The aim of the present paper is to present novel lattice results for the mass of the $K^*$ meson, as well as its vector and tensor couplings ($f_V$ and $f_T$ respectively), defined in Euclidean space-time as follows:

$$\langle 0|V_j|K^*;\lambda\rangle = -if_V\epsilon^j_\lambda m_{K^*},$$

$$\langle 0|T_{0j}|K^*;\lambda\rangle = -if_T\epsilon^j_\lambda m_{K^*}. $$

In the above expressions, $V_j = \bar{s}\gamma_j d$ is the vector current (spatial components only; $j = 1,2,3$), $T_{0j} = i\bar{s}\sigma_{0j}d$ is the tensor bilinear operator (temporal component), and $\epsilon^j_\lambda$ denotes the polarization vector.

Our results are based on simulations of the ETM Collaboration (ETMC) [1], with $N_f = 2$ dynamical flavors (sea quarks) and “lightish” pseudoscalar meson masses in the range $280 \text{ MeV} < m_{PS} < 550 \text{ MeV}$. With three lattice spacings ($a = 0.065 \text{ fm}, 0.085 \text{ fm}, \text{ and } 0.1 \text{ fm}$) we are able to extrapolate our results to the continuum limit. Our simulations are performed with the tree-level Symanzik improved gauge action. For the quark fields we adopt a somewhat different regularization for sea and valence quarks. The sea quark lattice action is the so-called maximally twisted standard tmQCD (referred to as “standard twisted mass regularization” [2] and matrix elements) [7]. Thus unitarity violation, which plagues any partially quenched theory at finite lattice spacing, is in $O(a^2)$ effect.

The sign of $\mu_s$ may be that of $\mu_d$ or its opposite. We conventionally refer to the setup in which $\text{sign}(\mu_d) = -\text{sign}(\mu_s)$ as the “standard twisted mass regularization” (denoted by tm) and the setup with $\text{sign}(\mu_d) = \text{sign}(\mu_s)$ as the “Osterwalder-Seiler regularization” (OS). Quenched pseudoscalar masses and decay constants in tm and OS setups have already been studied [8,9]. Having at our disposal two lattice regularizations in the valence quark sector, we study the impact of the valence quark as well as the ratio of the tensor to vector coupling. For the valence quarks we use the so-called Osterwalder-Seiler variant of tmQCD, which consists in maximally twisted flavors which, unlike the standard tmQCD case, doublets: $\bar{s}/C_{31}/C_{21}X = \{\bar{s}/C_{31}/C_{21}u/\bar{s}/C_{31}/C_{21}d\}$ doublets in the so-called “twisted basis,” is given by

$$L_{OS} = \sum_{f=d,s}\bar{q}_f[D_W + i\mu_f\gamma_5\tau^1]q_f,$$

where $\tau^3$ is the isospin Pauli matrix and $D_W$ denotes the critical Wilson-Dirac operator. By “critical” we mean that, besides the standard kinetic and Wilson terms, the operator also includes a standard, nontwisted mass term, tuned at the critical value of the quark mass ($m_{cr}$ in the language of the hopping parameter), so as to ensure maximal twist. With only two light dynamical flavors, strangeness clearly enters the game in a partially quenched context. For the valence quarks we use the so-called Osterwalder-Seiler variant of tmQCD, which consists in maximally twisted flavors which, unlike the standard tmQCD case, doublets: $\bar{s}/C_{31}/C_{21}X = \{\bar{s}/C_{31}/C_{21}u/\bar{s}/C_{31}/C_{21}d\}$ doublets in the so-called “twisted basis,” is given by

$$L_{OS} = \sum_{f=d,s}\bar{q}_f[D_W + i\mu_f\gamma_5\tau^1]q_f,$$

with $\text{sign}(\mu_f) = \pm 1$ (see below for details). This action, introduced in Ref. [4] and implicitly used in [5], has been studied in detail in Ref. [6]. For the case in hand (i.e. $K^*$-related quantities) we only need down- and strange-quark flavors in the valence sector. Note that the choice of maximally twisted sea and valence quarks implies $O(a)$ improvement of the physical quantities (i.e. the so-called automatic improvement of masses, correlation functions, and matrix elements) [7]. Thus unitarity violation, which plagues any partially quenched theory at finite lattice spacing, is an $O(a^2)$ effect.

