Supertubes

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It is shown that a IIA superstring carrying D0-brane charge can be “blown up,” in a Minkowski vacuum background, to a (1/4)-supersymmetric tubular D2-brane, supported against collapse by the angular momentum generated by crossed electric and magnetic Born-Infeld fields. This “supertube” can be viewed as a world-volume realization of the sigma-model Q lump.

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(1) Introduction.—M theory can be viewed as an extension of string theory to a “democratic” theory of branes, with dualities that allow the whole theory to be constructed from any brane. One aspect of this is that a given brane may often be considered as a partially collapsed version of another, higher-dimensional, brane. Conversely, there may be circumstances in which the lower-dimensional brane is “blown up” into the higher-dimensional brane. One example of this is the observation that a IIA superstring can be blown up to a tubular D2-brane by placing it in an appropriate (nonsupersymmetric) background [1]; the background fields impose an external force that prevents the collapse of the D2-brane. Another example is the observation that a similar nontrivial background can blow up a collection of D0-branes into a fuzzy 2-sphere [2]; this can again be considered as a D2-brane prevented from collapse by an external force [3]. Another way to support a brane against collapse to a lower-dimensional one is to allow it to carry angular momentum. The case of branes on spheres, first analyzed in [4], has recently found an application to “giant gravitons” [5]; these partially preserve the supersymmetry of an AdS background, to a (1/4)-supersymmetric tubular D2-brane, supported against collapse by the angular momentum generated by crossed electric and magnetic Born-Infeld fields. This “supertube” can be viewed as a world-volume realization of the sigma-model Q lump.

At first sight it would appear unlikely that a brane configuration stabilized by angular momentum could be supersymmetric. For a start, supersymmetry requires a time-independent energy density, which is certainly non-generic for configurations with nonzero angular momentum. Even if this can be arranged, there is still the fact that a supersymmetric configuration in Minkowski space minimizes the energy subject to fixed values of the central charges appearing in the supersymmetry algebra, but angular momentum is not one of these charges. However, although these considerations may make supersymmetric stabilization by angular momentum unlikely, they do not make it impossible. Indeed, in the somewhat different context of field theory solitons there are known examples of supersymmetric solitons that are stabilized by angular momentum. One such example is the Q lump of certain massive D = 3 supersymmetric sigma models [7], and the D2-brane supertube is essentially its world-volume realization.

The Q lump saturates an energy bound of the form $M = |L| + |Q|$, where L is a topological (lump) charge and Q is a Noether charge [8]. When viewed as a string-like solution of the maximally supersymmetric $D = 4$ sigma model, it preserves one-fourth of the supersymmetry of the sigma-model vacuum [9]. The simplest Q lump string has cylindrical symmetry [10], but for large Q its energy density is not concentrated on the axis of symmetry but rather in a hollow tube of radius Q. Therefore the Q lump-string tube can be interpreted, for large Q, as a lump string that has been blown up into a cylindrical tube of kink-2-brane [11]. This tube is prevented from collapse by a centrifugal force, generated by an angular momentum proportional to the charge Q. Because the effective action for the kink-2-brane is a Dirac-Born-Infeld (DBI) action [9], there should be a world-volume description of the blown-up sigma-model lump string as a (1/4)-supersymmetric solution of the $D = 4$ DBI equations. Any such solution would also solve the DBI equations of the $D = 10$ IIA D2-brane (in a IIA Minkowski vacuum) and must preserve at least two supersymmetries (this being one-fourth of the eight supersymmetries of the sigma-model vacuum). This solution is precisely the D2-brane...
supertube to be discussed below; it actually preserves eight of the thirty-two supersymmetries of the IIA vacuum and is therefore still one-fourth supersymmetric.

