Quantization of AdS_3 Black Holes in External Fields: Semiclassical Results from Pure Gravity

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(2 + 1)-dimensional anti–de Sitter (AdS) gravity is quantized in the presence of an external scalar field. We find that the coupling between the scalar field and gravity is equivalently described by a perturbed conformal field theory at the boundary of AdS_3. This allows us to perform a microscopic computation of the transition rates between black hole states due to absorption and induced emission of the scalar field. Detailed thermodynamic balance then yields Hawking radiation as spontaneous emission, and we find agreement with the semiclassical result, including greybody factors. This result also has application to four and five-dimensional black holes in supergravity. [S0031-9007(98)07124-5]

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The idea that black hole radiation should have its origin in transitions between discrete states of a thermally excited system has a long history [1]. The successful development of this picture, however, has always been hampered by the lack of a proper quantum description of the black hole. Ideally, one would like to quantize a classical system described by an action of the form

$$I[g, \Psi] = I_{\text{grav}}[g] + I_{\text{matter}}[\Psi; g]. \tag{1}$$

In view of the difficulty to quantize this system in a complete way. Hawking proposed to treat g as a classical, fixed background, in the presence of which the field Ψ is quantized [2]. The drawback in this approach is that the black hole is unaffected by the emission of radiation. No reference to black hole microstates is made, and accounting for back reaction has proven to be a notoriously difficult problem.

In this paper we suggest a different route, in which the microstates of the black hole play an explicit role. This approach is more akin to the old fashioned treatment of radiation from, say, an atom. The quantized atom can be excited by absorbing energy from the external radiation field, or it can also decay via induced emission of radiation by giving away energy to the field. Existence of a thermodynamical equilibrium then implies that spontaneous emission must occur, with rate given in terms of the coefficients for absorption and induced emission. it is a variation of that approach that we aim to develop here. That is, we treat the gravitational field g as quantum degrees of freedom, whereas the matter field Ψ will remain classical.

In view of the lack of a consistent quantum theory of four-dimensional gravity we will work in the framework of anti–de Sitter (AdS) gravity in 2 + 1 dimensions. (2 + 1)-dimensional gravity with a negative cosmological constant is know to have black hole solutions [3] which have proved to be a useful laboratory for the study of the microscopical properties of black holes. At the same time (2 + 1)-dimensional gravity is almost trivial. More precisely, it is topological, at least in the absence of matter fields. As a consequence the dynamics of the gravitational degrees of freedom is described by a conformal field theory (CFT) at the boundary, i.e., the asymptotic region at infinity in AdS_3.

The coupling of matter fields to (2 + 1)-gravity is not topological, however. But since we treat Ψ classically, matter will be *on shell* in the bulk of the black hole geometry. As we shall see, this reduces the coupling to gravity to a perturbation of the boundary CFT. This coupling to the boundary degrees of freedom is the analog of the coupling of an electric field to the dipole moment of the atom. Here, we choose to work with the simplest example: a scalar field with minimal coupling to gravity. The approach, however, can be readily extended to other fields.

Our results have also bearings for certain higher dimensional black holes, namely those for which the near horizon geometry reduces to an AdS_3 black hole. These are precisely the generalized four- and five-dimensional black holes for which a microscopic description of the low energy dynamics in terms of string theory has been found recently [4]. One may therefore speculate that the important structure present in these higher dimensional black holes is the near horizon AdS_3 gravity, which has a natural conformal field theory associated with it. String theory may be but one way to describe it.

The picture of black hole radiation that emerges from our approach is "holographic," in that all the interactions take place at the asymptotic boundary of AdS_3. It is closely related to (and in fact, inspired by) the extremely successful description of black hole radiation in string theory [4,5], in which the microscopic theory is fully quantum. The latter, however, relies essentially on a conjectured correspondence between AdS gravity and the CFT on its boundary [6]. In contrast, in the present approach this correspondence is an automatic consequence of the topological nature of (2 + 1)-gravity. This enables us to present what, to our knowledge, is the first *explicit*

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derivation of the coupling of the external field to the CFT on the boundary of AdS. The microscopic theory of black hole entropy and radiance has also been considered recently in, e.g., [7,8,9].