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sector provides a cross-check of the reliability of our results in the continuum limit.

The continuum operators of interest are expressed, in terms of their lattice counterparts, as follows:

\begin{equation}
V_{\mu}^{\text{cont}} = Z_A A_{\mu}^{a\mu} + O(a^2) = Z_V V_{\mu}^{a\mu} + O(a^2),
\end{equation}

\begin{equation}
T_{\mu\nu}^{\text{cont}} = Z_T T_{\mu\nu}^{a\mu} + O(a^2) = Z_T T_{\mu\nu}^{a\mu} + O(a^2),
\end{equation}

where \( \tilde{T}_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} T_{\rho\sigma} \). The vector and axial currents are normalized by the scale independent factors \( Z_V \) and \( Z_A \), while \( Z_T = Z_T(\mu) \) runs with a renormalization scale \( \mu \) (i.e. it is defined in a given renormalization scheme).

The vector boson mass, \( m_V \), as well as \( f_V \) and \( f_T \), are obtained from two-point correlation functions at zero spatial momenta and large time separations. These are defined in the continuum (Euclidean space-time) as

\[
C_V(x^0) = \frac{1}{3} \int d^3 \chi \langle V(x) V^\dagger(0) \rangle_{\text{cont}}
- \frac{f_V^2 m_V}{2} \exp[-m_V T/2] \cosh[m_V(T/2 - x^0)],
\]

\[
C_T(x^0) = \frac{1}{3} \int d^3 \chi \langle T_{0j}(x) T_{0j}^\dagger(0) \rangle_{\text{cont}}
- \frac{f_T^2 m_V}{2} \exp[-m_V T/2] \cosh[m_V(T/2 - x^0)],
\]

The asymptotic expressions of the above equations correspond to the large time limit of the correlation functions (symmetrized in time), with periodic boundary conditions for the gauge fields and (anti)periodic ones for the fermion fields in the (time)space directions (i.e. \( 0 \ll x^0 \ll T/2 \)). These are actually the boundary conditions of our lattice simulations. The lattice correlation functions are related to the continuum ones as suggested by Eqs. (1.5) and (1.6):

\[
C_V^{\text{cont}}(x^0) = Z_V^2 C_V^{a\mu}(x^0) + O(a^2) = Z_V^2 C_V^{a\mu}(x^0) + O(a^2),
\]

\[
C_T^{\text{cont}}(x^0) = Z_T^2 C_T^{a\mu}(x^0) + O(a^2) = Z_T^2 C_T^{a\mu}(x^0) + O(a^2).
\]

The meaning of the notation \( C_V^{a\mu}, C_T^{a\mu} \), etc. should be transparent to the reader. The ratio \( f_T/f_V \) is computed from the square root of the ratio of correlation functions \( C_T^{\text{cont}}/C_V^{\text{cont}} \), in which systematic effects cancel. We compute the vector meson mass and decay constant from \( C_V^{\text{cont}} \) and the ratio \( f_T/f_V \) from the ratio of correlation functions \( C_T^{\text{cont}}/C_V^{\text{cont}} \). The tensor coupling \( f_T \) is then obtained by multiplying \( f_T/f_V \) by \( f_V \).

Note that \( f_V \) is a scale independent quantity, while \( f_T(\mu) \) depends on the renormalization scale \( \mu \), as well as the renormalization scheme. The scale and scheme dependence of the latter quantity is carried by the renormalization factor \( Z_T(\mu) \); we opt for the \( \overline{\text{MS}} \) scheme and for \( \mu = 2 \, \text{GeV} \).

II. RESULTS

ETMC has generated \( N_f = 2 \) configuration ensembles at four values of the inverse gauge coupling; in this work we make use of only three of them. Light mesons consist of a valence quark doublet, with twisted mass \( a \mu_\ell = a \mu_{\text{sea}} \). Heavy-light mesons consist of a valence quark pair \( (a \mu_\ell = a \mu_{\text{sea}}, a \mu_\ell) \). As already stated, these bare quark mass parameters are chosen so as to have light pseudoscalar mesons (‘‘pions’’) in the range of \( 280 \leq m_{\text{PS}} \leq 550 \, \text{MeV} \) and heavy-light pseudoscalar mesons (‘‘Kaons’’) in the range \( 450 \leq m_{\text{PS}} \leq 650 \, \text{MeV} \). The simulation parameters are gathered in Table I.

