Soliton Collision in General Relativity

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An exact solution of the Einstein equations in vacuum representing two pairs of gravitational solitons propagating on an expanding universe is given and studied. It is suggested that the solitons evolve from quasiparticles to pure gravitational waves. Two of the four solitons collide and the focusing produced on null rays is studied. Although the space-time following the collision is highly distorted, null rays do not focus to a singularity.

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We shall be interested in two main problems: first, the characterization of the solitons through the Riemann tensor and their scalar invariants; second, the collision of solitons which will be studied by means of the expansion and shear of null congruences propagating on the metric field, and which provides an example of the interaction of nonlinear waves in general relativity.

The metric considered admits two commuting orthogonal spacelike Killing vectors; it is essentially an Einstein-Rosen metric but differs in its global structure and interpretation. It involves only three "soliton" parameters and a "seed" parameter ($\delta$). It is the diagonal, vacuum, four-soliton solution

$$ds^2 = f(t,z)(dz^2 - dt^2) + g_{11}(t,z)dx_1^2 + g_{22}(t,z)dx_2^2,$$

with

$$g_{11} = g_{22} = t^2 / g_{11},$$

$$f = t^{(t^2 - 1)/2 - \delta} \frac{\sigma_1^2 \sigma_2^5}{H_1 H_2 (1 - \sigma_1)(1 - \sigma_2)^2} \left( \frac{(\sigma_1^{\gamma} + \sigma_2^{\gamma})^2}{(1 + \sigma_1)(1 + \sigma_2)} - \frac{6 \omega^2 \sigma_1^2 \sigma_2^2}{(1 - \sigma_1^2)(1 - \sigma_2^2)^2} \right)^2,$$

(1)

The parameter $\delta \geq 0$ comes from the Kasner seed metric:

$$ds^2 = -t^{(t^2 - 1)/2}(dz^2 - dt^2) + t^{1 + 5} dx_1^2 + t^{1 - 5} dy_2^2.$$

(2)

The value $\delta = 0$ corresponds to the axisymmetric Kasner metric. For $\delta > 1$ the $z$ axis is expanding and for $\delta < 1$ it is contracting. Flat space is given by $\delta = 1$; we shall not consider that possibility in what follows.

In order to understand the behavior of the metric it is convenient to define the following asymptotic space-time regions: the "far region," $t \ll |z| \rightarrow \infty$; the "light-cone region," $|z| \approx t \rightarrow \infty$, and the "interaction region," $|z| \ll t \rightarrow \infty$. In all those asymptotic regions the metric tends towards the Kasner background. However, the longitudinal expansion, which is given by the coefficient $f_1$, is the Kasner background value $f_1$ in the far region only. In the light-cone and interaction regions the longitudinal expansion is $f_1$.
multiplied by a factor depending on the soliton parameters. For instance, for the solutions for which \( w_1^2 \sim w_2^2 \ll 1 \), in the interaction region the asymptotic values are \( f^{-f_0}/w^4 \), while in the light-cone region they are \( f^{-f_b}/w^4 \) along the light cones of the inner solitons (those which collided) and \( f^{-f_b}/w^4 \) along the light cones of the outer solitons.

The soliton structure and the intrinsic properties of the metric (1) can be seen from the Riemann tensor. This tensor can be easily calculated by iterative use of the formulas for the derivatives of \( \sigma_i \), which can be deduced from the pole trajectory equations.\(^1\) For a metric of type (1) the Riemann tensor has only three independent components: One can write it in terms of an “electric,” \( E \), and a “magnetic,” \( B \), 3 \( \times \) 3 matrices. The only nonzero components are \( E_{11} = e_1 \), \( E_{22} = e_2 \), \( E_{33} = e_3 \).\(^1\) and \( B_{12} = B_{21} = k \) (for the Kasner metric the Riemann tensor depends only on \( t \), and \( b = 0 \)). The soliton structure appears in the Riemann components: For small values of \( w^2 \) the four solitons are narrow pulses and the limit \( w^2 \to 0 \) produces \( \delta \) functions.

