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ABOUT SOME QUESTIONS OF DIFFERENTIAL ALGEBRA CONCERNING TO ELEMENTARY FUNCTIONS

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The study of elementary functions, that is, of those functions built up by using rational functions over $\mathbb{C}$, Exponentials, Logarithms and Algebraic operations, began somewhat sistematically with the diverse works that Joseph Liouville did in the 1830's. Although his capital aim was to obtain some result about the integration by means of elementary functions, along - his way he had to study aspects more circunscribed to the structure of these functions. About the first point we must say that, certainly, Liouville obtained a result that, even improved since that time, has not changed essentially. It sufices, for exemple, to compare the work of Liouville given in [2] and Rosenlicht's - in [6]. We can observe then that the introduction of new language and new thecnique has only linealized the problem, making clear which properties of elementary functions characterize them.

Using this new language and new thecnique it is possible deal more clearly with some questions about elementary functions. For exemple, their irredundancy, that was already established by Liouville himself in [2] and Hardy in [1]. Now, the irredundancy is a consequence of an Structure theorem (1.2) that we can quikly give by using Rosenlicht's thecniques. Also, the not solvability by means of elementary functions of certain classical transcendental equations can be easily established. We study this in Section 2. Another treatment of the Structure theorem and irredundance questions (but not using Rosenlicht's thecniques) can be found in the Risch's paper [5].

1.- AN STRUCTURE THEOREM.

First some definitions. All fields will be conmutative and of characteristic zero.

1.A. Remember that if E is a field, a map $D:E \longrightarrow E$ is a <u>derivation</u> of E if: (1) $\forall x, y \in E$ $D(x+y) = D(x) + D(y)$, (2) $\forall x, y \in E$ $D(xy) = xD(y) + yD(x)$. It follows from (2) that $D(1) = 0$, hence by (1) $D(z) = 0$ $\forall z \in Z$. The set $C_D = \{x \in E / D(x) = 0\}$ is the set of <u>constants of D</u>. Given that $\forall x \in E$ $D(x^n) = nD(x)x^{n-1}$, $D(x^{-1}) = -D(x)x^{-1}$ we have that C_D is a subfield of E.

A field E with a family of derivations Δ is a <u>differential field</u>; then, $C = \bigcap_{D \in \Delta} C_D$ is the <u>field of constants</u> of the differential field E. Let $E \subset F$ two differential fields. The <u>extension $E \subset F$ is differential</u> if $\forall D \in \Delta_F$, $D_{|E} \in \Delta_E$. Although two different derivations of Δ_F can coincide over E, we won't distinguish betwen Δ_E and Δ_F. Let C_E, C_F be the respective constant fields. We have $C_E \subset C_F$. When the equality holds we say that the extension <u>is with the same field of constants</u>.

EXEMPLES: $\mathbb{C}(X_1, \ldots, X_n)$ with $\Delta = (\delta/_{\delta x_i})_{i=1\ldots n}$ is a differential field. $\mathbb{C}(X) \subset \mathbb{C}(X, e^X)$ is a differential extension with the same field of constants.

1.B. The elementary nature is then formulated in the next way: let E be a differential field; $x, y \in E$. Then

- $y = \mathrm{Log}(x) \iff Dy = Dx/_x \ \forall D \in \Delta$ (y is Logarithm of x)
- $y = \mathrm{Exp}(x) \iff Dy/_y = Dx \ \forall D \in \Delta$ (y is Exponential of x)

If $E \subset F$ is a differential extension, $y \in F$ is <u>Elementary over E</u> if and only if

- either y is algebraic over E
- or $y = \mathrm{Log}(x)$ being $x \in E$
- or $y = \mathrm{Exp}(x)$ being $x \in E$.

The differential extension <u>$E \subset F$ is Elementary</u> if $F = E(\theta_1, \ldots, \theta_n)$ with θ_1 elementary over E, and θ_i elementary over $E(\theta_1, \ldots, \theta_{i-1})$ $\forall i \geq 2$. Then, $\mathrm{Card}\Delta_E = \mathrm{Card}\Delta_F$.

1.C. The tool wich allow us to liearize the arguments is the Module of Differentials. A fast construction of it (sufficient for us) is the following:

let $E \subset F$ be fields and consider the F-vector space generated by the symbols $\{dx\}_{x \in F}$. Let us impose them the following relations:

(1) $\forall\, x, y \in F \quad d(x+y) = dx + dy$

(2) $\forall\, x, y \in F \quad d(xy) = xdy + ydx$

(3) $\forall\, x \in E \quad dx = 0.$

Then we get a F-vector space called the <u>Module of the Differentials of $E \subset F$</u>. Its symbol is $\Omega_{F/E}$.

