Quantum-to-Classical Crossover in Full Counting Statistics

Eugene V. Sukhorukov[1] and Oleg M. Bulashenko[2]
[1]Département de Physique Théorique, Université de Genève, CH-1211 Genève 4, Switzerland
[2]Departament de Física Fonamental, Universitat de Barcelona, Diagonal 647, E-08028 Barcelona, Spain
(Received 3 August 2004; published 23 March 2005)

The reduction of quantum scattering leads to the suppression of shot noise. In this Letter, we analyze the crossover from the quantum transport regime with universal shot noise to the classical regime where noise vanishes. By making use of the stochastic path integral approach, we find the statistics of transport and the transmission properties of a chaotic cavity as a function of a system parameter controlling the crossover. We identify three different scenarios of the crossover.

DOI: 10.1103/PhysRevLett.94.116803 PACS numbers: 73.23.–b, 05.45.Mt, 73.50.Td, 74.40.+k

Random transfer of charge in electrical conductors leads to time-dependent fluctuations of the current, a phenomenon called shot noise. In recent years, the shot noise has been extensively studied in mesoscopic conductors [1], small degenerate electron systems of a size comparable to the coherence length of electrons. In contrast to the classical shot noise in vacuum tubes, which was explained by Schottky already in 1918 [2], the shot noise in mesoscopic conductors originates from the quantum-mechanical scattering of electrons. Consequently, in a noninteracting mesoscopic conductor biased by the chemical potential difference $\Delta\mu$, the average current $\langle I\rangle = \Delta\mu\sum_n T_n$ [3] (setting electron charge and the Planck constant $e = h = 1$), the noise power $S \equiv \langle\langle I^2\rangle\rangle = \Delta\mu\sum_n T_n(1-T_n)$ at zero temperature [1], and, in general, the higher cumulants of current $\langle\langle I^m\rangle\rangle$ [4] are determined by the transmission matrix $\hat{t}$, namely, by the eigenvalues T_n, $n = 1, \ldots, N$, of $\hat{t}^\dagger\hat{t}$. The current cumulants $\langle\langle I^m\rangle\rangle = \Delta\mu N C_m$ can be expressed via the cumulant generating function (CGF) $C_m = \partial^m \mathcal{H}(\lambda)/\partial\lambda^m|_{\lambda=0}$. In the semiclassical limit, $N \gg 1$, the CGF is given by [5]

$$\mathcal{H}(\lambda) = \int_0^1 dT\rho(T)\ln[1 + T(e^\lambda - 1)], \qquad (1)$$

where $\rho(T) = N^{-1}\sum_n\delta(T - T_n)$ is the transmission eigenvalue distribution. Equation (1) generalizes the binomial statistics, and together with the inverse formula (12), provides a connection between the full counting statistics (FCS) and the scattering properties of a mesoscopic system to leading order in $1/N$.

The quantum origin of shot noise in mesoscopic conductors implies that, regardless of the character of disorder, current can flow without noise if quantum scattering is suppressed [6]. Therefore, in the classical limit the noise should vanish even in a chaotic system, such as a mesoscopic cavity, where the transport in the quantum regime is universally described by random matrix theory (RMT) [7]. It has been predicted [8] that in a mesoscopic cavity with a long-range disorder the noise power shows an exponential crossover $S = S_{\mathrm{RMT}}\exp(-\tau_{\mathrm{E}}/\tau_D)$ as a function of the ratio of the Ehrenfest (diffraction) time τ_{E} to the average dwell time of electrons τ_D. Reference [9] suggested that this crossover results from a sharp cutoff introduced by the Ehrenfest time in the exponential distribution $\mathcal{P}(t) = \tau_D^{-1}\exp(-t/\tau_D)$ of the dwell times of classical trajectories. Recent numerical analysis [10] has demonstrated that the cutoff leads to a complete separation of the cavity's phase space into the quantum universal part of relative volume $v = \exp(-\tau_{\mathrm{E}}/\tau_D)$ and the classical noiseless part of the volume $1 - v$. As a result, the eigenvalue distribution splits into two terms, $\rho = v\rho_{\mathrm{RMT}} + (1 - v)\rho_{\mathrm{cl}}$, where

$$\rho_{\mathrm{RMT}}(T) = \frac{1}{\pi\sqrt{T(1-T)}} \qquad (2)$$

is the universal RMT result, and $\rho_{\mathrm{cl}}(T) = [\delta(T) + \delta(1 - T)]/2$ is the classical distribution. The onset of the quantum-to-classical crossover has been observed in the experiment on the shot noise of a mesoscopic cavity [11]. Since then interest in the physics of the crossover has grown dramatically and brought new results in the context of the shot noise suppression [9,12], the proximity effect in Andreev billiards [13], mesoscopic conductance fluctuations [10,14], and many other phenomena.

