Defect Dynamics in Viscous Fingering

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We study the dynamics of Saffman-Taylor fingering in terms of topological defects of the flow field. The defects are created and/or annihilated at the interface. The route towards the single-finger steady state is characterized by a detailed mechanism for defect annihilation. For small viscosity contrast this mechanism is impeded, and creation of new defects leads the system away from the single-finger solution. Strong evidence for a drastic reduction of the basin of attraction of the Saffman-Taylor finger is presented.

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The Saffman-Taylor problem [1] has played a central role in the study of interfacial pattern formation in nonequilibrium dissipative systems. Because of its relative simplicity, it has become a prototype system for the study of diffusion-controlled growth, with relevance for a wide variety of systems of current interest such as dendritic growth, directional solidification, chemical electrodeposition, and flame propagation [2]. More specifically, it deals with the dynamics of the interface separating two immiscible viscous fluids in a rectangular Hele-Shaw cell, and, in particular, with the emergence of a steady single-finger propagating solution.

Previous studies of finger competition have analyzed the global instability of parallel arrays of fingers in the limit of high viscosity contrast, in terms of a linear instability of the envelope of the finger front [3]. Consistent with experiments [4], the current qualitative understanding of the competition regime is that this global instability leads to coarsening, as longer fingers grow at the expense of smaller ones, giving rise to a cascade into large length scales ending up with a single finger. Here we will show that this scenario needs to be modified depending, in general, on the viscosity contrast.

The idea of topological defects has proved fruitful in different areas of nonequilibrium physics. Particularly in the context of pattern formation, it has been useful in the context of amplitude equations [5]. Our aim here is to introduce a defect-dynamics approach to the study of viscous fingering and related problems, and to illustrate the usefulness of this viewpoint. The Letter is organized in three parts. First, we introduce the basics of the topological approach, and the general rules governing the defect dynamics. Second, we obtain the explicit sequence of topology changes induced by the defect dynamics and characterizing the competition mechanism. Finally, we investigate the lack of competition in the low-viscosity-contrast limit of the problem.

In a Hele-Shaw cell with a gap b between plates, the two-dimensional velocity obeys Darcy's law, $\mathbf{v}=\nabla\phi$, where the velocity potential in phases $i=1,2$ is related to the pressure p as $\phi_i=-(p+\rho_i gy\cos\alpha)b^2/12\mu_i$, where μ_i are the viscosities, ρ_i the densities, and α the angle of the propagation direction y to the vertical. Incompressibility implies then $\nabla^2\phi=0$ in the bulk. The problem is then specified by two boundary conditions at the interface: the continuity condition $v_n=\mathbf{n}\cdot\nabla\phi_1=\mathbf{n}\cdot\nabla\phi_2$ and the pressure drop, usually taken as $p_1-p_2=\sigma\kappa$, where σ is surface tension and κ is curvature. The two dimensionless parameters of the problem [6] are the viscosity contrast $c=(\mu_1-\mu_2)/(\mu_1+\mu_2)$ and a dimensionless surface tension $d_0=12b^2\sigma/UW^2(\mu_1+\mu_2)$, where W is the width in the x direction. $U=V_G+cV_\infty$ is assumed positive, with $V_G=12b^2g\cos\alpha(\rho_1-\rho_2)/(\mu_1+\mu_2)$, and V_∞ the velocity at $y\rightarrow\pm\infty$. Without loss of generality we will take $V_\infty=0$ (so hereafter we will formally be considering the frame-independent velocity field $\mathbf{v}-\mathbf{V}_\infty$, with no sources or sinks at infinity).

The study of the Saffman-Taylor problem has recently concentrated on the role of surface tension in the selection of the steady-state single-finger solution [7]. Although the microscopic solvability scenario [7] for selection has not been explicitly applied for arbitrary viscosity contrast, numerical evidence indicates that the sensitivity of the steady state to c is very weak [6]. However, both simulations [6] and experiments [4,8] have shown that in the transient nonlinear regime, far from the two well-understood limits of the problem (i.e., the linear instability of the planar interface and the linear relaxation in a neighborhood of the single-finger solution), the viscosity contrast can play a major role.