(II) Energetics of the supertube.—The D2-brane Lagrangian, for unit surface tension, is
\[ \mathcal{L} = -\Delta = -\sqrt{\det(g + F)}, \]
where \( g \) is the induced world-volume 3-metric and \( F \) is the BI 2-form field strength. We shall choose spacetime coordinates such that the \( \Phi_1 = 10 \) Minkowski metric is
\[ ds^2 = -dt^2 + dx^2 + dR^2 + R^2 d\Phi^2 + ds^2(E^6), \]
with \( \Phi \sim \Phi + 1 \). If we take the world-volume coordinates to be \((t, x, \Phi)\) with \( \Phi \sim \Phi + 1 \), then we may fix the world-volume diffeomorphisms for a D2-brane of cylindrical topology by the “physical” gauge choice
\[ T = t, \quad X = x, \quad \Phi = \varphi. \]
For a cylindrical D2-brane of (possibly varying, but time-independent) radius \( R(x, \varphi) \) at a fixed position in \( E^6 \) (and with the \( X \) axis as the axis of symmetry), the induced metric is
\[ ds^2(g) = -dt^2 + dx^2 + R^2 d\varphi^2 + (R_x dx + R_\varphi d\varphi)^2, \]
where \( R_x = \partial_x R \) and \( R_\varphi = \partial_\varphi R \). We will allow for a time-independent electric field \( E \) in the \( x \) direction, and a time-independent magnetic field \( B \), so that the BI 2-form field strength is
\[ F = Edt \wedge dx + Bdx \wedge d\varphi. \]
Under these conditions the Lagrangian becomes
\[ \mathcal{L} = -\sqrt{(R_x^2 + R_\varphi^2)(1 - E^2) + B^2 + R^2 R_\varphi^2}. \]
The corresponding Hamiltonian density is defined as
\[ \mathcal{H} = \Pi E - \mathcal{L}, \]
where \( \Pi = \partial \mathcal{L} / \partial \dot{E} \) is the “electric displacement” subject to the Gauss law constraint \( \partial_\varphi \Pi = 0 \).

Let us now focus on the D2-brane supertube, for which \( R \) is constant. In this case, the relation between the electric field and the electric displacement takes the form
\[ E = \frac{\Pi}{R} \sqrt{\frac{B^2 + R_\varphi^2}{\Pi^2 + R^2}}, \]
so the Hamiltonian density becomes
\[ \mathcal{H} = R^{-1} \sqrt{(\Pi^2 + R_\varphi^2)(B^2 + R^2)}, \]
where it should be noted that \( B = \nabla \times A \) for BI 2-vector potential \( A \). The Gauss law constraint implies that \( \Pi \) is \( x \)-independent. In addition, the equation of motion for \( A \) forces \( B \) to be \( x \)-independent when \( R \) is constant, as we are now assuming. Under these conditions, \( \mathcal{H} \) is \( x \)-independent, and its integral over the circle parametrized by \( \varphi \) yields a constant energy per unit length, namely, the tube tension
\[ \tau = \int d\varphi \mathcal{H}. \]
This is a function of \( R \) and a functional of \( \Pi(\varphi) \) and \( B(\varphi) \).

For an appropriate choice of units, the integrals
\[ q_0 = \int d\varphi \Pi \quad \text{and} \quad q_s = \int d\varphi B \]
are, respectively, the conserved IIA string charge and the D0-brane charge per unit length carried by the tube. The total D0-brane charge is also conserved, so imposing periodic boundary conditions on the tube, with period \( L \), implies conservation of \( q_0 \), and also that \( q_0 \) is quantized in multiples of some unit proportional to \( L^{-1} \). For fixed values of these charges, the tube tension is bounded from below:
\[ \tau \geq |q_s| + |q_0|, \]
with equality if, and only if,
\[ \Pi = q_s, \quad B = q_0, \quad R = \sqrt{|q_s q_0|}. \]
The crossed electric and magnetic fields generate a Poynting 2-vector density with
\[ \mathcal{P}_{\varphi} = \Pi B \]
as its only nonzero component. The integral of \( \mathcal{P}_\varphi \) over \( \varphi \) yields an angular momentum per unit length \( J = \Pi B = q_s q_0 \) along the axis of the cylinder. It is this angular momentum that supports the tube at the constant radius \( \sqrt{|q_s q_0|} \).

This would be the “critical” electric field if the magnetic field were absent, as is shown by the fact that \( \Delta = B \) when \(|E| = 1 \).