Our calibrations are based on earlier collaboration results. The ratio \( r_0/a \), known at each value of the gauge coupling \( \beta \) from Ref. [10], allows us to express our raw dimensionless data (quark masses, meson masses, and decay constants) in units of \( r_0 \). Knowledge of the renormalization constant \( Z_T \) in the \( \overline{\text{MS}} \) scheme at 2 GeV (see Ref. [11]) enables us to pass from bare quark masses to renormalized ones (again in \( r_0 \) units). Using only data with light valence quarks in the tm setup, we have applied the procedure described in Refs. [1,10] for the determination of the physical continuum light quark mass \( \mu_{\text{PS}}^{\overline{\text{MS}}}(2 \, \text{GeV}) \). These quark mass values are listed in Table II. The Sommer scale we use, based on the simulation parameters are gathered in Table I.

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>( a^{-1}(L^3 	imes T) )</th>
<th>( a \mu_\ell = a \mu_{\text{sea}} )</th>
<th>( a \mu_\ell )</th>
<th>( N_{\text{mean}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.80</td>
<td>( 24^3 \times 48 )</td>
<td>0.0080, 0.0110</td>
<td>0.0165, 0.0200</td>
<td>180</td>
</tr>
<tr>
<td>( (a - 0.1 , \text{fm}) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.90</td>
<td>( 24^3 \times 48 )</td>
<td>0.0040</td>
<td>0.0250</td>
<td>400</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>0.0270</td>
<td>200</td>
</tr>
<tr>
<td>3.90</td>
<td>( 32^3 \times 64 )</td>
<td>0.0064, 0.0085</td>
<td>0.0150, 0.0220</td>
<td>270/170</td>
</tr>
<tr>
<td>( (a - 0.085 , \text{fm}) )</td>
<td></td>
<td>0.0100</td>
<td>0.0270</td>
<td>200</td>
</tr>
<tr>
<td>4.05</td>
<td>( 32^3 \times 64 )</td>
<td>0.0030, 0.0060</td>
<td>0.0120, 0.0150</td>
<td>200</td>
</tr>
<tr>
<td>( (a - 0.065 , \text{fm}) )</td>
<td></td>
<td>0.0080</td>
<td>0.0180</td>
<td>200</td>
</tr>
</tbody>
</table>

TABLE II. The quark mass values (in the \( \overline{\text{MS}} \) scheme), used in our analysis; see Ref. [10].

<table>
<thead>
<tr>
<th>( \mu_{\text{PS}}^{\overline{\text{MS}}}(2 , \text{GeV}) )</th>
<th>( \mu_{\text{PS}}^{\overline{\text{MS}}}(2 , \text{GeV}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.6(2) , \text{MeV}</td>
<td>95(6) , \text{MeV}</td>
</tr>
</tbody>
</table>
an analysis with three values of the lattice spacing, is $r_0 = 0.448(5)$ fm. This updates our previous $r_0$ computation, derived with two $\beta$'s, cf. Ref. [1]. In short, the physical values of $m_{\pi}$, $m_K$ and $f_{\pi}$ have been used for the calibration of the bare parameters.

We see from Eqs. (1.9) and (1.10) that we need to know the renormalization parameters $Z_V$, $Z_A$, and $Z_T$. These quantities, as well as $Z_P$, have been computed in Ref. [11], in the regularization independent scheme in momentum space (RI/MOM); $Z_P$ and $Z_T$ are perturbatively converted to $\overline{\text{MS}}$. In the same work a $Z_V$ estimate, obtained from a Ward identity, is also provided. In Table III we gather the most reliable estimates of Ref. [11], which we have used in the present analysis, as well as our estimates of the $r_0/a$ ratio.

