\cup T\cup L$, and $S\cup T\subseteq S\cup T\cup L'$. Then, L and L' are not comparable. Let us assume that $R = L\cap L'\neq\emptyset$ and $L\setminus L'\neq\emptyset$. By Lemma 3.2, $P_{-(T\cup R)}\bigwedge Q_{-(S\cup R)} = M_{-R}$, then $(S\cup T\cup R; M_{-R})$ is also an upper bound of $(S;P)$ and $(T;Q)$, but $(S\cup T\cup R; M_{-R})\sqsubset(S\cup T\cup L; P_{-(T\cup L)})$. And this fact contradicts the minimality condition of $(S\cup T\cup L; P_{-(T\cup L)})$. Then, $L\subseteq N\setminus L'$ or $L\subseteq L'$.

   On the other hand, let us take $(S\cup T\cup L; P_{-(T\cup L)})$ such that $P_{-(T\cup L)} = Q_{-(S\cup L)} = M_{-L}$ and for every $L'\subseteq N\setminus(S\cup T)$ with $P_{-(T\cup L')} = Q_{-(S\cup L')} = M_{-L'}$ it holds $L\subseteq L'$ or $L\subseteq N\setminus L'$. Clearly, $(S;P),\ (T;Q)\sqsubseteq(S\cup T\cup L; P_{-(T\cup L)})$. Let us assume that there is $(L';M')$ such that $(S;P),\ (T;Q)\sqsubset(L';M')\sqsubset(S\cup T\cup L; P_{-T\cup L})$. Then, $L' = S\cup T\cup R'$ with $R'\subset L$. Using Item 1 above and Lemma 3.2, we have $M' = (P_{-T}\bigwedge Q_{-S})_{-R'} = M_{-R'}$. Thus, $(L';M')$ is an upper bound of $(S;P)$ and $(T;Q)$. Then, $L\subseteq R'$ or $L\subseteq N\setminus R'$, getting a contradiction with the fact that

$R' \subset L$.

$\square$

An example of the result above is given next.

Example 3.8. *Let* $N = \{1,2,3,4\}$, $(S;P) = (\emptyset; \{\{1,2\}, \lfloor 3,4 \rfloor\})$ *and* $(T;Q) = (\emptyset; \{\{1,3\}, \lfloor 2,4 \rfloor\})$. *Notice that*

$$M = \{\{1,2\}, \lfloor 3,4 \rfloor\} \bigwedge \{\{1,3\}, \lfloor 2,4 \rfloor\} = \lfloor 1,2,3,4 \rfloor.$$

Take $L = \{1\}$. *We obtain* $P_{-(T\cup L)} = Q_{-(S\cup L)} = M_{-L} = \lfloor 2,3,4 \rfloor$. *If we take* $L' = \{2,3\}$, $M_{-L'} = \lfloor 1,4 \rfloor = P_{-(T\cup L')} = Q_{-(S\cup L')}$. *Then, the set of minimal upper bounds of* $(S;P)$ *and* $(T;Q)$ *is given by*

$$\{(\{1\}; \lfloor 2,3,4 \rfloor), (\{2,3\}; \lfloor 1,4 \rfloor)\}.$$

We can generalize the result above to every finite subset of $\mathcal{F}_N$.

Corollary 3.2. *Let* N *be a finite set and* $A \subseteq \mathcal{F}_N$.

1. *Every upper bound of* A *is of the form* $((\cup_{(S;P)\in A} S) \cup L; M')$ *with* $L \subseteq N \setminus (\cup_{(S;P)\in A} S)$ *and* $M' = P_{-((\cup_{(T;Q)\in A} T)\cup L)}$.

2. $((\cup_{(S;P)\in A} S) \cup L; P_{-((\cup_{(S;P)\in A} S)\cup L)})$ *is a minimal upper bound of* A *if and only if* $P_{-((\cup_{(T;Q)\in A} T)\cup L)} = M_{-L}$ *and for every* $L' \subseteq N \setminus (\cup_{(S;P)\in A} S)$ *with* $P_{-((\cup_{(T;Q)\in A} T)\cup L')} = M_{-L'}$, *it holds* $L \subseteq L'$ *or* $L \subseteq N \setminus L'$, *being* $M = \bigwedge_{(T;Q)\in A} Q_{-(\cup_{(S;P)\in A} S)} \in \Pi(N \setminus \cup_{(S;P)\in A} S)$.

The next example illustrate Item 2 in Corollary 3.2

Example 3.9. *Let* $N = \{1,2,3,4,5\}$ *and* $A = \{(S;P),(T;Q),(H;L)\}$ *with* $(S;P) = (\{2\}; \{\{1,3,4\}, \{5\}\})$, $(T;Q) = (\{3,4\}; \{\{1,5\}, \{2\}\})$, *and* $(H;L) = (\{2,4\}; \{\{1\}, \{3,5\}\})$. *Clearly,* $M = \lfloor 1,5 \rfloor \bigwedge \{1,5\} \bigwedge \lfloor 1,5 \rfloor = \lfloor 1,5 \rfloor$ *and* $S \cup T \cup H = \{2,3,4\}$. *Then, the set of minimal upper bounds of* A *is the set*

$$\{(\{1,2,3,4\}; \{5\}),\ (\{2,3,4,5\}; \{1\})\}.$$

Corollary 3.3. *Let N be a finite set and $(S;P),\ (T;Q) \in \mathcal{F}_N$ such that $P_{-T} = Q_{-S}$. Then, $\{(S;P),(T;Q)\}$ has a unique minimal upper bound given by $sup\{(S;P),(T;Q)\} = (S \cup T; P_{-T})$.*