(III) Supersymmetry.—We now aim to show that the tubular D2-brane configuration just described is one-fourth supersymmetric, but as we also aim to relate it to some previously discussed D2-brane configurations we shall now drop the assumption that the world-volume fields are independent of \( x \) and \( \varphi \). The number of supersymmetries preserved by any D2-brane configuration is the number of independent Killing spinors \( \epsilon \) of the background for which
\[ \Gamma \epsilon = \epsilon, \]
where \( \Gamma \) is the matrix appearing in the “\( \kappa \)-symmetry” transformation of the world-volume spinors [12]. By introducing \( \Gamma_\hbar \) as the constant matrix, with unit square which anticommutes with all ten spacetime Dirac matrices, and \( (\gamma_1, \gamma_5, \gamma_6) \) as the induced world-volume Dirac matrices, we have [13]
\[ \Gamma = \Delta^{-1}(\gamma_{1 \varphi} + E \gamma_\varphi \Gamma_{\hbar}) + B \gamma_1 \Gamma_{\hbar}. \]
For the D2-brane configuration of interest here
\[
\gamma_1 = \Gamma_+ \gamma_x = \Gamma_+ + R \Gamma_5
\]
\[
\gamma_5 = R \Gamma_\Phi + \Gamma_5 \Gamma_5
\]
where \( \Gamma_+ \), \( \Gamma_5 \), and \( \Gamma_\Phi \) are the constant Minkowski spacetime Dirac matrices (with \( \Gamma_5^2 = 1 \)). For the spacetime coordinates that we have chosen, any Killing spinor \( \epsilon \) can be written as
\[
\epsilon = M_+ \epsilon_0, \quad M_+ = \exp(\frac{i}{2} \Phi R \Gamma_5),
\]
where \( \epsilon_0 \) is a constant 32-component spinor of spin \( 1, 9 \). The condition for preservation of supersymmetry can now be written as
\[
0 = M_+ (R \Gamma_+ \Gamma_\Phi + B \Gamma_+ \Gamma_5 + R_+ R_5 \Gamma_5 - \Delta) \epsilon_0
\]
\[
+ M_- \gamma_5 \Gamma_1 (\Gamma_+ \gamma_5 + E) \epsilon_0.
\]
It is clear that in order to satisfy this equation for all values of \( \phi \) (equal to \( \Phi \) for our gauge choice) both terms on the right-hand side must vanish independently. From the vanishing of the second term we recover the condition that \( E = \pm 1 \), and we further deduce that \( \epsilon_0 \) must satisfy
\[
\Gamma_+ \Gamma_1 \epsilon_0 = -\text{sgn}(E) \epsilon_0.
\]
The vanishing of the first term leads to \( R = 0 \) and
\[
(R \Gamma_+ \Gamma_\Phi + B \Gamma_+ \Gamma_5) \epsilon_0 = \sqrt{R^2 R_5^2 + B^2} \epsilon_0.
\]
For the supertube configuration \( R_+ = 0 \), and \( B \) is constant, so the constraint above becomes simply
\[
\Gamma_+ \Gamma_1 \epsilon_0 = \text{sgn}(B) \epsilon_0.
\]
The two conditions (21) and (23) are compatible and imply preservation of one-fourth supersymmetry; the minimal energy tubular D2-brane configuration is a supertube. Note that the constraints (21) and (23) are those associated with, respectively, a IIA superstring (in the X direction) and D0-brane charge; in particular, there is no trace of the D2-brane in these conditions. The physical reason for this is that a cylindrical D2-brane carries no net D2-brane charge.

When \( R_+ \neq 0 \), the vanishing of the first term on the right-hand side of (20) implies that
\[
B = B_0 R R_+
\]
for some constant \( B_0 \), and the constraint on the spinor becomes
\[
(\Gamma_\Phi + B_0 \Gamma_+ \Gamma_5) \epsilon_0 = \sqrt{1 + B_0^2} \epsilon_0.
\]
The Gauss law now implies that
\[
R(\chi) = C e^{x/E_0}
\]
for some constants \( C \) and \( E_0 \). This is just the “Blon in a magnetic background” solution of [9] representing a IIA string ending on a bound state of D2-branes and D0-branes. To see this more clearly, first note that the constraint (25) indeed corresponds to that of a D2-D0 bound state in the \( (R, \Phi) \) plane. Second, we can invert (26) to find
\[
x = E_0 \ln(R/C),
\]
which shows that \( x \) is a harmonic function on the D2-brane two-dimensional world space, as expected for the Blon [14,15]. Finally, by using (27) we can rewrite the BI field strength (5) as
\[
F = \frac{E_0}{R} dt \wedge dR + B_0 RdR \wedge d\phi.
\]
This corresponds to a radial Coulomb-like electric field on the D2-brane world space, as expected from the charge at the end point of the string, and to a constant density of D0-brane charge per unit world space area, precisely as in the solution of [9].