As can be seen in Table I, at $\beta = 3.90$ we have performed more extensive simulations, which enable us to check in some detail the quality and stability of the measured physical quantities. We wish to highlight straightaway the two problems we have encountered in these tests, performed for the tm setup: (i) For all sea quark masses, when the valence quark attains its lightest value $a\mu_\ell = 0.0040$, the vector meson effective mass has a poor plateau. The situation already improves at the next quark mass $a\mu_\ell = 0.0064$. Nevertheless, since the signal-to-noise ratio behaves as expected (i.e. it drops like $\exp[-(m_V - m_{\overline{\text{PS}}})a^0]$) the $\rho$-meson mass and decay constant can still be extracted (see results presented in Ref. [12]). (ii) A poor quality vector meson effective mass is also seen when $\mu_\ell < \mu_{\text{sea}}$. This problem is absent in the pseudoscalar channel. The above problems are easily avoided in the present work, since the quantities of interest are related to the $K^*$-meson, consisting of a down and a strange valence quark mass ($\mu_{u/d} < \mu_s$).

In unquenched simulations, depending on the values of quark masses, the resonance decay into a two meson $|K, \pi\rangle$ state may mix with $|K^*\rangle$. The threshold for decay is $E_K + E_\pi$, with $E_{K,\pi} = [m_{K,\pi}^2 + (\mathbf{p}_{K,\pi})^2]^{1/2}$ ($\mathbf{p}$ is the lattice linear extension). Our simulations are performed above threshold, so that the lowest energy state is $|K^*\rangle$ at zero spatial momentum.

We proceed as follows: at each $\beta$ value, we compute the necessary observables (vector meson mass $m_V$, vector decay constant $f_V$, and the ratio $f_T/f_V$), for all combinations of $a\mu_\ell = a\mu_{\text{sea}}$ and $a\mu_h$ (with $\mu_\ell < \mu_h$). In this way unitarity holds in the light quark sector, while the heavy valence quark mass, in a partially quenched rationale, spans a range around the physical value $\mu_s$. Examples of the quality of our signal are given in Figs. 1 and 2; the lightest mass is $a\mu_{\min}$ and the heavy mass, corresponding to the physical strange value $a\mu_s$, is obtained by interpolation, as will be explained below.

![FIG. 1 (color online). Effective vector meson mass $r_0 m_V$ at three values of the lattice spacing. The light quark mass is $a\mu_{\min}$ (see Table I) and the heavy quark mass $a\mu_h$ is close to that of the physical strange-quark; (a) tm setup, (b) OS setup. Plateau intervals are indicated by straight lines.](014505-3)

### Table III. The renormalization parameters used in our analysis and the $r_0/a$ values at each gauge coupling. $Z_V$ is obtained from a lattice vector Ward identity, while the other renormalization constants are obtained from the regularization independent scheme in momentum space; for details see Ref. [11].

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$Z_V$</th>
<th>$Z_A$</th>
<th>$Z_T^{\overline{\text{MS}}}(2 \text{ GeV})$</th>
<th>$Z_P^{\overline{\text{MS}}}(2 \text{ GeV})$</th>
<th>$r_0/a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.80</td>
<td>0.5816(02)</td>
<td>0.746(11)</td>
<td>0.733(09)</td>
<td>0.411(12)</td>
<td>4.54(07)</td>
</tr>
<tr>
<td>3.90</td>
<td>0.6103(03)</td>
<td>0.746(06)</td>
<td>0.743(05)</td>
<td>0.437(07)</td>
<td>5.35(04)</td>
</tr>
<tr>
<td>4.05</td>
<td>0.6451(03)</td>
<td>0.772(06)</td>
<td>0.777(06)</td>
<td>0.477(06)</td>
<td>6.71(04)</td>
</tr>
</tbody>
</table>
Statistical errors are estimated with the bootstrap method, employing 1000 bootstrap samples. A reliable direct determination of the ratio $f_T/f_V$ in the OS setup is not possible, because the ratio of correlation functions $C_{OS}^{T}/C_{OS}^{V}$ do not display satisfactory plateaux, due to big statistical fluctuations of the tensor correlator $C_{OS}^{T}$. We only present $f_T/f_V$ results in the tm setup, obtained from the better behaved correlation function $C_{TM}^{T}$. In Fig. 3 we show results for this ratio at $a\mu_{min}$ and also at a heavier light quark mass.