Proof. Let N be a finite set and $(S;P),\ (T;Q) \in \mathcal{F}_N$ such that $P_{-T} = Q_{-S}$. Then, $P_{-T} \bigwedge Q_{-S} = P_{-T} = Q_{-S}$. It is clear that $(S \cup T; P_{-T})$ is an upper bound of $(S;P)$ and $(T;Q)$. Using Proposition 3.2, any upper bound is given by $(S \cup T \cup L; P_{-(T\cup L)})$ with $L \subseteq N \setminus (S \cup T)$, and $P_{-(T\cup L)} = Q_{-(S\cup L)}$. If $L \neq \emptyset$, we have $(S \cup T; P_{-T}) \sqsubset (S \cup T \cup L; P_{-(T\cup L)})$ and $(S \cup T \cup L; P_{-(T\cup L)})$ is not a minimal upper bound. Thus, $(S \cup T; P_{-T}) = sup\{(S;P),(T;Q)\}$. □

In what follows, given a finite set $A \subseteq \mathcal{F}_N$, we denote by $\vee_{(S;P)\in A}(S;P)$ the set of minimal upper bounds of A. The operation $\vee$ does not satisfy the associative property.

Example 3.10. *Let $N = \{1,2,3,4,5,6\}$, $(S;P) = (\{1,2\};\{\{3,4\},\{5,6\}\})$, $(T;Q) = (\{1,3\};\{\{2,5\},\{4,6\}\})$, $(L;M) = (\{2,3\};\{\{1,6\},\{4,5\}\})$. According to Corollary 3.2 and taking into account*

$$\{\{4\},\{5,6\}\} \bigwedge \{\{5\},\{4,6\}\} \bigwedge \{\{4,5\},\{6\}\} = \lfloor 4,5,6 \rfloor,$$

the set of minimal upper bounds is given by

$$\{(\{1,2,3,4,5\};\{6\}),(\{1,2,3,4,6\};\{5\}),(\{1,2,3,5,6\};\{4\})\}.$$

Now we evaluate $(S;P) \vee [(T;Q) \vee (L;M)]$. First,

$$(T;Q) \vee (L;M) = \{(\{1,2,3,4\};\lfloor 5,6 \rfloor),(\{1,2,3,5,6\};\{4\}\}.$$

Second, $(S;P) \vee \{(\{1,2,3,4\};\lfloor 5,6 \rfloor),(\{1,2,3,5,6\};\{4\}\} = (N;\emptyset)$.

The operations $\vee$ and $\wedge$ do not satisfy the distributive properties.

Example 3.11. *Let $N = \{1,2,3,4,5,6\}$, $(S;P) = (\{1,2\};\{\{3,4\},\{5,6\}\})$, $(T;Q) = (\{1,4\};\{\{3\},\{2,5,6\}\})$, and $(L;M) = (\{2,3\};\{\lfloor 1,4 \rfloor,\{5,6\}\})$. First, we check that*

$(S;P) \vee [(T;Q) \wedge (L;M)] \neq [(S;P) \vee (T;Q)] \wedge [(S;P) \vee (L;M)]$. *Notice that*

$$(T;Q) \wedge (L;M) =$$
$$\{(\emptyset; \{\lfloor 1,3,4 \rfloor, \{2,5,6\}\}),\ (\emptyset; \{\{1,3\}, \{4\}, \{2,5,6\}\}),\ (\emptyset; \{\{3,4\}, \{1\}, \{2,5,6\}\})\},$$

$$(S;P) \vee [(T;Q) \wedge (L;M)] =$$
$$\{(\{1,2,3\}; \{\{4\}, \{5,6\}\}),\ (\{1,2,4\}; \{\{3\}, \{5,6\}\})\},$$

but

$$[(S;P) \vee (T;Q)] \wedge [(S;P) \vee (L;M)] =$$
$$\{(\{1,2,4\}; \{\{3\}, \{5,6\}\})\} \wedge \{(\{1,2,3\}; \{\{4\}, \{5,6\}\})\} =$$
$$\{(\{1,2\}; \{\lfloor 3,4 \rfloor, \{5,6\}\}),\ (\{1,2\}; \{\{3,4\}, \{5,6\}\})\}.$$

Second, we also check that $(T;Q) \wedge [(S;P) \vee (L;M)] \neq [(T;Q) \wedge (S;P)] \vee [(T;Q) \wedge (L;M)]$. *Notice that*

$$\begin{aligned} (T;Q) \wedge [(S;P) \vee (L;M)] &= (T;Q) \wedge \{(\{1,2,3\}; \{\{4\}, \{5,6\}\})\} \\ &= \{(\{1\}; \{\{2,5,6\}, \lfloor 3,4 \rfloor\}), (\{1\}; \{\{2,5,6\}, \{3,4\}\})\} \end{aligned}$$

$$(T;Q) \wedge (S;P) = (\{1\}; \{\{3,4\}, \{2,5,6\}\}),$$

Thus, $(T;Q) \wedge [(S;P) \vee (L;M)] \neq [(T;Q) \wedge (S;P)] \vee [(T;Q) \wedge (L;M)] = \{(\{1,3\}; \{\{2,5,6\}, \{4\}\}), (\{1,4\}; \{\{2,5,$

Proposition 3.3. *Let* $(S;P) \in \mathcal{F}_N \setminus \{(N;\emptyset)\}$. *Then, the number of elements in* $\mathcal{F}_N$ *that cover* $(S;P)$ *is* $|N \setminus S|$.

Proof. Let $(S;P) \in \mathcal{F}_N \setminus \{(N;\emptyset)\}$ and $i \in N \setminus S$. Then, $(S;P) \sqsubseteq (S \cup \{i\}; P_{-\{i\}})$ and there is no $(L;M) \in \mathcal{F}_N$ such that $(S;P) \sqsubset (L;M) \sqsubset (S \cup \{i\}; P_{-\{i\}})$. $\square$

Proposition 3.4. *Let* $(S;P) \in EC^N$.