In this Letter, we demonstrate that the presence of the homogeneous short-range disorder in a chaotic cavity dramatically changes the quantum-to-classical crossover. It leads to the large-angle quantum scattering of electrons and results in the relaxation of the deterministic occupation function $f_{\mathbf{p}}$, which takes values 0 and 1, to its fully quantum isotropic value $f_C < 1$. The relaxation with the constant rate τ_Q^{-1}, where τ_Q is the quantum scattering time, does not introduce a sharp cut-off in the dwell time distribution. As a result, all cumulants have a power-law dependence on the crossover parameter $\gamma = \tau_Q/\tau_D$. In particular, in contrast to the case of the long-range disorder discussed above, the noise power shows the power-law crossover [15]

$$S = S_{\mathrm{RMT}}/(1 + \gamma), \qquad \gamma = \tau_Q/\tau_D. \qquad (3)$$

The distribution $\rho(T)$ gradually evolves as a function of the parameter γ from its RMT limit (2) in the quantum regime to the classical limit $\rho_{\rm cl}$ with two δ peaks [16].

The model.—The mesoscopic chaotic cavity is a metallic island connected to the leads L and R through ballistic point contacts. This system has several characteristic time scales. The time of the ballistic flight of electrons across the cavity $\tau_{\rm F}$ is much shorter than the average dwell time $\tau_D = n_{\rm F}/N$, where $n_{\rm F}$ is the density of states in the cavity at Fermi level, and N is the number of modes in each point contact (symmetric cavity). We consider the quasiballistic regime, $\tau_Q \geq \tau_{\rm F}$, where τ_Q is the time of *quantum* scattering off a short-range disorder, and neglect inelastic processes. The temperature is smaller than the bias and set here to zero.

In the semiclassical limit considered here, $N \gg 1$ and $\Delta\mu n_{\rm F} \gg 1$, the transport to leading order in number of modes N can be described classically. We use the classical approach of Refs. [17,18] based on the principle of minimal correlations [19]. According to this principle, the currents through the left and right point contacts $I_{L,R}$ are considered to be noise sources which are correlated solely via the conservation of the charge $\Delta\mu n_{\rm F} f_C$ in the cavity. Here $f_C = \langle f_{\mathbf{p}}\rangle$ is the isotropic part of the occupation function $f_{\mathbf{p}}$, and $\langle\cdots\rangle$ denotes the averaging over the momentum on the Fermi surface. The statistics of sources can be obtained by taking into account the fermionic statistics of electrons, which leads to binomial fluctuations of the occupation function in each semiclassical state with the cumulant generator $\ln[1 + f_{\mathbf{p}}(e^{\lambda} - 1)]$. Multiplying this function by the electron velocity $\mathbf{v}$, summing over $\mathbf{p}$, and integrating over the area of the contacts, we obtain the generators $\Delta\mu N \mathcal{H}_{L,R}$ of the left and right currents

$$\mathcal{H}_l(\lambda_l, f_C) = \langle \ln[1 + f_{\mathbf{p}}(e^{\lambda_l} - 1)]\rangle - f_l\lambda_l, \qquad l = L, R, \tag{4}$$

where $f_L = 1$ and $f_R = 0$ are the occupations in the left and right leads. This expression is the semiclassical limit of the result of Ref. [4]. The charge conservation can be taken into account nonperturbatively in fluctuations δf_C using the stochastic path integral [17,18]. In the stationary limit $t \gg \tau_D$, the saddle-point evaluation amounts to the minimization of the function

$$\mathcal{H} = \mathcal{H}_L(\lambda_C - \lambda/2, f_C) + \mathcal{H}_R(\lambda_C + \lambda/2, f_C) \tag{5}$$

with respect to the occupation f_C and variable λ_C, a Lagrange multiplier conserving charge. The result of this procedure gives the CGF (1).