Regarding vector meson masses $m_V$ and couplings $f_V$, both tm and OS results display a similar plateau quality and statistical accuracy. At finite lattice spacing and for equal bare quark masses, tm and OS estimates of $m_V$ are compatible within errors. Agreement is also very good for $f_V$, with occasional discrepancies, interpreted as cutoff effects, showing up at the coarsest lattice. Contrary to the well-known large $O(a^2)$ isospin breaking effects in the neutral to charged pion splitting mass, no numerically large differences are observed between tm and OS results for $f_V$ and $m_V$. This fact is in agreement with theoretical expectations, see Ref. [13].

The extrapolation to the physical quark masses is carried out in two steps. First, for fixed values of the gauge coupling $\beta$ and light quark mass $a\mu_h$, we perform linear interpolations of $r_0m_V$, $r_0f_V$, and $f_T/f_V$ to the physical strange-quark mass $\mu_s$. The second step consists in using these interpolated results for a combined fit of our

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$^2$Given the large fluctuations of $f_T/f_V$ in the OS setup at the finer lattice spacing, we only quote results for this ratio in the tm-setup.
data at three lattice spacings and all available light quark masses, in order to determine the continuum value of the quantity of interest ($m_{V}r_{0}$, $r_{0}f_{V}$, and $f_{T}/f_{V}$; see Figs. 4–6, respectively). We model our data with the following ansatz:

$$m_{V}r_{0} = C_{0}(\mu_{s}r_{0}) + C_{1}(\mu_{s}r_{0})\mu_{\ell}r_{0} + D(\mu_{s}r_{0})\frac{a^{2}}{r_{0}^{2}},$$  \hspace{1cm} (2.1)

and similarly for $f_{V}r_{0}$ and $f_{T}/f_{V}$.

The results of the interpolations in the heavy quark mass $\mu_{h}$ to the physical value $\mu_{s}$, at each $\beta$ and $a\mu_{\ell}$, are gathered in Table IV. Errors are due to (i) statistical fluctuations, (ii) the uncertainty of the strange-quark mass $\mu_{s}$ when interpolating in $\mu_{h}$, (iii) the uncertainty of $r_{0}/a$, and (iv) the errors of renormalization constants $Z_{A}$ and $Z_{T}$ (where applicable).

In the same table we also display the results of the combined chiral and continuum extrapolations. The uncertainty of these estimates includes, besides the fitting error, also that of $\chi^{2}/d.o.f.$ is less than unity. The linear dependence of our data on the light quark mass agrees with the predictions of chiral perturbation theory (extended by the inclusion of tensor external sources) for the ratio $f_{T}/f_{V}$ in the $K^{+}$ mass range; see Refs. [14,15].

Our final results, extracted in the tm setup, are

$$m_{K^{+}} = 981(31)(10)[33] \text{ MeV},$$  \hspace{1cm} (2.2)

TABLE IV. Results for three values of lattice spacing and several light quark masses $a\mu_{\ell}$, interpolated to the physical strange mass $\mu_{s}$. Vector mass and vector decay constant results are presented for both tm and OS setups. The ratio $f_{T}/f_{V}$ results are given only in the tm setup. Our extrapolations at the $\mu_{u/d}$ physical point and in the continuum limit are also shown. In the last row the experimental results for the vector mass and the vector decay constant, in units of $r_{0}$, have been added.