1. *If* $(S;P) = (N;\emptyset)$, *then the number of elements in* $\mathcal{F}_N$ *covered by* $(S;P)$ *is* n.
2. *If* $(S;P) \neq (N;\emptyset)$, *then the number of elements in* $\mathcal{F}_N$ *covered by* $(S;P)$ *is* $|S|(|P|+1)$.

Proof. Let $(S;P) \in EC^N$.

1. Let us consider $(S;P)=(N;\emptyset)$. Then, for every $i\in N$, $(N\setminus\{i\};\{i\})\sqsubseteq(N;\emptyset)$ and there is no $(T;Q)\in\mathcal{F}_N$ such that $(N\setminus\{i\};\{i\})\sqsubset(T;Q)\sqsubset(N;\emptyset)$.

2. Let us consider $(S;P)\neq(N;\emptyset)$. For every $i\in S$ and $R\in P$, we have $(S\setminus\{i\};P_{-R}\cup\{R\cup\{i\}\})\sqsubseteq(S;P)$ and there is no $(T;Q)\in\mathcal{F}_N$ such that $(S\setminus\{i\};P_{-R}\cup\{R\cup\{i\}\})\sqsubset(T;Q)\sqsubset(S;P)$. Additionally, for every $i\in S$, $(S\setminus\{i\};P\cup\{i\})\sqsubseteq(S;P)$ and there is no $(T;Q)\in\mathcal{F}_N$ such that $(S\setminus\{i\};P\cup\{i\})\sqsubset(T;Q)\sqsubset(S;P)$. As a consequence of all this we have the result.

□

Proposition 3.5. *$(\mathcal{F}_N,\sqsubseteq)$ is a graded poset.*

Proof. Let $h:\mathcal{F}_N\longrightarrow\mathbb{N}$ defined by $h(S;P)=|S|+1$. It is clear that h is a height function since if $(S;P)\sqsubset(T;Q)$ then $|S|<|T|$ and $h(S;P)<h(T;Q)$. Additionally, if $(T;Q)$ covers $(S;P)$ we have $|T|=|S|+1$. Thus, $h(T;Q)=h(S;P)+1$. □

This function h is called the grading function. In our case $h(S;P)=1$ if and only if $S=\emptyset$. Given $1\leq k\leq n$, we say that $(S;P)$ is in level k if $h(S;P)=k+1$. Hence, $\mathcal{F}_N$ has $n+1$ levels. From the above result, $(\mathcal{F}_N,\sqsubseteq)$ has the Jordan-Dedekind property. Now we compute the number of elements in each level.

Proposition 3.6. *Let N be a finite set and $1\leq k\leq|N|+1$. Let $(\mathcal{F}_N,\sqsubseteq)$. Then, there are $B_{|N\setminus S|}\binom{|N|}{k-1}$ elements of $\mathcal{F}_N$ at level k being $|S|=k-1$ and $B_{|N\setminus S|}$ the Bell number of $|N\setminus S|$.*

Proof. Let $1\leq k\leq n+1$. Using Proposition 3.5, an element $(S;P)\in\mathcal{F}_N$ at level k satisfies $|S|=k-1$ and $P\in\Pi(N\setminus S)$. Then, counting all the possibilities we get the number $|\Pi(N\setminus S)|\binom{|N|}{k-1}=B_{|N\setminus S|}\binom{|N|}{k-1}$. □

Proposition 3.7. *Let N be a finite set with $|N|\geq 2$.*

i) The set of coatoms of $(\mathcal{F}_N, \sqsubseteq)$ *is given by* $\mathcal{C} = \{(N \setminus \{i\}; \{i\}) : \; i \in N\}$.

ii) The set of meet-irreducible elements of $(\mathcal{F}_N, \sqsubseteq)$ *is given by*

$$\mathcal{M} = \mathcal{C} \cup \{(S;P) \in \mathcal{F}_N \;:\; |S| = n-2\}$$

Proof. Let N be a finite set with $|N| \geq 2$.

i) Let $(S;P) \in \mathcal{F}_N$. A coatom $(S;P)$ belongs to the n^{th} level. Then, $|S| = n-1$ and $(S;P) = (N \setminus \{i\}; \{i\})$ for some $i \in N$.

ii) Let us take $i \in N$ and $(S;P) = (N \setminus \{i\}; \{i\})$. Let us consider $A \subseteq \mathcal{F}_N$ with $|A| \geq 2$ such that $(S;P) = inf(A)$. Thus, $S = \cap_{(T;Q)\in A} T$. This implies that $N \setminus \{i\} \subseteq T$ for every $(T;Q) \in A$. Then, there is some $(T;Q) \in A$ such that $(T;Q) = (S;P)$.

Let $i, j \in N$, $(S;P) = (N \setminus \{i,j\}; \{i,j\})$. Let us consider $A \subseteq \mathcal{F}_N$ with $|A| \geq 2$ such that $(S;P) = inf(A)$. Then, $S = \cap_{(T;Q)\in A} T$. Let us assume that $(S;P) \notin A$. Hence, we have $(N \setminus \{i\}; \{i\})$, $(N \setminus \{j\}; \{j\}) \in A$ and $(S;P)$ and $(S; \lfloor i,j \rfloor)$ are maximal lower bounds of A. Then, there is no infimum of A, which is a contradiction. A similar reasoning proves that $(S; \lfloor i,j \rfloor)$ is meet-irreducible.

Now it remains to check that only elements in $\mathcal{M}$ are meet-irreducible. If $|N| = 2$, $\mathcal{M} = \mathcal{F}_N \setminus \{(N; \emptyset)\}$ and then all elements are meet-irreducible. Let us take a finite set N with $|N| \geq 3$ and $(S;P) \in \mathcal{F}_N$ such that $(S;P) \notin \mathcal{M}$. In particular, $|S| \leq n-3$. We distinguish two cases.