Counting statistics.—The crossover from the classical to the quantum transport regime may be viewed as being caused by the relaxation of the classical occupation $f_{\mathbf{p}} = 0, 1$ to the quantum isotropic value f_C as a result of scattering off the short-range disorder. This process can be described by the Boltzmann equation

$$\mathbf{v}\nabla f_{\mathbf{p}} + \tau_Q^{-1}(f_{\mathbf{p}} - f_C) = 0, \tag{6}$$

where the classical chaotic dynamics of electrons is taken into account by the "gradient" term and the quantum scattering is described by the second, the collision integral in the scattering time approximation. In the classical limit $\gamma = \tau_Q/\tau_D \gg 1$ the second term can be neglected, and the solution of Eq. (6) takes one of the boundary values $f_{L,R} = 0, 1$ giving $\mathcal{H}_l = (f_C - f_l)\lambda_l$. Then the minimization of the function (5) leads to $\mathcal{H} = \lambda/2$ giving the average current $\langle I\rangle = \Delta\mu N/2$ and no noise. In the quantum limit $\tau_Q/\tau_D \ll 1$ the second term in Eq. (6) dominates; therefore, in Eq. (4) $f_{\mathbf{p}}$ may be replaced with f_C. Minimizing $\mathcal{H}$ given by (5), we obtain the result [17]

$$\mathcal{H} = 2\ln(1 + e^{\lambda/2}) - 2\ln 2, \qquad \gamma = 0, \tag{7}$$

which agrees with the RMT result [20].

In order to obtain the coarse-grained value of the logarithm in Eq. (4) we multiply Eq. (6) by $(f_{\mathbf{p}} - f_C)^{k-1}$ and integrate the resulting equation

$$\nabla[\mathbf{v}(f_{\mathbf{p}} - f_C)^k] + (k/\tau_Q)(f_{\mathbf{p}} - f_C)^k = 0, \tag{8}$$

over the phase space $(\mathbf{p}, \mathbf{r})$ of the cavity. Using the identity $\int d\mathbf{r}\nabla[\mathbf{v}(\ldots)] = \int d\mathbf{s}\mathbf{v}(\ldots)$, we reduce the volume integral in the first term to the surface integral over the cavity openings and arrive at the following expression:

$$\langle (f_{\mathbf{p}} - f_C)^k\rangle = \frac{\gamma}{k + 2\gamma}[(1 - f_C)^k + (-f_C)^k]. \tag{9}$$

Expanding the logarithm in Eq. (4) in powers of $f_{\mathbf{p}} - f_C$, using the result (9) and resumming the logarithm, we obtain the integral representation

$$\mathcal{H}_l(\lambda_l) = -f_l\lambda_l + \gamma\int_0^1 du u^{2\gamma-1}(\ln\{1 + [u + (1 - u)f_C] \times (e^{\lambda_l} - 1)\} + \ln\{1 + [(1 - u)f_C](e^{\lambda_l} - 1)\}). \tag{10}$$

This expression has to be substituted into the variation function (5). Surprisingly, the stationary point is given by $\lambda_C = 0$ and $f_C = 1/2$ independent of γ, implying the absence of cascade corrections [21] to the FCS. Evaluating Eq. (10) at the stationary point, we obtain the current generator for a symmetric cavity as a function of the crossover parameter γ:

$$\mathcal{H}(\lambda, \gamma) = \lambda/2 + 2\int_0^1 du \frac{u^{2\gamma+1}}{\coth^2(\lambda/4) - u^2}. \tag{11}$$

This equation is one of our main results. It correctly reproduces the quantum limit (7) at $\gamma \to 0$ and has an asymptotic form $\mathcal{H} = \lambda/2 + \gamma^{-1}\sinh^2(\lambda/4) + O(\gamma^{-2})$ in the classical limit $\gamma \to \infty$ [valid for $\lambda \lesssim \ln(\gamma)$]. Thus the crossover has a power-law character in contrast to the case of a long-range disorder: High cumulants are suppressed as $\sim\gamma^{-1}$ in the classical limit.

Normalized current cumulants $C_m = \langle\langle I^m\rangle\rangle/(\Delta\mu N)$ may be obtained by differentiating the CGF (11) with respect to λ. Odd cumulants vanish $C_m = 0$ for $m \geq 3$ as a consequence of the zero temperature limit and of the fact that the cavity is symmetric. The first three nonvanishing cumulants are $C_1 = 1/2$, which determines the mean current, $C_2 = 1/[8(\gamma + 1)]$, which determines the noise power and agrees with the result (3), and $C_4 = (\gamma - 1)/[32(\gamma + 1)(\gamma + 2)]$.