Remember too that if $\{x_i\}_{i=1\ldots r}$ are elements of F, then they are algebraicaly independent over E if and only if the family $\{dx_i\}_{i=1\ldots r}$ is F-lineary independent on $\Omega_{F/E}$. So $\mathrm{Tr.deg.}_E F = \dim_F(\Omega_{F/E})$. (see [6], Prop.3)

The next result, due to Rosenlicht, is a fundamental one for this work:

1.1.- <u>THEOREM.</u> Let $E \subset F$ be a differential extension with the same field of constants. Let C be this field and take $y_1, \ldots, y_n \in F$, $z_1, \ldots, z_r \in F-\{0\}$ and $\{c_{ij}\}_{\substack{i=1\ldots n \\ j=1\ldots r}} \subset C$ such that $\forall\, i = 1, \ldots, n$, $\forall\, D \in \Delta$

(1) $\displaystyle\sum_{j=1}^{r} c_{ij} Dz_j / z_j + Dy_i \in E$. Then

- either $\mathrm{Tr.deg.}_E E(y_1, \ldots, y_n, z_1, \ldots, z_r) \geq n$

- or the n elements of $\Omega_{F/E}$:

$\displaystyle\sum_{j=1}^{r} c_{ij} 1/_z dz_j + dy_i$, $i = 1, \ldots, n$ are C-lineary dependent.

<u>Proof:</u> see Theorem 1. of [6].

I.D. Let F be a differential field. We say that the <u>equality $Y = \mathrm{Log}\, X$</u> has a solution in F if there are elements $x, y \in F$ verifying it. It is natural, then, to ask haw many solutions of this equality there are in an elementary extension $E \subset F$. The following theorem, from wich Risch gives another version in [5], answers this question. Previously some notation:

let $E \subset F$ be an elementary differential field extension with the same field of constants: $E \subset F = E(\theta_1, \ldots, \theta_n)$. Let

$y_1 = \mathrm{Log}\, x_1, \ldots, y_r = \mathrm{Log}\, x_r$

the not algebraic cases among the θ_i's; that is, $r = \mathrm{Tr.deg.}_E F$ and for each Θ_i not algebraic (over the preceeding subextension) there exists x_j or y_j such that $\Theta_i = x_j$ or y_j depending on whether Θ_i is Exponential or Logarithm. Suposse they are arranged according to their order of appearance and that $\overline{E}$ is an algebraic closure of E.

1.2.- Theorem. On the abovementioned hypothesis if the equality Y = = Log X holds in F, for any solution x, y there exist $c_1, \ldots, c_r \in C$, f, $g \in \overline{E} \cap F$, and $n_1, \ldots, n_r, n \in Z$ such that

$$y + c_1 y_1 + \ldots + c_r y_r = f \ , \quad x^n x_1^{n_1} \ldots x_r^{n_r} = g \ .$$

Proof: if the equality holds in F we can consider the system

$$\begin{cases} \forall i \quad Dy_i - Dx_i/_{x_i} = 0 \in E \\ Dy - 1/_x Dx = 0 \in E \qquad \forall \ D \in \Delta \ . \end{cases}$$

By Theorem 1.1 we get

- either $\mathrm{Tr.deg.}_E E(y_1, \ldots, y_r, y, x_1, \ldots, x_r, x) \geqslant r+1$

- or the elements of $\Omega_{F/E}$: $(dy_i - 1/_{x_i} dx_i)$, $i = 1, \ldots, r$,

$(dy - 1/_x dx)$ are C-lineary dependent.

Here it is clear that only the second condition is possible. So there exist $c_1, \ldots, c_r, c \in C$ not all zero such that

$$(1) \ c(dy - 1/_x dx) + \sum_{i=1}^{r} c_i (dy_i - 1/_{x_i} dx_i) = 0.$$

We can also take $c \neq 0$ since otherwise

$$\sum_{i=1}^{r} c_i (dy_i - 1/_{x_i} dx_i) = 0.$$

But if $y_r = \Theta_j$ for some j, because of the elementarity of $E \subset F$, each dy_i, dx_j except dy_r is a linear combination of the preceeding r-1 $d\Theta_s$ with coefficients in F. But they are F-lineary independent beacuse of 1.C. So $c_r = 0$. The same happens if $x_r = \Theta_i$ for some i. Appliyng repeatdly this argument we get $c_1 = \ldots = c_r = 0$, not possible.