To illustrate the situation, we show in Fig. 1 two representative simulations which capture the essence of two different scenarios of finger competition for the limiting cases $c=1$ and 0. More details will be presented elsewhere [9]. Periodic boundary conditions are assumed in the x direction, so that these configurations simultaneously account for the alternating mode of the global instability. For $c=1$ [Fig. 1(a)], one sees that the small finger is already out of the competition, since it is receding. In contrast, having started with the same intitial condition for $c=0$ [Fig. 1(b)], the small finger is not only advancing but somehow growing at the expense of the long one, since the latter, though faster, is narrowing as the smaller finger widens.

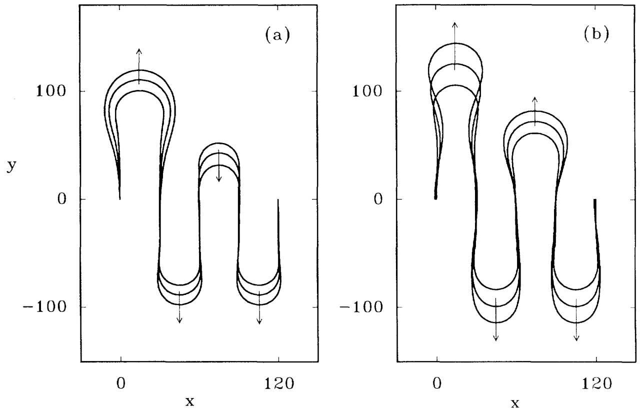

FIG. 1. Two-finger competition for (a) $c=1$ and (b) $c=0$ with $d_0=\frac{1}{360}$. Periodicity with $W=120$ is assumed. Corresponding times in dimensionless units are (a) $t=2,6,10$, and (b) $t=5,15,25$. Arrows indicate direction of tip velocities.

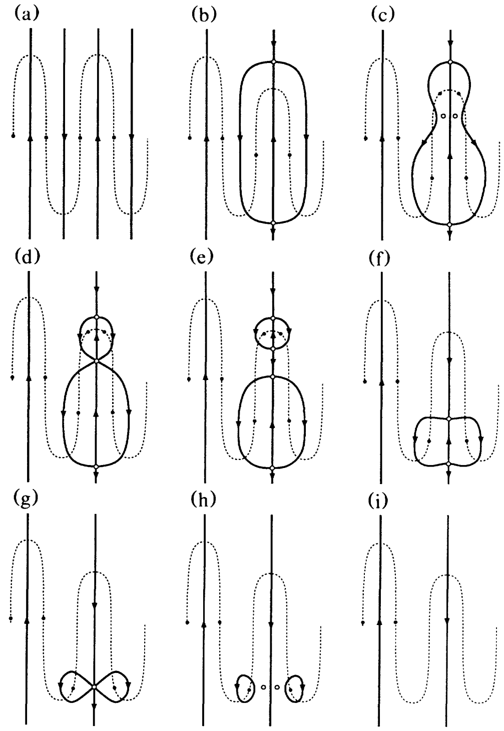

FIG. 2. Schematic flow diagrams during finger competition for $c=1$. Dotted line, interface; solid lines, streamlines $\psi=0$; black dots, extrema of ψ; open circles, defects. [With parameters of Fig. 1(a), for (b)-(i) $\Delta t\approx 3$.]

The general idea of our approach is to focus on global properties of the flow structure in the bulk. The competition can then be pictured by realizing that neighboring fingers generate flows with opposite sense of circulation. (Streamlines, $\psi=$const, are closed and cross the interface.) Periodic arrays of fingers define stripes with alternate circulation [see Fig. 2(a)]. The single-finger steady state has only two such stripes. Defects in the flow structure, occurring as vertices of streamlines separating domains of circulation, must therefore play a central role in the approach to the steady state. The separatrices of the flow defined by the streamlines connected to defects provide a useful diagrammatic characterization of the flow structure (see examples in Fig. 2).