The logarithm of the distribution of transmitted charge in the stationary phase approximation is given by $\ln P(Q) = Q_0 \min_\lambda\{\mathcal{H}(\lambda) - \mathcal{Q}\lambda\}$ [18], where $\mathcal{Q} = Q/Q_0$ is the transmitted charge normalized to its maximum value $Q_0 = \Delta\mu N t$. The result of the evaluation using Eq. (11) is shown in Fig. 1 for different values of γ. In the quantum limit, we use Eq. (7) to obtain $\ln P(Q)/Q_0 = -2\ln 2 - 2[\mathcal{Q}\ln\mathcal{Q} + (1-\mathcal{Q})\ln(1-\mathcal{Q})]$, which vanishes at the average value of charge $\mathcal{Q} = 1/2$, giving the correct normalization of $P(Q)$. In the classical limit $\gamma \gg 1$ the noise is Gaussian, $\ln P(Q)/Q_0 = -4\gamma(\mathcal{Q} - 1/2)^2$, for $|\mathcal{Q} - 1/2| \lesssim \gamma^{-1}$. Surprisingly, the extreme value distribution in the range $\gamma^{-1} \lesssim |\mathcal{Q} - 1/2| \lesssim 1/2$ shows a weak γ dependence: $\ln P(Q)/Q_0 = -2|\mathcal{Q} - 1/2|\{\ln(8\gamma|\mathcal{Q} - 1/2|) - 1\}$; see Fig. 1. This remarkable behavior may be attributed to the formation of almost open (closed) quantum channels, the situation specific to the short-range disorder considered here. The number of such channels is nearly independent of γ and close to the total number of modes N (see the discussion below).

Distribution of transmission eigenvalues.—Having found the CGF, we now invert Eq. (1) in order to obtain the distribution of transmission eigenvalues $\rho(T)$. We note that $\mathcal{H}$ as a function of the variable $\Lambda \equiv e^\lambda - 1$ has a branch cut in the complex plane at $-\infty < \Lambda < -1$. Analytically continuing from $\Lambda > 1$ to the branch cut [22], we obtain

$$\rho(T) = \frac{1}{\pi T^2}\,\mathrm{Im}(\partial\mathcal{H}/\partial\Lambda|_{\Lambda\to -1/T - i0}). \tag{12}$$

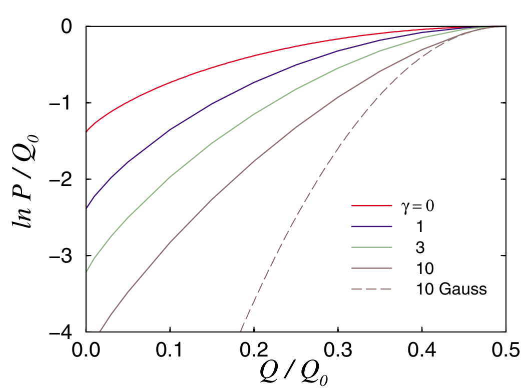

FIG. 1 (color online). The logarithm of the distribution of transmitted charge Q plotted versus charge normalized to its maximum value $Q_0 = \Delta\mu N t$. It is symmetric around the average value $Q/Q_0 = 0.5$. Note a relatively weak dependence of the extreme value statistics on the crossover parameter $\gamma = \tau_Q/\tau_D$. The dashed line is the Gaussian distribution shown for a comparison.

Using this relation together with Eq. (11), we arrive at the following result for $\rho(T)$ in the crossover regime:

$$\rho(T) = \frac{\gamma}{\pi\sqrt{T(1-T)}}\int_{-1}^{1} du\,\frac{(1-u^2)|u|^{2\gamma-1}}{(1+u)^2 - 4Tu}. \tag{13}$$

The distribution is symmetric with respect to $T \to 1 - T$ and properly normalized: one can verify that $\int_0^1 dT\rho(T) = 1$. In the quantum limit $\gamma \to 0$, Eq. (13) leads to the RMT result (2). In the classical limit we obtain the asymptotic formula

$$\rho(T)|_{\gamma\to\infty} = \frac{1}{8\pi\gamma[T(1-T)]^{3/2}} + O(\gamma^{-2}), \tag{14}$$

which is valid away from the points $T = 0$ and $T = 1$ where $\rho(T)$ in (14) is divergent. This divergence, being cut at $T, 1 - T \sim \gamma^{-2}$ in Eq. (13), gives the main contribution to the normalization of $\rho(T)$, as well as to the average current, and determines the extreme value statistics discussed above. However, it is integrable for high cumulants of current.

The distribution $\rho(T)$ for several values of γ is illustrated in Fig. 2. The quantum-to-classical crossover appears as a gradual transition from the RMT distribution at $\gamma = 0$ to two δ functions at $T = 0$ and $T = 1$. Following Ref. [10] we plot the integrated distribution $\mathcal{I}(T) = \int_0^T \rho(T')dT'$, which turns out to be a smooth function of T. This is in contrast to the case of a long-range disorder, where $\mathcal{I}(T)$ shows an offset at $T = 0$ [10], indicating the separation of phase space into a classical and a quantum part. Our result implies that such a separation does not occur in the case of a homogeneous short-range disorder.