1. $|S| = 0$. If $|P| = n$ or $|P| = 1$, take $A = \{(\{i\}; P_{-\{i\}}) : \; i \in N \setminus S\}$. Then, $(S;P) = inf(A)$, but $(S;P) \notin A$ which implies that $(S;P)$ is not meet-irreducible. If $1 < |P| < n$, there is some $R \in P$ with $|R| \geq 2$. Take $i \in R$, $j \in N \setminus R$, and $A = \{(\{i\}; P_{-\{i\}}), (\{j\}; P_{-\{j\}})\}$. It is clear that $(S;P) = inf(A)$, but $(S;P) \notin A$. Thus $(S;P)$ is not meet-irreducible.
2. $|S| > 0$. Let us take $A = \{(S \cup \{i\}; P_{-\{i\}}) : \; i \in N \setminus S\}$. We have $(S;P) = inf(A)$, but $(S;P) \notin A$. Then, $(S;P)$ is not meet-irreducible.

□

The set of atoms is empty because there is no bottom element.

Proposition 3.8. *Let N be a finite set. The set of join-irreducible elements of $(\mathcal{F}_N, \sqsubseteq)$ is given by*

$$\mathcal{I} = \{(S;P) \in \mathcal{F}_N : \; |S| \leq 1\}.$$

Proof. Let $(S;P) \in \mathcal{I}$. If $|S| = 0$, then $S = \emptyset$ and $(S;P)$ is join-irreducible. If $|S| = 1$, then $(S;P) = (\{i\};P)$ with $P \in \Pi(N \setminus \{i\})$, for some $i \in N$. Let $A \subset \mathcal{F}_N$ such that $(\{i\};P) = sup(A)$. If $(\{i\};P) \notin A$, then $A \subseteq \{(\emptyset;Q) : \; Q \in \Pi(N), Q_{-\{i\}} = P\}$. Then, $(N \setminus \{i\};\{i\})$ is also an upper bound of A, but $(\{i\};P)$ and $(N \setminus \{i\};\{i\})$ are not comparable. Then, the set of minimal upper bounds of A has more than one element and there is no supremum, reaching a contradiction. Then, $(\{i\};P)$ is join-irreducible. Finally, we check that every $(S;P)$ with $|S| \geq 2$ and $P \in \Pi(N \setminus S)$ is not a join-irreducible embedded coalition. Take $(S;P)$ with $|S| \geq 2$ and $P \in \Pi(N \setminus S)$. Let $A = \{(S \setminus \{i\}; P \cup \{i\}), (S \setminus \{j\}; P \cup \{j\})\}$, for some $i, j \in S$. Then, $(S;P) = sup(A)$ according to Corollary 3.3, but $(S;P) \notin A$. □

We obtain an isomorphism between chains of $(\mathcal{F}_N, \sqsubseteq)$ and $(\mathcal{B}(N), \subseteq)$.

Proposition 3.9. *Let N be a finite set. Let $(S;P)$, $(T;Q) \in \mathcal{F}_N$ such that $(S;P) \sqsubseteq (T;Q)$. Then, $[(S;P),(T;Q)]$ is isomorphic to $[S,T]_{\mathcal{B}(N)}$.*

Proof. Let $(S;P)$, $(T;Q) \in \mathcal{F}_N$ such that $(S;P) \sqsubseteq (T;Q)$. Notice that $S \subseteq T$ and $Q = P_{-T}$. We define the mapping ϕ from $[(S;P),(T;Q)]$ to $[S,T]_{\mathcal{B}(N)}$ as follows: $\phi(L;P_{-L}) = L$ for every $(L;P_{-L}) \in [(S;P),(T;Q)]$. It is clear that if $(L;P_{-L}),(L';P_{-L'}) \in [(S;P),(T;Q)]$ with $(L;P_{-L}) \sqsubseteq (L';P_{-L'})$ we have, in particular, $L \subseteq L'$ and then, $\phi(L;P_{-L}) \subseteq \phi(L';P_{-L'})$. Take the mapping ϕ^{-1} from $[S,T]_{\mathcal{B}(N)}$ to $[(S;P),(T;Q)]$ defined by $\phi^{-1}(L) = (L;P_{-L})$ for every $L \in [S,T]_{\mathcal{B}(N)}$. ϕ and ϕ^{-1} are inverse maps and if $L \subseteq L'$ we have $\phi^{-1}(L) \sqsubseteq \phi^{-1}(L')$. □

We can obtain Proposition 3.5 and Proposition 3.6 as a consequence of Proposition 3.9.

Proposition 3.10. *Let N be a finite set and $i \in N$. Let $P \in \Pi(N)$, $(\{i\}; P_{-i})$, $(T;Q) \in \mathcal{F}_N$ with $(\{i\}; P_{-i}) \sqsubseteq (T;Q)$.*

1. *The number of chains $[(\{i\}; P_{-\{i\}}), (T;Q)]$ is $(|T|-1)!$*
2. *The number of chains $[(\emptyset; P), (T;Q)]$ is $|T|!$*
3. *The total number of chains in $(\mathcal{F}_N, \sqsubseteq)$ is $|N|!B_n$, being B_n the Bell number of size $|N| = n$.*

Proof. Items 1 and 2 follow directly from the isomorphism presented in Proposition 3.9.