The central object in the present discussion will be the stream function ψ, defined as the harmonic conjugate of ϕ so that the complex potential $\Phi(z)=\phi(x,y)+i\psi(x,y)$ is analytic in $z=x+iy$ in the bulk. Unlike ϕ, the stream function ψ is continuous at the interface, though nondifferentiable. Our approach relies on two observations. First, we exploit the continuity properties of the flow field under the compactification of the infinite strip with periodic boundary conditions in the x direction [10] into a sphere [11]. This leads to the existence of topological invariants of the flow field. More specifically, in the reference frame with no sinks or sources at infinity, ψ remains a continuous function after this compactification, with M local maxima and m local minima. Under these conditions, the quantity $M+m-S$, where S is the total weight of saddle points (see below) of ψ, is a topological invariant which equals the Euler characteristic χ of the surface, in our case $M+m-S=\chi=2$ [12]. The second crucial fact is that ψ, being harmonic in the bulk of both phases, can have local extrema only on the interface (where all the vorticity is confined). The saddle points, however, can occur in the bulk of both phases as zeros of the complex velocity field $\Omega(z)=d\Phi/dz=v_x-iv_y$. The saddle points are precisely the point defects anticipated above as vertices of streamlines. The order of the zeros of Ω defines point defects of corresponding order. The local structure of Φ around a defect of order of n at $z_i(t)$ in the bulk is thus given by

$$\Phi_i(z,t)=A_i(t)+B_i(t)[z-z_i(t)]^{n+1}+\cdots \qquad (1)$$

defining a generalized saddle-point flow with $2(n+1)$ hyperbolic sectors of angle $\pi/(n+1)$, and which contributes an amount n to S. [See, for example, a configuration with $S=4$ in Fig. 2(d).] These point defects correspond to phase singularities of the complex velocity field Ω, since the phase of Ω changes by an amount $\pm 2n\pi$ along any path enclosing the defect [13]. Physically, defects in the bulk correspond to (nonstationary) stagnation points of the flow, i.e., $\mathbf{v}(x_i(t),y_i(t))=\mathbf{0}$ [5].

The general idea is to characterize the dynamics in terms of the motion of these point defects. We can attribute a "charge" (the winding number [5,14]) $Q=+1$ to each extremum and $Q=-n$ to each defect of order n. The conservation of the total Q by the dynamics and the localization of the extrema of ψ at the interface assure us that S cannot decrease as long as the defects stay away from the interface; however, single defects (i.e., $n=1$)

can be annihilated by merging with extrema at the interface. Extremum–single-defect pairs can also be created at the interface, with only the defect being able to move into the bulk. On the other hand, for a single-finger flow, we have $M=m=1$ and $S=0$, whereas multifinger configurations will have in general $S>0$. Therefore, the positive integer S, which gives the weight of defects, can be seen as a topological characterization of the "complexity" of the flow structure, and as a measure of the "distance" from the single-finger flow structure. The time dependence of the integer S gives a global characterization of the dynamics. A systematic decrease of S signals the progression toward the steady state.

As a direct application of these ideas, we have first investigated the competition of two fingers in the limit of high viscosity contrast, $c=1$. In Figs. 2(b)–2(i) we have, for clarity, *schematically* depicted the sequence of flow diagrams describing the topological evolution during the dynamical elimination of a smaller finger by a bigger one. (Note only the topological properties of the process are displayed in Fig. 2 and not the precise location of defects, streamlines, and interface.) The diagrams can be inferred, for instance, from global features of the stream function $\psi(s)$ as a function of arclength s along the interface [such as number of zeros of $\psi(s)$ and $\psi'(s)=v_n(s)$ [9]], which we have obtained using standard boundary integral methods [2,15]. (In the actual sequence, for initial fingers differing by about 10% in height, the interface shape does not change significantly, so we have indicated the same qualitative shape for a better visualization.)