Hence, dividing by c, we can assume

$$(2) \ dy + c_1 dy_1 + \ldots + c_r dy_r = 1/_x dx + c_1 1/_{x_1} dx_1 + \ldots + c_r 1/_{x_r} dx_r \ .$$

Consider now a maximal Q-lineary independent system among the

$\{1, c_1, \ldots, c_r\} : \{e_1, \ldots, e_k\}$ such that $e_1 = 1$. Then

$$\forall i : c_i = \sum_{j=1}^{k} q_{ij} e_j \ , \ q_{ij} \in Q \ \forall i, j. \text{ Therefore}$$

$$1/_x dx + c_1 1/_{x_1} dx_1 + \ldots + c_r 1/_{x_r} dx_r = e_1 1/_x dx + \sum_{j=1}^{k} q_{1j} e_j dx_1 + \ldots +$$

$$+ \sum_{j=1}^{k} q_{rj} e_j 1/_{x_r} dx_r = e_1(1/_x dx + \sum_{i=1}^{r} q_{i1} 1/x_i dx_i) + \ldots + e_k(\sum_{i=1}^{r} q_{ik} 1/_{x_i} dx_i) =$$

$$= e_1 1/_{f_1} df_1 + \ldots + e_k 1/_{f_k} df_k \ , \quad \text{being} \quad f_1 = x x_1^{q_{11}} \ldots x_r^{q_{r1}}$$

$$\vdots$$

$$f_k = x_1^{q_{1k}} \ldots x_r^{q_{rk}}.$$

Then

(2') $d(y + c_1 dy_1 + \ldots + c_r dy_r) = e_1 1/_{f_1} df_1 + \ldots + e_k 1/_{f_k} df_k$.

By Prop 4. of [6] we have

- $y + c_1 y_1 + \ldots + c_r y_r = g \in \overline{E} \cap F$
- $f_i \in \overline{E} \cap F \ \forall i$.

So $x x_1^{q_{11}} \ldots . x_r^{q_{r1}} \in \overline{E} \cap F$. But if $\forall i \ \ q_{i1} = m_{i1}/_m$, m_{i1}, $m \in Z$ we get $x^m x_1^{m_{11}} \ldots x_r^{m_{r1}} = f \in \overline{E} \cap F$, q.e.d.

Sometimes it is possible to give a complete description for the solution of $Y = \text{Log } X$. This happens when E is a classical differential field :

1.3.- Theorem. On the hypothesis of Theorem 1.2, supose moreover that $E = C(z)$, C the field of constants of E and $z \notin C$ such that $\forall D \in \Delta \ \ Dz \in C$. Then, any solution of the equality can be written in the form

$$y = c_1 y_1 + \ldots + c_r y_r + c$$

$$x = x_1^{c_1} \ldots x_r^{c_r} c' \ , \text{ being } c_1, \ldots, c_r \in Q, \ c, c' \in C.$$

Proof: applying the same argument used in 1.2 and taking the system

$$\begin{cases} Dz \in C \\ \forall i \ Dy_i - Dx_i/_{x_i} = 0 \in C \\ Dy - Dx/_x = 0 \in C \ \ \forall D \in \Delta \ , \end{cases}$$

we get there exist $q_1, \ldots, q_r \in Q$ such that

$$x x_1^{q_1} \ldots x_r^{q_r} \in \overline{C} \cap F.$$

But any derivation has only one extension for an algebraic extension of E ([8] Cap. 2, 17, Cor. 2). So $\overline{C} \cap F$ is a field of constants and given that $E \subset F$ is an extension with the same field of constants we have $C = \overline{C} \cap F$. Therefore

(1) $x = x_1^{q_1} \ldots x_r^{q_r} c'$, $c' \in C$. Deriving (1) yields

$\forall D \in \Delta$, $Dy = Dx/_x = D(x_1^{q_1} \ldots x_r^{q_r})/_{(x_1^{q_1} \ldots x_r^{q_r})} =$

$q_1 Dx_1/_{x_1} + \ldots + q_r Dx_r/_{x_r}$. So

$y = q_1 y_1 + \ldots + q_r y_r + c$, $c \in C$, q.e.d.

Remark: it can happen that $x_1^{q_1} \ldots x_r^{q_r} \notin F$. However, it is an algebraic point that doesn't disturb the elementarity of the process.