Let us prove Item 3. Taking into account the first item, the total number of chains in $(\mathcal{F}_N, \sqsubseteq)$ from $(\{i\}; P)$ to $(N; \emptyset)$ is $(|N|-1)!$, for every $i \in N$ and $P \in \Pi(N \setminus \{i\})$. Additionally, there are $|P|+1$ elements in level 1 linked to $(\{i\}; P)$. Thus, the total number of chains is

$$\begin{aligned} \textstyle\sum_{i\in N} \sum_{P\in\Pi(N\setminus\{i\})}(|N|-1)!(|P|+1) &= |N|! \textstyle\sum_{P\in\Pi(N\setminus\{i\}}(|P|+1) \\ &= |N|! \textstyle\sum_{r=1}^{n-1}(r+1)S_{n-1,r} = |N|!B_n, \end{aligned}$$

using the generalized recurrence expression provided in Spivey (2008) applied to $n-1$ and 1. □

The isomorphism presented in Proposition 3.9 also allows us to characterize the Möbius function of $(\mathcal{F}_N, \sqsubseteq)$. Next, we recall the definition of the Möbius function of a finite poset. Let $(\mathcal{A}, \leq)$ be a finite poset. The *Möbius function* of $(\mathcal{A}, \leq)$, μ, is given by

$$\mu(x,y) = \begin{cases} 1 & \text{if } x = y \\ -\sum_{x\leq z<y} \mu(x,z) = -\sum_{x<z\leq y} \mu(z,y) & \text{if } x < y \end{cases}$$

for every $x, y \in \mathcal{A}$ with $x \leq y$. The Möbius function of $(\mathcal{B}(N), \subseteq)$ is given by $\hat{\mu}(S,T) = (-1)^{|T|-|S|}$, for every $S \subseteq T \subseteq N$. Using Proposition 3.9 we obtain

$$\mu((S;P),(T;Q)) = \begin{cases} (-1)^{|T|-|S|} & \text{if } (S;P) \sqsubseteq (T;Q) \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

4 Partition function form games on $(\mathcal{F}_N, \sqsubseteq)$

Let N be a finite set. Let $v : \mathcal{F}_N \longrightarrow \mathbb{R}$ be a function such that $v(\emptyset; P) = 0$, for every $P \in \Pi(N)$. This type of function is usually called a *partition function form game* with player set N. The family of all partition function form games with player set N will be denoted by $\mathcal{G}_N$. Let $(S; P) \in EC^N$. The *unanimity game* associated to $(S; P)$ is defined as

$$u_{(S;P)}(T;Q) = \begin{cases} 1 & \text{if } (S;P) \sqsubseteq (T;Q) \\ 0 & \text{otherwise} \end{cases}$$

for every $(T; Q) \in \mathcal{F}_N$. The family of unanimity games $\mathcal{U} = \{u_{(S;P)} : \ (S; P) \in EC^N\}$ is a basis of the vector space of partition function form games. Then,

$$v = \sum_{(S;P) \in EC^N} \delta_{(S;P)} u_{(S;P)}$$

for every partition function form game v. Using the Möbius function characterized in Equation (3), we can obtain the coefficients of a game in the basis $\mathcal{U}$.

Proposition 4.1. *Let v be a partition function form game and $(S; P) \in EC^N$. Then,*

$$\delta_{(S;P)} = \sum_{(T;Q) \sqsubseteq (S;P)} (-1)^{|S|-|T|} v(T;Q)$$

Proof. Let $(S; P) \in EC^N$. We obtain the coefficient $\delta_{(S;P)}$ through the Möbius inversion formula as follows

$$\delta_{(S;P)} = \sum_{(T;Q) \sqsubseteq (S;P)} \mu((T;Q),(S;P)) v(T;Q) = \sum_{(T;Q) \sqsubseteq (S;P)} (-1)^{|S|-|T|} v(T;Q).$$

□

A *cooperative game in characteristic function form* (in short, classic game) is a partition function form game $v \in \mathcal{G}_N$ such that $v(S; P) = v(S; Q)$ for every $(S; P), (S; Q) \in EC^N$. That is the worth of a coalition does not depend on how the remaining players are organized. *Superadditivity* and *convexity* are two well-known and interesting properties for games in characteristic function form. Nevertheless, there are several alternative ways to generalize these properties for partition function form games in the literature. Here we present a new version of these two properties and analyze their relationship with similar concepts already presented. First, we extend the concept of an essential classic game to the framework of partition function form games.

Definition 4.1. *Let $v \in \mathcal{G}_N$. We say that v is* essential *if and only if*

$$\sum_{i \in N} v(\{i\}; P_{-\{i\}}) \leq v(N; \emptyset), \quad \textit{for every } P \in \Pi(N).$$

This concept is less demanding that the notion of efficiency for the grand coalition as it was presented in Hafalir (2007). A game in partition function form $v \in \mathcal{G}_N$ is efficient for the grand coalition if and only if

$$\sum_{S \in P} v(S; P \setminus S) \leq v(N; \emptyset), \quad \text{for every } P \in \Pi(N), \tag{4}$$

It is clear that if $v \in \mathcal{G}_N$ is efficient for the grand coalition, it is also essential. The reverse implication does not hold in general, as we illustrate next.

Example 4.1. *Let us take $N = \{1, 2, 3\}$. We consider $v \in \mathcal{G}_N$ as follows:*

$$\begin{aligned} &v(N; \emptyset) = 7,\ v(\{1\}; \{2, 3\}) = 1,\ v(\{1\}; \lfloor 2, 3 \rfloor) = 0, \\ &v(\{i\}; P_{-\{i\}}) = 2,\ \textit{for every } i \in N \setminus \{1\},\ P \in \Pi(N),\ \textit{and} \\ &v(\{j, k\}; \{i\}) = 6,\ \textit{for every } j, k \in N \setminus \{i\},\ j \neq k, i \in N. \end{aligned}$$

Then, this game is essential, but it is not efficient for the grand coalition. For instance,

$$v(\{1, 3\}; \{2\}) + v(\{2\}; \{1, 3\}) = 6 + 2 > 7 = v(N; \emptyset).$$

Definition 4.2. *Let $v \in \mathcal{G}_N$. We say that v is* superadditive *if and only if*

$$v(S \cup T; P_{-(S \cup T)}) \geq v(S; P) + v(T; Q)$$

for every $(S; P), (T; Q) \in \mathcal{F}_N$ such that $S \cap T = \emptyset$ and $P_{-T} = Q_{-S}$.