Inhomogeneous disorder.—So far we have considered a relatively weak homogeneous disorder with the strength characterized by the scattering time τ_Q. Another experimentally relevant situation is the case of a strong inhomogeneous short-range disorder. For instance, a few strong impurities, sharp openings to the leads or irregularities at

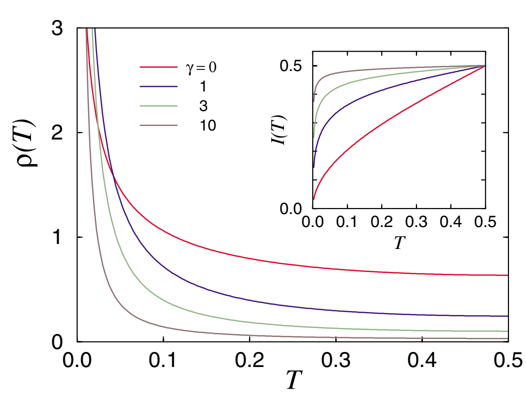

FIG. 2 (color online). The crossover of the distribution of transmission eigenvalues $\rho(T)$ between the quantum ($\gamma = 0$) and classical ($\gamma \to \infty$) regimes. $\rho(T)$ is symmetric around $T = 0.5$. Inset: Integrated probability distribution $\mathcal{I}(T) = \int_0^T \rho(T')dT'$ for the same set of parameters.

the boundary of the cavity belong to this class of disorder. The inhomogeneity implies that some trajectories do not enter the disordered region and remain classical with $f_{\mathbf{p}} = 0, 1$. The fact that the disorder is strong means that all trajectories entering the disordered region acquire the isotropic occupation f_C. For the coarse-grained occupation function $\langle f_{\mathbf{p}} \rangle$ this leads to the relaxation with the collision rate $\tau_{\rm imp}$. This process is described by Eq. (6) with τ_Q replaced by $\tau_{\rm imp}$. In the present case the solution of this equation determines the relative volume of the quantum phase space $v = 1/(1+\gamma)$, where $\gamma = \tau_{\rm imp}/\tau_D$ is the new crossover parameter. Therefore we conclude that inhomogeneous strong disorder leads to the complete separation of the phase space on the classical and quantum parts with the consequence that $\rho = v\rho_Q + (1-v)\rho_{\rm cl}$, where ρ_Q is a quantum (nonuniversal) distribution. However, in contrast to the case of long-range disorder, the FCS is a power-law function of the crossover parameter: $C_n \sim 1/\gamma$.

Asymmetric cavity.—From the above analysis it follows that the noise power has the same dependence (3) on the crossover parameter γ for both types of a short-range disorder. Moreover, since the odd cumulants of current for the symmetric cavity vanish at zero temperature, the difference in noise appears starting from the fourth cumulant. The recent progress in the measurement of a third-order cumulant [23] motivates us to analyze the counting statistics for asymmetric cavities with a nonequal number of modes N_L and N_R in the point contacts, for which odd cumulants are expected to be finite [24].

To obtain the third cumulant for the case of a homogeneous disorder, we utilize the operator approach of Ref. [18], which in the case of a single variable represents a convenient alternative to the cascade diagrammatics [18,21]. Omitting lengthy calculations we present the result

$$C_3(\gamma) = \frac{3C_3(0)}{(1+\gamma)(3+2\gamma)}, \tag{15}$$

where $C_3(0) = -2[N_L N_R(N_L - N_R)/(N_L + N_R)^3]^2$ is the RMT value of the third cumulant. We note that the third cumulant (15) vanishes as γ^{-2} in the classical limit, i.e., faster than the one for the inhomogeneous disorder. Therefore the measurement of the third cumulant may help to distinguish the character of disorder.

In conclusion, we have analyzed the FCS and transmission properties of a mesoscopic cavity at the crossover from the universal quantum to the classical transport regimes. We have found new different scenarios of the crossover in a cavity with short-range disorder. In case of homogeneous disorder, the crossover occurs via the formation of almost open (closed) quantum channels, which determine the extreme value statistics. In the case of an inhomogeneous strong disorder, the phase space of the cavity splits into two parts: classical noiseless channels and quantum channels. In both cases the FCS has a power-law dependence on the crossover parameter.

We thank Ph. Jacquod for helpful discussions. This work was supported by the SNF, and by INTAS (Project No. 0014, open call 2001). O. M. B. acknowledges support by the MCyT of Spain through the "Ramón y Cajal" program.

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