2.- *SOME CONSEQUENCES.*

2.A. The first conclusion we draw from 1. is that we'll name The Irredundance of Elementary Functions. This means that building up elementary extensions by means of algebraic elements, logarithm elements or exponential elements are completly independent processes: no one of them can be obtained from the others.

In order to set the problem we'll use an adecuate language; we say that the differential extension $E \subset F$ is Algebraic if the field extension $E \subset F$ so is; it is Logarithmic if $F = E(\Theta_1, \ldots, \Theta_n)$ such that $\Theta_1 = \mathrm{Log}\Psi_1$, $\Psi_1 \in E$, $\Theta_i = \mathrm{Log}\Psi_i$, $\Psi_i \in E(\Theta_1, \ldots, \Theta_{i-1})$ $\forall i \geq 2$. Changing Log by Exp we have an Exponential extension.

2.1.- Lemma. Let $E \subset F = E(\Theta)$ be a differential extension with the same field of constants C and $\Theta \notin E$.

(1) If $\forall D \in \Delta$ $D\Theta \in E$, then Θ is transcendental over E.

(2) If $\forall D \in \Delta$ $D\Theta/_\Theta \in E$, then Θ is algebraic over E if and only if there exists $n \in N$ such that $\Theta^n \in E$, and the irreducible polynomial of Θ over E is $X^n - \Theta^n$, n being the least of these naturals.

Proof: assume Θ to be algebraic over E and let $P(X) = X^n + a_1X^{n-1} + \ldots + a_{n-1}X + a_n$ be the irreducible polynomial of Θ. Then

$$(*)\quad \Theta^n + a_1\Theta^{n-1} + \ldots + a_{n-1}\Theta + a_n = 0.$$

(1) Deriving (*) we get $\forall D\in\Delta$, $(Da_1 + nD\Theta)\Theta^{n-1} + \ldots = 0$. Given that P(X) is the irreducible polynomial of Θ over E we have that $\forall D\in\Delta$ $Da_1 + nD\Theta = 0$. So $\forall D\in\Delta$ $D\Theta = D(-a_1/n)$ and $\Theta + a_1/n$ is a constant. Due to $E\subset F$ is with the same field of constants we get $\Theta\in E$, not possible.

(2) Now it suffices to prove that $\Theta^n\in E$. Deriving (*) we get $\forall D\in\Delta$ $nD\Theta/_{\Theta}\Theta^n + (Da_1 + (n-1)D\Theta/\Theta)\Theta^{n-1} + \ldots + Da_n = 0$. But $a_n \neq 0$, so $Da_n = nD\Theta/_{\Theta}a_n$ $\forall D\in\Delta$. Hence $Da_n/a_n = nD\Theta/_{\Theta} \Rightarrow Da_n/a_n = D\Theta^n/_{\Theta}n \Rightarrow D(a_n/_{\Theta}n) = 0$ $\forall D\in\Delta$. So $a_n/_{\Theta}n\in C\subset E$, and $\Theta^n\in E$, q.e.d.

2.2.- Theorem. Let E be a differential field with field of constants C. Let $E\subset F = E(\Theta_1, \ldots, \Theta_r)$ be an elemental differential extension with the same field of constants. Then:

(a) When F is Logarithmic, $E\subset F$ is a purely transcendental extension. If $E\subset F$ is Exponential, $E\subset F$ is purely transcendental unless there exist $n_1, \ldots, n_r\in Z$ such that $\Theta_1^{n_1}\ldots\Theta_r^{n_r}\in E$.

Let $x\in E$.

(b) The equality $Y = \mathrm{Log}(x)$ never holds in F-E if $E\subset F$ is Algebraic or Exponential.

(c) The equality $Y = \mathrm{Exp}(x)$ never holds in F-E if $E\subset F$ is Logarithmic, and if there is a solution when $E\subset F$ is Algebraic then there exists $n\in N$ such that $y^n\in E$.

Moreover, if $E = C(z)$, $z\notin C$, $Dz\in C$ $\forall D\in\Delta$ being C the field of constants of E, C algebraically closed, then there are not exceptions for the case (c).

Proof: (a) The stament is an easy consequence of Lemma 2.1 for the Logarithmic case. Assume that $E\subset F$ is Exponential and not purely transcendental extension. By Lemma 2.1 there exists Θ_s, $p\in N$ such that $\Theta_s^p\in E(\Theta_1, \ldots, \Theta_{s-1})$. Let Θ_k the first of them, that is, $\Theta_1, \ldots, \Theta_{k-1}$ are algebraic independent over E and $\Theta_k^p\in E(\Theta_1, \ldots, \Theta_{k-1})$. Then by Theorem 1.2 we get the stament.