If $(S; P), (T; Q) \in \mathcal{F}_N$ such that $P_{-T} = Q_{-S}$ and $S \cap T = \emptyset$, then the set of maximal lower bounds is non-empty. Indeed, all of them are of the type $(\emptyset; M)$ for some $M \in \Pi(N)$ (see Proposition 3.1), and $sup(\{(S;P),(T;Q))\}) = (S \cup T; P_{-(S\cup T)})$ (see Corollary 3.3). Any superadditive game is also essential. Since every classic game is a game in partition function form and the definitions of essentiality and superadditivity in this framework extend the homonymous properties for classic games, we can assert that there are some essential partition function form games that are not superadditive. Additionally, the result is true if we take partition function form games wich are not classic games as we illustrate next.

Example 4.2. *Let us take $N = \{1,2,3\}$. We consider $v \in \mathcal{G}_N$ as follows:*

$$v(N;\emptyset) = 7,\ v(\{1\};\{2,3\}) = 1,\ v(\{1\};\lfloor 2,3 \rfloor) = 0,$$
$$v(\{i\};P_{-i}) = 2,\ \text{for every } i \in N \setminus \{1\},\ P \in \Pi(N),\ \text{and}$$
$$v(\{j,k\};\{i\}) = 0,\ \text{for every } j,k \in N \setminus \{i\},\ j \neq k, i \in N.$$

Then, the game is essential, but it is not superadditive as we can see by taking $(S;P) = (\{2\};\{1,3\})$ and $(T;Q) = (\{1\};\lfloor 2,3 \rfloor)$.

Maskin (2003) and Hafalir (2007) use the following definition of superadditivity.

Definition 4.3. *Let $v \in \mathcal{G}_N$. A game v is* superadditive *in Maskin's sense if and only if*

$$v(S \cup T;P) \geq v(S;P \cup \{T\}) + v(T;P \cup \{S\})$$

for every $S,T \subseteq N$ such that $S \cap T = \emptyset$ and $P \in \Pi(N \setminus (S \cup T))$.

Our notion of superadditiviy is different from Maskin's notion of superadditivity. We see this by revisiting Example 1 in Hafalir (2007).

Example 4.3. *(Hafalir, 2007) Let $N = \{1,2,3\}$ and $v \in \mathcal{G}_N$ defined by*

$$v(N;\emptyset) = 11,$$
$$v(\{i\};\lfloor j,k \rfloor) = 4, \text{for every } j,k \in N \setminus \{i\},\ j \neq k, i \in N,$$
$$v(\{j,k\};\{i\}) = 9,\ \text{for every } j,k \in N \setminus \{i\},\ j \neq k, i \in N,\ \text{and}$$
$$v(\{i\};\{j,k\}) = 1,\ \text{for every } j,k \in N \setminus \{i\},\ j \neq k, i \in N.$$

This game is superadditive in Maskin's sense, but it is not superadditive according to Definition 4.2. This follows directly from the fact that this game is not essential using Definition 4.1.

Now we present our definition of convexity for partition function form games.

Definition 4.4. *Let $v \in \mathcal{G}_N$. We say that v is* convex *if and only if*

$$v(S \cup T; P_{-T}) + v(S \cap T; M) \geq v(S; P) + v(T; Q) \tag{5}$$

for every $(S; P), (T; Q) \in \mathcal{F}_N$ with $P_{-T} = Q_{-S}$ and $(S \cap T; M)$ a maximal lower bound of $\{(S; P),\ (T; Q)\}$.

It is clear that if $v \in \mathcal{G}_N$ is convex, then it is also superadditive.

Remark 4.1. *Hafalir (2007) provides the following definition of convexity. Let $v \in \mathcal{G}_N$. A game v is* convex *if and only if*

$$v(S \cup T; P) + v(S \cap T; P \cup \{S \setminus T, T \setminus S\}) \geq v(S; P \cup \{T \setminus S\}) + v(T; P \cup \{S \setminus T\})$$

for every $S, T \subseteq N$ and $P \in \Pi(N \setminus (S \cup T))$.

Notice that for every $S, T \subseteq N$ and $P \in \Pi(N \setminus (S \cup T))$, we have $(P \cup \{T \setminus S\})_{-T} = P = (P \cup \{S \setminus T\})_{-S}$. Moreover,

$$(P \cup \{T \setminus S\} \cup \lfloor S \setminus T \rfloor) \bigvee (P \cup \{S \setminus T\} \cup \lfloor T \setminus S \rfloor) = P \cup \{S \setminus T, T \setminus S\}.$$

Then, the set of maximal lower bounds for $(S; P \cup \{T \setminus S\})$ and $(T; P \cup \{S \setminus T\})$, using Proposition 3.1, is given by

$$\{(S \cap T; P \cup \{S \setminus T, T \setminus S\}), (S \cap T; P \cup \{S \Delta T\})\},$$

being $S \Delta T = (S \cup T) \setminus (S \cap T)$. This implies that if v satisfies our notion of convexity, then it also satisfies Hafalir's condition. In the example below we show that both concepts are not the same.

Example 4.4. *Let* $N = \{1,2,3\}$ *and* $v \in \mathcal{G}_N$ *defined as follows:*

$$v(\{i\};\{j,k\}) = 1,\ v(\{i\};\lfloor j,k\rfloor) = 2,\ v(\{i,j\};\{k\}) = 4,\ v(N;\emptyset) = 6.$$

Clearly, this game is superadditive according to Definition 4.2 and convex in Hafalir's sense. Nevertheless, it is not convex according to Definition 4.4. For instance, take $(S;P) = (\{1,3\};\{2\})$ *and* $(T;Q) = (\{1,2\};\{3\})$*. The embedded coalition* $(\{1\};\{2,3\})$ *is a maximal lower bound of* $\{(S;P),(T;Q)\}$*, but*

$$v(N;\emptyset) + v(\{1\};\{2,3\}) = 6 + 1 < 4 + 4 = v(\{1,3\};\{2\}) + v(\{1,2\};\{3\}).$$

Thus, this game is not convex according to Definition 4.4.