In this paper we take gravitational action in (1) to be the standard three-dimensional Einstein-Hilbert action with a negative cosmological constant,

$$I_{\mathrm{EH}} = -\frac{1}{16\pi G}\int d^3x\sqrt{-g}\left(R + \frac{2}{\ell^2}\right), \qquad (2)$$

where $\Lambda = -1/\ell^2$ is the cosmological constant. The identification of this theory with a boundary conformal field theory has been described by several authors [10–13], and our description of it will accordingly be rather cursory. Three-dimensional gravity can be mapped to a Chern-Simons (CS) theory [14] by expressing the triad e^a_μ and spin connection $\omega^a = \varepsilon^a_{bc}\omega^{bc}$ in terms of two $\mathrm{SL}(2,\mathrm{IR})$ Chern-Simons gauge potentials A and $\tilde{A}$,

$$A^a_\mu = \omega^a_\mu + \frac{e^a_\mu}{\ell}, \qquad \tilde{A}^a_\mu = \omega^a_\mu - \frac{e^a_\mu}{\ell}. \qquad (3)$$

The Einstein-Hilbert action (2) is then equivalent to the difference of two Chern-Simons (CS) actions, $I_{\mathrm{EH}} = I[A] - I[\tilde{A}]$, where (our conventions agree with [12])

$$I[A] = \frac{k}{4\pi}\int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right), \qquad (4)$$

with $k = -\frac{\ell}{4G}$. Gauge transformations in this theory correspond to diffeomorphisms in (2). However, if the manifold has a boundary, only gauge transformations that vanish at the boundary leave the CS-action invariant. The dynamics of the residual degrees of freedom is, in turn, described by a CFT. We follow the analyses in [11,12], and work within the canonical formalism. We choose, as our radial coordinate, the proper radius ρ, rescaled by ℓ to make it dimensionless. The boundary, which is at very large ρ, is parametrized by $u = \frac{t}{\ell} + \varphi$ and $v = \frac{t}{\ell} - \varphi$. Furthermore, we choose the boundary conditions $A_v = \tilde{A}_u = 0$ for the CS potentials. We will see that these boundary conditions are compatible with the existence of black hole solutions, but still leave too much freedom. In order to have a variational principle compatible with these boundary conditions a boundary term must be added to (4) [10].

To continue we choose the gauge

$$A_\rho = b(\rho)^{-1}\partial_\rho b(\rho), \qquad \tilde{A}_\rho = b(\rho)\partial_\rho b(\rho)^{-1}, \qquad (5)$$

with $b(\rho) = \exp(\rho T_3)$. Solving the Gauss's constraint $F_{\rho\varphi} = 0$ we express

$$A_\varphi = b(\rho)^{-1}a(u)b(\rho) = \begin{pmatrix} a^3(u) & e^{-\rho}a^+(u) \\ e^\rho a^-(u) & -a^3(u) \end{pmatrix}, \qquad (6)$$

and similarly for the $\tilde{A}$. Upper indices in a^a (and $\tilde{a}^a$) correspond to group indices. The gauge transformations that preserve these boundary conditions and gauge choices have infinitesimal parameters of the form $\eta = b^{-1}\lambda(u)b$, $\tilde{\eta} = b\tilde{\lambda}(v)b^{-1}$. These, in turn, can be expressed in terms of diffeomorphisms $\xi^i(u)$, $\tilde{\xi}^i(v)$ $(i = \rho, \varphi)$ by means of the relations $\eta = \xi^i A_i$, $\tilde{\eta} = \tilde{\xi}^i\tilde{A}_i$. Hence

$$\delta A_\varphi = \begin{pmatrix} \frac{1}{2}\partial_\varphi\xi^\rho + \partial_\varphi(\xi^\varphi a^3) & e^{-\rho}[\partial_\varphi(\xi^\varphi a^+) - \xi^\rho a^+] \\ e^\rho[\partial_\varphi(\xi^\varphi a^-) + \xi^\rho a^-] & -\frac{1}{2}\partial_\varphi\xi^\rho - \partial_\varphi(\xi^\varphi a^3) \end{pmatrix}. \qquad (7)$$