(IV) Discussion.—By definition, a Dp-brane is a sink for IIA string charge. However, the D0-brane, being pointlike, is special because the IIA charge has nowhere to go and must exit on another IIA superstring. Thus D0-branes can appear as “beads” on a IIA superstring, breaking the one-half supersymmetry of the string to one-fourth supersymmetry. Of course, quantum mechanics will ensure that the ground state of such a superstring is one for which the D0-brane charge is uniformly distributed along the string. This “charged” IIA superstring will have a tension exactly equal to the D2-brane supertube discussed above, \( \tau = |q_s| + |q_0| \). What distinguishes it is the angular momentum; the charged superstring has zero angular momentum while the supertube has angular momentum per unit length \( J \) equal to \( q_s q_0 \). In principle, \( J \) can be specified independently of the string and D0-brane charges, so, given \( q_s \), and \( q_0 \), we might expect there to be some supersymmetric string/tube configuration with arbitrary \( J \). As long as \( |J| \leq |q_s q_0| \) it is not difficult to see what this will be: a supertube with angular momentum per unit length \( J \) together with a charged superstring along its central axis. Because \( J \) is quantized in the same units as \( B \), it is always possible for excess string and D0-charge to “condense” out of a tubular D2-brane to leave behind a supertube supported from collapse by an given \( |J| \) less than \( |q_s q_0| \). On the other hand, it is unclear what the ground state could be when \( |J| > |q_s q_0| \). It is conceivable that \( |q_s q_0| \) is an upper bound on the angular momentum of a supersymmetric IIA superstring configuration with charges \( q_s \) and \( q_0 \), and that supersymmetry is spontaneously broken when \( |J| > |q_s q_0| \).

There is some similarity here to the status of angular momentum in the context of black holes of \( D = 5 \) supergravity [16–18]. The black hole mass is determined entirely by its charge, if it is supersymmetric. For a given charge there is a one-parameter family of supersymmetric black hole spacetimes, parametrized by an angular momentum, but there is a critical value of the angular momentum beyond which the physics is qualitatively different. There is
also a similarity to the suggestion [6] that the “giant graviton” ground state is not supersymmetric above a critical value of the angular momentum, although the supersymmetry of relevance in that case is AdS rather than Poincaré.

Finally, we wish to comment on some $M$-theory configurations dual to the D2-brane supertube. The first one involves the M2-brane: the D2-brane action is equivalent to the action for the $D_{11}$ supermembrane, the equivalence involving an exchange of the BI 1-form potential for a periodically identified scalar field $\Psi$ (with unit period) representing position in the 11th dimension [13]. Specifically, one has

$$\partial_i \Psi = \frac{g_{ij} \epsilon^{jkl} F_{kl}}{2\sqrt{-\det(g + \tilde{F})}}.$$  

(29)

For the D2-brane supertube this yields

$$\Psi = \Pi \varphi - t.$$  

(30)

First, this implies that the $M2$-brane is wound $\Pi$ times around the 11th dimension, as expected given the identification of $\Pi$ with IIA string charge. Second, it implies that $\Pi$ must be an integer, obviously equal to the number of IIA strings dissolved in the original D2-brane. Third, it implies that there is a wave at the speed of light in the 11th dimension, as expected from the D0-brane charge; the momentum of this wave is proportional to $\bf{B}$. Note that if the $x$ dimension is periodically identified with period $L$ then one can take $L \to 0$ to arrive at a new (1/4)-supersymmetric IIA configuration in which a helical string rotates, producing a net momentum along the axis of the helix [19].

Another dual $M$-theory configuration consists of an $M5$-brane with topology $\mathbb{R}^4 \times S^1$, carrying dissolved membrane charges oriented along orthogonal planes in $\mathbb{R}^4$. To see this, let the $S^1$ be parametrized by $\varphi$, and let the two planes span the directions 1–2 and 3–4. Reducing to the IIA theory by compactifying the 4-direction, and $T$-dualizing along 2 and 3, leads to the D2-brane supertube.

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10. This is an oversimplification because the integral of the charge density of the cylindrically symmetric solution has an infrared logarithmic divergence; for this reason the charge $Q$ mentioned below should be regarded only as a measure of scale.
19. We owe this observation to Barak Kol and Juan Maldacena; note that $T$ duality along the axis of the helix interchanges the string with the wave, yielding a IIB version of the rotating helical string.