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$a\mu_{\ell}$</th>
<th>$r_{0}m_{V}^{\text{tm}}(\ell, s)$</th>
<th>$r_{0}m_{V}^{\text{OS}}(\ell, s)$</th>
<th>$r_{0}f_{V}^{\text{tm}}(\ell, s)$</th>
<th>$r_{0}f_{V}^{\text{OS}}(\ell, s)$</th>
<th>$[f_{T}/f_{V}]^{\text{tm}}(\ell, s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.80</td>
<td>0.0080</td>
<td>2.443(41)</td>
<td>2.471(30)</td>
<td>0.642(18)</td>
<td>0.700(13)</td>
<td>0.764(38)</td>
</tr>
<tr>
<td></td>
<td>0.0110</td>
<td>2.508(32)</td>
<td>2.500(23)</td>
<td>0.651(14)</td>
<td>0.706(15)</td>
<td>0.792(35)</td>
</tr>
<tr>
<td>3.90</td>
<td>0.0040</td>
<td>2.410(41)</td>
<td>2.381(38)</td>
<td>0.610(21)</td>
<td>0.643(17)</td>
<td>0.755(19)</td>
</tr>
<tr>
<td></td>
<td>0.0064</td>
<td>2.441(32)</td>
<td>2.427(35)</td>
<td>0.626(22)</td>
<td>0.659(12)</td>
<td>0.726(20)</td>
</tr>
<tr>
<td></td>
<td>0.0085</td>
<td>2.484(48)</td>
<td>2.441(33)</td>
<td>0.628(16)</td>
<td>0.652(16)</td>
<td>0.776(27)</td>
</tr>
<tr>
<td></td>
<td>0.0100</td>
<td>2.468(54)</td>
<td>2.481(32)</td>
<td>0.619(20)</td>
<td>0.657(16)</td>
<td>0.774(31)</td>
</tr>
<tr>
<td>4.05</td>
<td>0.0030</td>
<td>2.305(86)</td>
<td>2.263(80)</td>
<td>0.568(49)</td>
<td>0.588(40)</td>
<td>0.742(27)</td>
</tr>
<tr>
<td></td>
<td>0.0060</td>
<td>2.439(67)</td>
<td>2.295(76)</td>
<td>0.618(41)</td>
<td>0.578(46)</td>
<td>0.768(30)</td>
</tr>
<tr>
<td></td>
<td>0.0080</td>
<td>2.512(65)</td>
<td>2.427(48)</td>
<td>0.649(31)</td>
<td>0.648(27)</td>
<td>0.741(31)</td>
</tr>
<tr>
<td>CL</td>
<td>$\mu_{u/d}$</td>
<td>2.227(71)</td>
<td>2.200(60)</td>
<td>0.545(41)</td>
<td>0.525(30)</td>
<td>0.701(46)</td>
</tr>
<tr>
<td>expt.</td>
<td></td>
<td>2.025</td>
<td>0.493</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

FIG. 4 (color online). $r_{0}m_{V}$ plotted against the renormalized light quark mass $r_{0}\bar{\mu}_{\ell}$; (a) tm setup, (b) OS setup. The continuous lines are combined chiral and continuum extrapolations to the physical point. The bottom (black) line corresponds to Eq. (2.1) at $a = 0$. The separation among the four lines in (a) is invisible to the naked eye (i.e. small scaling violations).
The first error includes the statistical uncertainty and the systematic effects related to the simultaneous chiral and continuum fits, mass interpolations and extrapolations, and uncertainties in the renormalization parameters. The second error arises from that of \( r_0 \). These two errors, combined in quadrature, give the total error in the square brackets. It is encouraging that these results agree with the ones obtained in the OS setup (which is a different regularization), namely,

\[
\frac{m_K}{\beta} = 240(18)(02)[18] \text{ MeV.} \tag{2.3}
\]

The overestimation of \( m_K \) is presumably due to the fitting ansatz, which does not take resonance effects into account. A more sophisticated treatment of the extrapolation, or a lattice study nearer the physical point could address these issues along the lines of Ref. [12]. This is beyond the scope of the present work.

Our final estimate (tm setup) for the ratio of vector meson couplings is

\[
\frac{f_T(2 \text{ GeV})}{f_V} \bigg|_{K^*} = 0.704(41). \tag{2.4}
\]

This is compatible with the continuum limit quenched result \( \frac{f_T(2 \text{ GeV})}{f_V} \bigg|_{K^*} = 0.739(17)(3) \) of Ref. [16]. We are also in agreement with the result of the RBC/UKQCD Collaboration [17]; using \( N_f = 2 + 1 \) dynamical fermions at a single lattice spacing, they quote \( \frac{f_T(2 \text{ GeV})}{f_V} \bigg|_{K^*} = 0.712(12) \). The lattice results are also in agreement with the sum rules’ estimate \( \frac{f_T(2 \text{ GeV})}{f_V} \bigg|_{K^*} = 0.73(4) \), quoted in [18].

Nevertheless, the difference between the physical vector mass \( m_{K^*} \) and our result (2.2) indicates that it is essential to perform simulations in the vicinity of the physical point, in order to make reliable predictions.

**ACKNOWLEDGMENTS**

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