For what follows it may be helpful to think of the diffeomorphisms along the boundary as infinitesimal conformal transformations $u \to u + \xi^\varphi(u)$, $v \to v - \tilde{\xi}^\varphi(v)$. Under these transformations the fields $a^a(u)$, $\tilde{a}^a(v)$ transform as conformal primary fields with weights $(1,0)$ and $(0,1)$, respectively. This is expected as Chern-Simons theory, upon imposing boundary conditions as above, reduces to a chiral Wess-Zumino-Witten (WZW) theory at the boundary [15]. The fields $a^a(u)$, $\tilde{a}^a(v)$ are then precisely the components of the corresponding level k, left/right Kac-Moody currents.

For later use, we now give the asymptotic form of the metrics that are described by the connections (5), (6). One has

$$ds^2 = \ell^2 d\rho^2 - \ell^2 e^{2\rho}a^-(u)\tilde{a}^+(v)\,du\,dv + \ldots, \qquad (8)$$

where, for the sake of brevity, we omit terms that are subleading at large ρ.

While the system presented so far could be taken as a starting point for quantization, it appears that it has to be further reduced in order to isolate the black hole degrees of freedom. In particular, the boundary WZW theory does not account properly for the Bekenstein-Hawking entropy [12,16]. A consistent restriction is obtained by imposing that the induced metric on the boundary remains fixed under the allowed diffeomorphisms (7). With these extra restrictions the algebra of asymptotic symmetry generators receives a classical central charge [13], which was used in [9] to argue that the geometrical entropy is reproduced with the boundary CFT (subtleties in the application of this formula to the present situation are discussed in [16]).

On the other hand, this constraint relates the diffeomorphisms along the boundary ξ^φ, $\tilde{\xi}^\varphi$ to the radial displacement $\rho \to \rho + \xi^\rho(u) + \tilde{\xi}^\rho(v)$ by

$$\xi^\rho = -\partial_\varphi\xi^\varphi, \qquad \tilde{\xi}^\rho = -\partial_\varphi\tilde{\xi}^\varphi. \qquad (9)$$

From the point of view of the WZW theory the relation (9) is implemented by the "improved" Virasoro generator [17]

$$L = L_{\mathrm{sug}} + k\partial_\varphi a^3, \qquad (10)$$

with classical central charge $c = 6k$. Here L_{sug} is the Sugawara stress-energy tensor associated to the Kac-Moody algebra of $a^\pm$, a^3. It is well known that the constraints described above are precisely those imposed in the WZW to Liouville reduction [10,17]. More details on this will be given elsewhere. At present we just note that the constraint (9) implies that under conformal transformations the proper distance ρ and the Liouville field ϕ transform in the same way. It is therefore natural to identify $\rho \to -\phi(u,v)$.

We now consider the external matter field Ψ, in the form of a minimally coupled scalar field. Matter fields perturb the dynamics of the metric by acting as sources of energy and momentum. The field Ψ is treated classically, i.e., taken to satisfy the classical wave equation in the bulk of AdS_3. One may think of this as the curved space equivalent of taking a homogeneous external field in the case of the atom in a radiation field. In this approximation one does not resolve the detailed structure of the bulk. The matter action then reduces to a boundary term

$$I_{\text{matter}} = -\frac{1}{16\pi G}\int \sqrt{-g}\, g^{\mu\nu}\partial_\mu \Psi^\dagger \partial_\nu \Psi \to -\frac{1}{16\pi G}\,\mathcal{B}\,,$$
$$\mathcal{B} = \frac{1}{2}\int_{\partial\mathcal{M}} \sqrt{-g}\, g^{\rho\mu}(\Psi^\dagger\partial_\mu\Psi + \Psi\partial_\mu\Psi^\dagger)\,. \quad (11)$$