Frame-independent soliton properties are seen from the two nonzero curvature scalar invariants: \( I_1 = (e_1^2 + e_2^2 + e_3^2 - 2k^2)/2 \) and \( I_2 = (e_1^3 + e_2^3 + e_3^3 + 3k^2 e_2)/6 \). The evolution in time of the ratio \( I_1/I_1^0 \) (\( I_1^0 \) is the Kasner value) is shown in Fig. 1 for a model in which \( \delta = 0 \). Before the collision four solitons with small tails are seen. After the collision the outer solitons are clear but the intensities of the inner solitons are very small because of the \( 1/w^4 \) factor of the longitudinal expansion. The same factor makes the value of \( I_1 \) in the interaction region very small too.

The behavior of the Riemann tensor and its scalar invariants indicates that metric (1) represents intrinsic inhomogeneities of the gravitational field propagating on a Kasner space-time.

From the algebraic classification of the metric in the light-cone region one can see that the solitons, which are localized there, seem to evolve towards pure gravitational waves. In fact, in that region the leading terms of the Riemann tensor are \( e_1 - e_2 - k \) which is typical of pure gravitational waves: type-N metrics in the Petrov classification [in general, however, metrics (1) are globally of Petrov type I].

Further features of metric (1) can be seen by projecting the Riemann tensor over a null tetrad \( \hat{n}, \hat{\imath}, \hat{m}, \text{and} \hat{\jmath} \): \( \hat{n} = 2^{-1/2} (\hat{\imath} + \hat{\jmath}) \), \( \hat{\imath} = 2^{-1/2} (\hat{n} + \hat{\jmath}) \), and \( \hat{m} = 2^{-1/2} (\hat{n} - \hat{\imath} + \hat{\jmath}) \) where \( \hat{n} \) is a geodesic null vector field that lies tangent to the rightward-traveling solitons. With that tetrad the nonnull Riemann components are \( \psi_0 = \frac{1}{2} [(e_2 - e_1)/2 + k]/\hat{\jmath}, \psi_2 = -e_3/2 \), \( \psi_3 = [(e_2 - e_1)/2 - b]/\hat{\jmath} \). \( \psi_0 \) and \( \psi_3 \) contain the radiative part of the field whereas \( \psi_2 \) contains its Coulomb part. Comparing with the homogeneous background we see that solitons induce inhomogeneities in those fields; in particular, \( \psi_2 \) will induce time-varying tidal forces to test particles on the plane orthogonal to the propagation of the rightward solitons.

In Fig. 2, the evolution of the field \( \psi_3 \) is shown. It indicates the gravitational strength of the rightward-traveling solitons. The soliton structure as a nondispersive wave is clear: The solitons are localized in their light-cone regions; the amplitude of \( \psi_3 \) decreases there as \( t^{1.2} \) but this is a consequence of the background expansion and

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**FIG. 1.** Time evolution of the ratio of the first curvature scalar to the Kasner value, \( I_1/I_1^0 \) for a model in which the background in the axially symmetric Kasner metric, \( \delta = 0 \). The curves are represented against the propagation axis (\( z \) axis). The numbers and origins of the solitons are \( w_1^2 = w_2^2 = 0.05 \) and \( z_2^0 = 1 \). The different curves from bottom to top of figure represent the time sequence \( t = 0.1, 0.3 \) (before collision), 0.5 (collision time: \( t_c \)), and 0.7 (after collision).

**FIG. 2.** Time evolution of the Riemann tensor component \( \psi_3 \) giving the gravitational strength of the rightward-traveling solitons. Same model and parameters as in Fig. 1. The Kasner background value is \( \psi_3^0 = 0 \). The three curves from left to right of figure represent the time sequence \( t = 0.1, t_c = 0.5, \) and \( t = 0.9 \).
shear on such a null congruence as a consequence of its overall expansion on the x-y plane. The solitons will produce inhomogeneities on θ and σ and as a consequence of their energy they will focus the null rays. In fact, in Fig. 3, the ratio of the expansions θ/θ* of the null rays is shown. Initially the focusing produced by the four solitons is clear. By the Raychaudhuri equation this can be interpreted as due to the gravitational "energy" of the solitons which can be measured by the θ² and σ² they produce. After the collision θ is getting small. This can be interpreted as focusing produced by the gravitational energy of the collision. This is quite different from what one would expect from the simple superposition of linear waves; it is a consequence of the nonlinear interaction of the solitons. The expansion, however, is still positive, θ > 0, and there will be no convergence of null rays to a singularity. The overall expansion of the background prevents the formation of singularities in the future.

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