(b) Let y be a solution. Then, $\forall D\in\Delta\ Dy = Dx/x$. Since $x\in E$ we can take the differential extension $E\subset E(y)$. By 2.1 y is not algebraic over E. Suposse now $E\subset F$ is Exponential. Then, by 1.2 we get there exist $c_{i_1}, \ldots, c_{i_k}\in C$, $n, n_{i_1}, \ldots, n_{i_k}\in Z$ such that

$$y + c_{i_1}\Psi_{i_1} + \ldots + c_{i_k}\Psi_{i_k}\in\bar{E}\cap F,\quad x^n\theta_{i_1}^{n_{i_1}}\ldots\ \theta_{i_k}^{n_{i_k}}\in\bar{E}\cap F,$$

being $\theta_{i_1}, \ldots, \theta_{i_k}$ a maximal algebraically independent system over E among $\theta_1, \ldots, \theta_r$ like in 1.D.

But $x\in E$: so $n_{i_1}, \ldots, n_{i_k}$ are 0, and looking in 1.2 for the construction of these naturals we have $c_{i_1} = \ldots = c_{i_k} = 0$. Hence $y\in\bar{E}\cap F$, not possible as we have proved above.

(c) The Lemma 2.1 assure us that if y is algebraic over E then there exist $n\in N$ such that $y^n\in E$. Suposse $E\subset F$ is Logarithmic. By 1.2 we have there exist $n_1, \ldots, n_r, n\in Z$, $c_1, \ldots, c_r\in C$ such that

$$x + c_1\theta_i + \ldots + c_r\theta_r\in\bar{E}\cap F\ ,\quad y^n\Psi_1^{n_1}\ldots\ \Psi_r^{n_r}\in\bar{E}\cap F.$$

Now, by Lemma 2.1, $\theta_1, \ldots, \theta_r$ is an algebraically independent system over E, so $c_1 = \ldots = c_r = 0$, and $n_1 = \ldots = n_r = 0$ (look for the construction of $n_1, \ldots, n_r$ in 1.2). Hence $y\in\bar{E}\cap F = E$, not possible.

On the assumtion that $E = C(z)$ ..... , the stament is consequence of aplying Theorem 1.3 to $y^n\in E = C(z)$.

Remark: an example that give us an exception for (a) is:

$E = \mathbb{C}(z)(\mathrm{Exp}(2z+2z^2))$, $F = E(\mathrm{Exp}z, \mathrm{Exp}z^2)$. Then, $\mathrm{Exp}(z+z^2)\in F-E$ and is algebraic over E.

2.B. The question of whether some transcendental equations can be solved by means of elementary functions sometimes can be answered using the Structure theorem 1.2. Let us see two classical examples:

assume $E\subset F$ is a diffrential extension with the same field of constants. Let C be this field and $E = C(z)$ such that $\forall D\in\Delta\ Dz\in C$, $z\notin C$. Suposse C is algebraically closed and $E\subset F$ Elementary.

Consider the equation $\alpha Y = \mathrm{Log}(\beta Y)\quad \alpha, \beta\in E.$

Suposse there is a solution in F, y. Using the same notation of 1.3 we get

$$y = cx_1^{c_1} \dots x_n^{c_n}, \quad y = \bar{c} + c_1 y_1 + \dots + c_n y_n,$$

$c, \bar{c} \in C$, $c_1, \dots, c_n \in Q$. Passing to the Module of differentials, $\Omega_{F/E}$, we have

$$\alpha'((x_1^{c_1} \dots x_n^{c_n}) c_1 1/_{x_1} dx_1 + \dots + (x_1^{c_1} \dots x_n^{c_n}) c_n 1/_{x_n} dx_n) =$$

$$= c_1 dy_1 + \dots + c_n dy_n, \quad \alpha' = \alpha c/\beta.$$

But taking the Module of differentials respect on the penultimate subextension not algebraic and taking into account 1.C we have that

$$\alpha'(x_1^{c_1} \dots x_n^{c_n}) c_n 1/_{x_n} dx_n = c_n dy_n$$, where $dx_n = 0$ or $dy_n = 0$ because of the elementarity of $E \subset F$, being one of them not zero. Therefore $c_n = 0$; repeating this argument we have $c_i = 0 \ \forall i$. Consequently, any solution is trivial.