Requiring Ψ to satisfy the classical wave equation in a background that is asymptotically of the form (8) fixes its asymptotic form to (up to a log term which is of higher order in the frequency [18])

$$\Psi(\rho,\varphi,t) = (1 - ie^{-2\rho})\psi_+(t,\varphi) + (1 + ie^{-2\rho})\psi_-(t,\varphi)\,. \quad (12)$$

We have decomposed the wave into components $+$, $-$ with positive (ingoing) and negative (outgoing) flux, respectively [18]. Substitution of this and the asymptotic metric (8) into (11) then leads to

$$\mathcal{B} = \frac{\ell}{i}\int du\,dv\,\mathcal{O}(u,v)\,(\psi_+\psi_-^\dagger - \psi_-\psi_+^\dagger)\,, \quad (13)$$

where $\mathcal{O}(u,v) = a^-(u)\tilde{a}^+(v)$. For definiteness, we take the dependence in t and ϕ to be of the form

$$\psi_\pm(t,\varphi) = e^{i(\omega_\pm t - m_\pm\varphi)}. \quad (14)$$

Then we find

$$\mathcal{B} = 2\ell\int du\,du\,\mathcal{O}(u,v)\sin(\omega t - m\varphi)\,, \quad (15)$$

where $\omega = \omega_+ - \omega_-$, $m = m_+ - m_-$. This is our main result: the external field introduces a perturbation of the CFT at the boundary at infinity by a primary operator (13) with conformal weight $(1,1)$.

Note that upon reduction to the Liouville theory one keeps $e^{2\rho}a^-\tilde{a}^+$ fixed. According to our remarks above one is then led to identify

$$\mathcal{O}(u,v) = e^{2\phi}. \quad (16)$$

Although this identification may need further clarification it suggests a simple geometrical picture: Think of the conformal field theory as a "string at infinity" which adjusts its proper radial position such as to keep its world-sheet volume constant. The scalar field couples to the position of the string. This is described in the conformal field theory language by the coupling (16), which is the gravitational analog of the coupling of an external electric field to the dipole moment operator of an atom. This approximation should be limited to transitions between neighboring black hole states, that is, with small energy differences, as the effect of the change in the geometry in the bulk on the scalar field is neglected.

We now apply our results to the specific case of interest, the BTZ black hole [3,19]. In lightcone coordinates u, v and proper radius ρ the black hole has metric

$$ds^2 = -\frac{\ell^2}{4}\sinh^2\rho(z_+du + z_-dv)^2 + \ell^2 d\rho^2 + \frac{\ell^2}{4}\cosh^2\rho(z_+du - z_-dv)^2. \quad (17)$$

This coordinate patch covers the region outside the (outer) horizon of a nonextremal black hole. Here,

$$z_\pm = \sqrt{8G(M \pm J\ell)}\,, \quad (18)$$

parametrize the family of nonextremal black hole solutions. For the black hole, the conformal operators a, $\tilde{a}$ introduced above take the expectation values $\langle a^\pm\rangle = z_+/2$, $\langle\tilde{a}^\pm\rangle = z_-/2$, $\langle a^3\rangle = \langle\tilde{a}^3\rangle = 0$.

Note that an arbitrary nonextremal black hole can be obtained from (17) by a constant rescaling $(u,v)\to(\lambda u,\tilde{\lambda}v)$ [3]. In the quantum theory $z_\pm$ are replaced by operators a, $\tilde{a}$ and conformal transformations change the eigenvalues of the mass and angular momentum operators in the usual manner.