If we consider $w \in \mathcal{G}_N$ *defined as* $w(S;P) = v(S;P)$ *if* $(S;P) \neq (N;\emptyset)$ *and* $w(N;\emptyset) = 10$*, we obtain a convex game according to Definition 4.4.*

Next, we present a characterization of convexity that is parallel to a well known result for games in characteristic function form.

Proposition 4.2. *Let* $v \in \mathcal{G}_N$*. The game* v *is* convex *if and only if for every* $i \in N$ *and* $(S;P) \sqsubseteq (T;Q) \sqsubseteq (N \setminus \{i\};\{i\})$ *we have*

$$v(T \cup \{i\};Q_{-\{i\}}) - v(T;Q) \geq v(S \cup \{i\};P_{-\{i\}}) - v(S;P). \tag{6}$$

Proof. Let $v \in \mathcal{G}_N$. Let us assume that v is convex. Take $i \in N$ and $(S;P) \sqsubseteq (T;Q) \sqsubseteq (N \setminus \{i\};\{i\})$. The elements $(S \cup \{i\};P_{-\{i\}})$ and $(T;Q)$ satisfy the conditions in Definition 4.4 since $(S;P) \sqsubseteq (T;Q)$ implies $S \subseteq T$ and $P_{-T} = Q$. Hence, $P_{-(T\cup\{i\})} = Q_{-\{i\}}$. Using Corollary 3.3, the supremum of $(S \cup \{i\};P_{-\{i\}})$ and $(T;Q)$ exists and it is given by $(T \cup \{i\};Q_{-\{i\}}) = (T \cup \{i\};P_{-(T\cup\{i\})})$. Notice that $(S;P)$ is a maximal lower bound of $\{(S \cup \{i\};P_{-\{i\}}),(T;Q)\}$. Applying Inequality (5) to $(S \cup \{i\};P_{-\{i\}}),(T;Q)$ we get

$$v(T \cup \{i\};Q_{-\{i\}}) + v(S;P) \geq v(S \cup \{i\};P_{-\{i\}}) + v(T;Q),$$

and Inequality (6) holds.

Now let us assume that v satisfies Inequality (6) for every $i \in N$ and $(S;P) \sqsubseteq (T;Q) \sqsubseteq (N \setminus \{i\};\{i\})$. Let us take $(S;P),(T;Q) \in \mathcal{F}_N$ such that $P_{-T} = Q_{-S}$ and

$(S\cap T;M')$ be a maximal lower bound of $(S;P)$ and $(T;Q)$. We proceed by induction on $|S\setminus T|$. If $|S\setminus T|=0$ then, $S\subseteq T$. Since $P_{-T}=Q$, we have $(S;P)\sqsubseteq(T;Q)$. Then, $(S\cap T;M')=(S;P)$, $(S\cup T;P_{-T})=(T;Q)$, and Inequality (5) holds. Let us assume that $|S\setminus T|=1$ and $S\setminus T=\{i\}$. Then, $S\cap T=S\setminus\{i\}$ and a maximal lower bound of $(S;P)$ and $(T;Q)$ belongs to the set

$$\{(S\setminus\{i\};M)\}\cup\{(S\setminus\{i\};M_{-(R\cup R')}\cup\{R\cup R'\})\ :R\subseteq T,R'\subseteq S,R,R'\in M\}$$

with $M=(P\cup\{i\})\bigvee(Q\cup\lfloor T\setminus S\rfloor)$. Let $(S\setminus\{i\};M')$ be a maximal lower bound of $(S;P)$ and $(T;Q)$. Then, $(S\setminus\{i\};M')\sqsubseteq(S;P)$ and $P=M'_{-S}=M'_{-\{i\}}$. Besides, $(S\setminus\{i\};M')\sqsubseteq(T;Q)\sqsubseteq(N\setminus\{i\};\{i\})$ which implies $M'_{-T}=Q$. Since $P_{-T}=Q_{-S}$ and the choice of i, we get $P_{-(T\cup\{i\})}=P_{-T}=Q_{-S}=Q_{-\{i\}}$. Applying Inequality (6) to $(S\setminus\{i\};M')$ and $(T;Q)$, we obtain

$$\begin{aligned}&v(T\cup i;P_{-(T\cup\{i\})})-v(T;Q)\geq v(S;P)-v(S\setminus\{i\};M'),\ \text{or}\\&v(S\cup T;P_{-T})+v(S\cap T;M')\geq v(S;P)+v(T;Q)\end{aligned}$$

and Inequality (5) holds.

Let us assume that the result holds for every $(S;P),(T;Q)\in\mathcal{F}_N$ with $|S\setminus T|\leq k$, $P_{-T}=Q_{-S}$, and $(S\cap T;M')$ a maximal lower bound of $(S;P)$ and $(T;Q)$. Now, take $(S;P),(T;Q)\in\mathcal{F}_N$ such that $|S\setminus T|=k+1$, $P_{-T}=Q_{-S}$, and $(S\cap T;M')$ a maximal lower bound of $(S;P)$ and $(T;Q)$. Let us take $i\in S\setminus T$ and $(T\cup(S\setminus\{i\});Q_{-(S\setminus\{i\})})$. We can assume that $(S\cap T;M')$ is not covered by $(S;P)$. Otherwise, $|S\setminus T|=1$ and we have just proved the result for this situation. Take $(S\setminus\{i\};\hat{M})$ such that $(S\cap T;M')\sqsubseteq(S\setminus\{i\};\hat{M})\sqsubseteq(S;P)$. Since $i\in S\setminus T$ and $(S\cap T;M')$ is a maximal lower bound of $\{(S;P),(T;Q)\}$, $(S\setminus\{i\};\hat{M})$ does not precede $(T;Q)$. Nevertheless, $(S\cap T;M')\sqsubseteq(T\cup(S\setminus\{i\});Q_{-(S\setminus\{i\})})$ because $S\cap T\subset T\cup(S\setminus\{i\})$, $M'_{-T}=Q$, and $M'_{-(T\cup(S\setminus\{i\}))}=Q_{-(S\setminus\{i\})}$. Besides, $(S\setminus\{i\};\hat{M})\sqsubseteq(T\cup(S\setminus\{i\});Q_{-(S\setminus\{i\})})$ because $S\setminus\{i\}\subseteq T\cup(S\setminus\{i\})$ and $\hat{M}_{-T}=M'_{-(T\cup(S\setminus\{i\}))}=Q_{-(S\setminus\{i\})}$ as a consequence of $M'_{-(S\setminus\{i\})}=\hat{M}$ and $M'_{-(T\cup(S\setminus\{i\}))}=Q_{-(S\setminus\{i\})}$. Applying Inequality (6) to $(S\setminus\{i\};\hat{M})$ and $(T\cup(S\setminus\{i\});Q_{-(S\setminus\{i\})})$, we obtain