As a particular case and taking $E = \mathbb{C}(z)$ we have that the equation $\underline{\mathrm{Log}(Y) = Y/_z}$ has not solution by means of elementary functions.

With the same hypotesis consider now the equation

(*) $\underline{Y + \alpha = \beta \mathrm{Exp}(\gamma Y) + \beta' \mathrm{Exp}(-\gamma Y), \quad \alpha, \beta, \beta', \gamma \in E.}$

Let y be a solution, $y \in F$. We can suposse also that $\mathrm{Exp}(\gamma y) \in F$. Then by 1.3 and with the same notation we have

(**) $\gamma y = c + c_1 y_1 + \dots + c_n y_n$, $c \in C$, $c_1, \dots, c_n \in Q$.

Substuing for γy in (*) we get that

$$y + \alpha = \beta x_1^{c_1} \dots x_n^{c_n} + \beta' x_1^{-c_1} \dots x_n^{-c_n}$$ (where we have operated adequatly β, β').

Passing now to the Module of differentials $\Omega_{F/E}$ we get

$$dy = \sum_i \beta(x_1^{c_1} \dots x_n^{c_n}) c_i 1/_{x_i} dx_i + \sum_i \beta'(x_1^{-c_1} \dots x_n^{-c_n})(-c_i) 1/_{x_i} dx_i .$$

But taking into account (**) we have that

$$1/_\gamma(c_1 dy_1 + \dots + c_n dy_n) = \sum_i \beta(x_1^{c_1} \dots x_n^{c_n}) c_i 1/_{x_i} dx_i + \sum_i \beta'(x_1^{-c_1} \dots x_n^{-c_n})$$
$$(-c_i) 1/_{x_i} dx_i .$$

If as above we take now the Module of differentials respect on the penultimate subexetension not algebraic we get only

$$c_n{}^1/_\gamma dy_n = \beta(x_1^{c_1}\dots x_n^{c_n})c_n{}^1/_{x_n}dx_n - \beta'(x_1^{-c_1}\dots x_n^{-c_n})c_n{}^1/_{x_n}dx_n .$$

It follows from the elementarity of $E\subset F$ that either dx_n or $dy_n = 0$, one of them being not zero. Then

- $dx_n = 0 \Rightarrow c_n = 0.$
- $dy_n = 0 \Rightarrow$ either $c_n = 0$ or $x_1^{2c_1}\dots x_n^{2c_n} = \beta'/_\beta \in E.$

The last equality can hold in F, but if we assum that $\beta, \beta', \gamma\in C$ then $x_1^{2c_1}\dots x_n^{2c_n}\in C \Rightarrow \gamma y\in C \Rightarrow y\in C$. Repeating this argument we conclude that if $\beta, \beta', \gamma\in C$ any solution of (*) is constant, that is, trivial.

Taking $E = \mathbb{C}(z)$, $\alpha = -z$, $\beta' = -\beta = -h/2i$, $\gamma = i$ we get that the equation $Y = z + h\sin(Y)$, $h\in\mathbb{C}$ (Kepler's equation) has not a solution by means of elementary functions.

2.C As a final aplication of 1.1 we give a result of Ostrowski proved in [4]; this is an example of how the methods purposed by Rosenlicht simplifie the arguments. The result permet, under certain conditions, to transform algebraic relations into linear relations.

2.3.-Proposition. Let $E\subset F$ be a differential extension with the same field of constants, C. Let $y_1, \dots, y_n$ be elements of F such that $\forall D\in\Delta$ $Dy_i\in E$ $\forall i$. Then, if $y_1, \dots, y_n$ are algebraically dependent over E there are $c_1, \dots, c_n\in C$ such that $c_1y_1 + \dots + c_ny_n\in E$.

Proof: given that $y_1, \dots, y_n$ are algebraically dependent over E, by 1.1 we get that $dy_1, \dots, dy_n \in\Omega_{F/E}$ are C-lineary dependent. So there exist $c_1, \dots, c_n\in C$ such that $c_1dy_1 + \dots + c_ndy_n = 0$. So $d(c_1y_1 + \dots + c_ny_n) = 0$ and by 1.C $c_1y_1 + \dots + c_ny_n\in\widetilde{E}\subset F$. But $D(c_1y_1 + \dots + c_ny_n)\in E$ (hyp), so by Lemma 2.1 we get that $c_1y_1 + \dots + c_ny_n\in E$ q.e.d..

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