The black hole corresponds to a thermal state of the left and right moving sectors of the CFT [20]. The effective temperature of each sector can be found from the energy and entropy formulas,

$$\varepsilon_R = \frac{2\pi}{V}L_0 = \frac{z_+^2}{16G}\,, \qquad s_R = 2\pi\sqrt{\frac{cN_R}{6}} = \frac{\pi\ell z_+}{4G}$$
$$\varepsilon_L = \frac{2\pi}{V}\tilde{L}_0 = \frac{z_-^2}{16G}\,, \qquad s_L = 2\pi\sqrt{\frac{cN_L}{6}} = \frac{\pi\ell z_-}{4G}\,, \quad (19)$$

where V is the volume of the boundary CFT and N_R, N_L are the eigenvalues of L_0, $\tilde{L}_0$, respectively. The corresponding left- and right-moving temperatures are therefore

$$T_{R,L}^{-1} = \frac{\partial s_{R,L}}{\partial\varepsilon_{R,L}} = \frac{2\pi\ell}{z_\pm}\,. \quad (20)$$

These are related to the Hawking temperature as $2T_H^{-1} = T_R^{-1} + T_L^{-1}$. After properly rotating to Euclidean time these effective temperatures correspond to the inverse periods of the lightcone variables [20]. Note that (20) is rather insensitive to the details of the concrete realization of the underlying boundary CFT. Indeed, only the relation between energy and entropy enters.

As explained above this interaction vertex should correctly describe the transition between black hole states with small energy difference. Note that it is not required that the initial state itself has low energy. In particular, it should describe correctly the low frequency decay rates of highly excited black holes.

The calculation will be similar to that in [21]. From (15), the transition amplitude between an initial and a final state in the presence of an external flux with frequency and angular momentum ω, m is then given by

$$\mathcal{M} = \ell \int du\, dv \langle f|\mathcal{O}(u,v)|i\rangle e^{-i(\omega\ell - m)(u/2)} \times e^{-i(\omega\ell + m)(v/2)}, \tag{21}$$

where i, f denote the initial and final black hole state, respectively. If this term corresponds to emission, then the term in (15) with the opposite frequency will give absorption, but at this moment this is still a matter of convention. The important point is that calculation of transition amplitudes is reduced to the computation of correlation functions of $(1,1)$ primary fields. In particular, it does not rely on the identification (16), which, to some, may seem a little far fetched.

We proceed to compute the decay rate. For simplicity we set $m = 0$. Squaring the amplitude $\mathcal{M}$ and summing over final states leads to

$$\sum_f |\mathcal{M}|^2 = \ell^2 \int du\, du' dv\, dv' \langle i|\mathcal{O}(u,v)\mathcal{O}(u',v')|i\rangle \times e^{-i\omega\ell(u-u'/2)} e^{-i\omega\ell(v-v'/2)}. \tag{22}$$

Since the black hole corresponds to a thermal state, we must average over initial states weighed by the Boltzmann factor, i.e., we take the finite temperature two point functions, which for fields of conformal weight one are given by

$$\langle \mathcal{O}(0,0)\mathcal{O}(u,v)\rangle_{T_R,T_L} = \left[\frac{\pi T_R}{\sinh(\pi T_R u)}\right]^2 \left[\frac{\pi T_L}{\sinh(\pi T_L v)}\right]^2, \tag{23}$$

provided $T \gg V^{-1}$. These have the right periodicity properties in the Euclidean section. The remaining integrals can be performed by contour techniques of common use in thermal field theory. Whether we deal with emission or absorption depends on how the poles at $u = 0$, $v = 0$ are dealt with. The resulting emission rate is then given by

$$\Gamma = \frac{\omega \pi^2 \ell^2}{(e^{(\omega/2T_L)} - 1)(e^{(\omega/2T_R)} - 1)}, \tag{24}$$

where we have included a factor ω^{-1} for the normalization of the outgoing scalar. Equation (24) reproduces correctly the semiclassical result [18,22], therefore providing a microscopical derivation of the decay of AdS_3 black holes relying exclusively on the gravitational degrees of freedom.

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