$$v(T\cup S;Q_{-S})-v(T\cup(S\setminus\{i\});Q_{-(S\setminus\{i\})})\geq v(S;P)-v(S\setminus\{i\};\hat{M}).\qquad(7)$$

Notice that $|S \setminus (T \cup \{i\})| = k$. We apply the induction hypothesis to $(S \setminus \{i\}; \hat{M})$, $(T; Q)$ and $(S \cap T; M')$ because $\hat{M}_{-T} = Q_{-(S \setminus \{i\})}$ and $(S \cap T; M')$ is also a maximal lower bound of $\{(S \setminus \{i\}; \hat{M}),\ (T; Q)\}$. Thus,

$$v(T \cup (S \setminus \{i\}); Q_{-(S \setminus \{i\})}) + v(S \cap T; M') \geq v(T; Q) + v(S \setminus \{i\}; \hat{M}). \tag{8}$$

Finally, adding up Inequalities (7)-(8), we obtain Inequality (5), concluding the proof. □

As a consequence of Proposition 4.2 we have the following result.

Proposition 4.3. *Let $v \in \mathcal{G}_N$. The game v is convex if and only if, for every $P \in \Pi(N)$, v^P is also a classic convex game with $v^P(S) = v(S; P_{-S})$, for every $S \subseteq N$.*

Proof. Let $v \in \mathcal{G}_N$. First, let us assume that v is convex. Let $P \in \Pi(N)$ and consider the classic game v^P defined as follows: $v^P(S) = v(S; P_{-S})$, for every $S \subseteq N$. Let $i \in N$ and $S \subseteq T \subseteq N \setminus \{i\}$. Take $(S; P_{-S}),\ (T; P_{-T}) \in \mathcal{F}_N$. It is clear that $(S; P_{-S}) \sqsubseteq (T; P_{-T}) \sqsubseteq (N \setminus \{i\}; \{i\})$. Since v is convex, using Inequality (6), we have

$$v(T \cup \{i\}; P_{-(T \cup \{i\})}) - v(T; P_{-T}) \geq v(S \cup \{i\}; P_{-(S \cup \{i\})}) - v(S; P_{-S}). \tag{9}$$

Rewriting both sides in Inequality (9), we have

$$v^P(T \cup \{i\}) - v^P(T) \geq v^P(S \cup \{i\}) - v^P(S),$$

concluding that v^P is a classic convex game.

Second, let us assume that v^P is a classic convex game for every $P \in \Pi(N)$. We check Inequality (6). Let $i \in N$, $(S; P),\ (T; Q) \in \mathcal{F}_N$ such that $(S; P) \sqsubseteq (T; Q) \sqsubseteq (N \setminus \{i\}; \{i\})$. Let $M \in \Pi(N)$ such that $M_{-S} = P$. Then, $M_{-T} = P_{-T} = Q$ because $(S; P) \sqsubseteq (T; Q)$ and the choice of M. Take the classic game v^M. Since v^M is a convex classic game, we get

$$v^M(T \cup \{i\}) - v^M(T) \geq v^M(S \cup \{i\}) - v^M(S).$$

Taking into account the definition of v^M, and the fact that $M_{-T} = P_{-T} = Q$ and $M_{-S} = P$, we have

$$v(T \cup \{i\}; Q_{-\{i\}}) - v(T; Q) \geq v(S \cup \{i\}; P_{-\{i\}}) - v(S; P).$$

We have proved Inequality (6) and conclude that v is convex. $\square$

We point out that convexity in Hafalir's sense does not satisfy an analogous result to the one in Proposition 4.3. We take Example 2 in Hafalir (2007) to illustrate this.

Example 4.5. *(Hafalir, 2007) Let $N = \{1, 2, 3\}$ and $v \in \mathcal{G}_N$ defined by*

$$\begin{aligned}
&v(N; \emptyset) = 16,\\
&v(\{i\}; \lfloor j, k \rfloor) = 4, \text{for every } j, k \in N \setminus \{i\},\ j \neq k, i \in N,\\
&v(\{j, k\}; \{i\}) = 9,\ \text{for every } j, k \in N \setminus \{i\},\ j \neq k, i \in N,\ \text{and}\\
&v(\{i\}; \{j, k\}) = 6,\ \text{for every } j, k \in N \setminus \{i\},\ j \neq k, i \in N.
\end{aligned}$$

This game is convex in Hafalir's sense, but it is not convex according to Definition 4.4. This can be checked by using the embedded coalitions $(S; P) = (1; \{2, 3\})$ and $(T; Q) = (\{2\}; \lfloor 1, 3 \rfloor)$. Let us take $P = \{N\}$. Define the classic game v^N as $v^N(\{i\}) = 6$, $v^N(\{i, j\}) = 9$, and $v^N(N) = 16$. This game has an empty core and, then, it is not convex. Thus, we see that an analogous result to the one established in Proposition 4.3 does not hold using Hafalir's notion of convexity.

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