The general result is that the dynamical elimination of a finger is characterized by a diagram of two connected defects, one in each phase, and located on the central separatrix of the finger to be eliminated [Fig. 2(b)]. These defects have to be annihilated, at the end of the process, merging with the maximum and minimum of ψ located at both sides of the finger [Figs. 2(h) and 2(i)]. To do so, the defects have to meet first in the less viscous (lower) phase [Fig. 2(g)], in order to disconnect from each other. Figures 2(c)–2(e) describe the intermediate steps defining the effective crossing of a defect through a finger tip. This happens in a delocalized way in both time and space, via the momentary creation [Figs. 2(b) and 2(c)] and annihilation [Figs. 2(e) and 2(f)] of two auxiliary extremum-defect pairs, in the neighborhood of the finger tip. The final diagram Fig. 2(i) corresponds to the trivial topology of a single-finger flow, with $S=0$. At this point, the smaller finger still exists from a morphological point of view, but the signature of it has been completely eliminated from the flow structure. On the other hand, the global instability [3] of $N+1$ equal fingers can now be reinterpreted as the unstable nature of an N defect at infinity. A perturbation in the periodic array breaks the defect at infinity, bringing single defects to finite distances [see Figs. 2(a) and 2(b)].

As a second application we now address the question of the low-contrast behavior. The flow diagram corresponding to the configurations of Fig. 1(b), for $c=0$, is also Fig. 2(b). Computing numerically [9] the corresponding positions of defects, we find that the defect velocities now have the signs reversed with respect to the case $c=1$, i.e., the connected defects are moving apart from each other. More interestingly, the increase in the number of zeros of $\psi'(s)=v_n(s)$ [9] produced by the narrowing of the longer finger indicates the creation of extremum-defect pairs. Therefore, not only is the mechanism of Fig. 2 not working, but the quantity S is *increasing* with time. Further evidence will be discussed elsewhere [9].

The observation that competition results in creation of new defects and the absence of a mechanism for the dynamical elimination of fingers suggests that, in the case $c=0$, the basin of attraction of the Saffman-Taylor steady state is drastically reduced essentially to single-finger initial conditions (i.e., $S=0$ or, more generally, configurations for which a decrease of S is not due to competition, but is due, for example, to the decay of short-wavelength perturbations because of surface tension). Notice that, since the single-finger stationary solution appears to be well behaved and linearly stable for arbitrary c including $c=0$ [6,9], the present analysis does not signal any failure of the current theory of steady-state selection [7], but suggests a richer nonlinear structure of the problem, in particular, opening the possibility of the existence of other attractors.

Finally, the inhibition of the competition mechanism of Fig. 2 for $c=0$ may be interpreted as a consequence of the symmetry of the dynamics under up-down reflection ($y\to -y$), given the intrinsic asymmetry of the competition mechanism under $c\to -c$ (defects hit finger tips from the more viscous phase [9]). (Notice that this reflection symmetry is by itself not incompatible with coarsening [15].) For $c\neq 0$ the up-down symmetry is broken, so, as far as the competition mechanism of Fig. 2 is concerned, one might expect that any $c\neq 0$ would eventually cross over to the high-contrast behavior. To our knowledge, there is no direct experimental or numerical evidence of such a crossover. This fact sets rough bounds on possible crossover times which lie beyond reach for any practical purposes for a wide range of c. An alternative scenario that is consistent with numerical [6] and experimental [4,8] evidence arises from the observation that, for a wide range of c, the pinchinglike mechanism for the creation of defects seems to be clearly dominant in the accessible time regimes [16]. This suggests that the basin of attraction of the single-finger steady state could be gradually eroded as one proceeds away from $c=1$, and its "measure" would be close to the full phase space only in a small neighborhood of $c=1$ [17]. Further investigation is required to explore this open question.

In summary, we have shown that a reduced characterization of nonlocal interface dynamics in terms of topological defects can capture essential dynamical information in a few degrees of freedom. Furthermore, this characterization provides a useful tool in addressing new nonlinear dynamical aspects which may appear in a variety of similar problems. Although the exact dynamics of the defects involves, in principle, the solution of the whole nonlocal problem, new approximation schemes can now be explored, based for instance on effective defect interactions or asymptotic local equations of motion for defects, in the spirit of Refs. [18]. To this point, the flow structure in the bulk has not been the focus of experiments, but defects can be found (and S obtained) by direct measure or visualization of the velocity field or indirectly, for example, from measurements of $v_n(s)$. The approach is also potentially generalizable to a variety of related problems, such as the study of fingering in radial Hele-Shaw flow.

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