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Exact Solution to the Mean Exit Time Problem for Free Inertial Processes Driven by Gaussian White Noise

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We obtain the exact analytical expression, up to a quadrature, for the mean exit time, $T(x,v)$, of a free inertial process driven by Gaussian white noise from a region $(0,L)$ in space. We obtain a completely explicit expression for $T(x,0)$ and discuss the dependence of $T(x,v)$ as a function of the size L of the region. We develop a new method that may be used to solve other exit time problems.

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The study of the statistics of extremes and especially mean first passage times and mean exit times is of great importance in a wide variety of problems in mathematics, physics, chemistry, and engineering. Perhaps one of the most relevant examples of an exit time problem in physics is the evaluation of the average time required for a given system to escape (due to noise) from a stable state. This problem, which is often referred to as "Kramers problem," appears in many physical phenomena [1], and it has been the object of intense research since it was first studied by Kramers in 1940 [2]. A classical example of a first passage time problem in engineering is the time required for a mechanical structure to first reach a critical amplitude exceeding a stability threshold and collapse due to random external vibrations (wind, ocean waves, earthquakes, etc.) [3]. Another classical example in communication theory is the so-called "false alarm" problem, where one tries to measure the time at which internal fluctuations cause the current or the voltage in an electrical circuit to attain some critical value for which an alarm is triggered [4].

The mean first passage time problem is well understood, and closed analytical results are available, for independent processes [5], for one-dimensional Markov processes such as one-dimensional diffusion processes [6], and for one-dimensional non-Markov processes such as dichotomous and shot noise processes [7]. Thus, for example, for the simple one-dimensional dynamical processes described by the Langevin equation $\dot{X} = \xi(t)$, where $\xi(t)$ is Gaussian white noise with the correlation function $\langle \xi(t)\xi(t')\rangle = D\delta(t - t')$, the mean first-passage time (MFPT) to a given label, say, L, is ∞, while the mean exit time (MET) out of an interval $(0,L)$ is given by the simple expression $T(x) = x(L - x)/D$. The extension of these results to general one-dimensional diffusion processes is not difficult [6]. Nevertheless, the evaluation of MFPT and MET for higher order systems is an extremely difficult problem and, to our knowledge, no exact analytical solution exists even in the simplest cases. There are, however, situations where it is possible to obtain approximate solutions to MFPT or MET problems for higher order systems. This is the case of weakly damped physical systems where there are nearly conserved quantities whose variation in time is slower than that of the dynamical variables (e.g., the energy or the amplitude of an oscillation). In this situation the mean exit time problem reduces to that of first-order processes for this nearly conserved quantity and an approximate solution can be obtained. Another situation is that of strongly damped inertial systems. In this case the so-called "adiabatic approximation," which roughly consists in neglecting the inertial effects, reduces the system to a one-dimensional description and MFPT and MET are readily obtained [2,5,8].

Our aim in this Letter is to present the exact analytical solution to the mean exit time out of an interval $(0,L)$ for the displacement of an undamped free particle under the influence of a random acceleration

$$\ddot{X}(t) = \xi(t), \tag{1}$$

where $\xi(t)$ is zero-centered Gaussian white noise with correlation function $\langle \xi(t)\xi(t')\rangle = D\delta(t - t')$. Note that here no damping term is present and, in consequence, we cannot apply the adiabatic approximation mentioned above. Let us denote by v the velocity of our inertial processes, then the mean exit time of process (1) is a function of the displacement and the velocity, $T = T(x, v)$, and obeys the partial differential equation [8,9]

$$\frac{1}{2} D \frac{\partial^2 T}{\partial v^2} + v \frac{\partial T}{\partial x} = -1, \tag{2}$$

with boundary conditions

$$T(L, v) = 0 \quad \text{if } v \geq 0, \quad T(0, v) = 0 \quad \text{if } v \leq 0. \tag{3}$$

Boundary value problems like this are known in mathematics literature as a "problem of Fichera," and it was shown in the late fifties that they are well-posed boundary value problems [10]. In physics, boundary conditions similar to those given by Eq. (3) were first introduced by Wang and Uhlenbeck [11] for the joint probability density function of the displacement and the velocity. Nevertheless, any attempt to solve them, even in the simple case (2) and (3), has failed to result in closed and exact expressions for $T(x, v)$ [12]. The reason for this difficulty lies in the special form of the boundary conditions (3) with data on a nonsmooth boundary at $v = 0$ (see Fig. 1).

We first observe from Eqs. (2) and (3) that $T(x, v)$ satisfies the fundamental symmetry relation

$$T(x, v) = T(L - x, -v). \tag{4}$$

This equation implies the following continuity conditions at $v = 0$:

$$T(x, 0) = T(L - x, 0),$$
$$\left.\frac{\partial T(x, v)}{\partial v}\right|_{v=0} = -\left.\frac{\partial T(L - x, v)}{\partial v}\right|_{v=0}. \tag{5}$$

We note that Eq. (4) allows us to write the solution $T(x, v)$ for all v once we know the solution of (2) and (3) for, say, $v \leq 0$. We thus assume that $v \leq 0$ and write

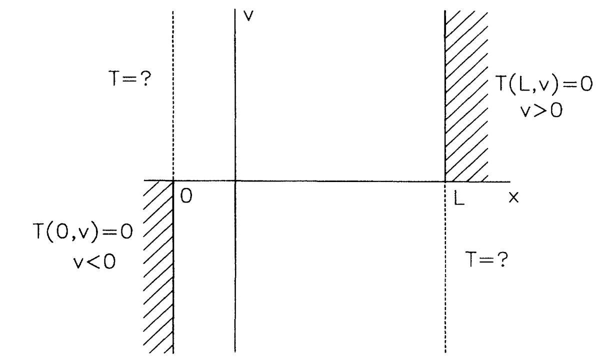

FIG. 1. Boundary conditions (3) for $T(x, v)$ in the position-velocity (x, v) phase space.

$T_1(x, v) \equiv T(x, v)$, if $v \leq 0$. Hence, in dimensionless units defined by

$$u = x/L, \quad y = -(2/LD)^{1/3} v,$$
$$T_1'(u, y) = (D/2L^2)^{1/3} T_1(x, v), \tag{6}$$

the MET $T_1'(u, y)$ obeys the equation

$$\frac{\partial^2 T_1'}{\partial y^2} - y \frac{\partial T_1'}{\partial u} = -1, \tag{7}$$

with boundary conditions

$$T_1'(0, y) = 0 \quad (y \geq 0), \quad T_1'(u, \infty) = 0 \quad (0 \leq u \leq 1). \tag{8}$$

Although the range of u is bounded by an upper bound, it is still permissible to define the Laplace transform of $T_1'(u, y)$ that results in the following Airy equation:

$$\frac{d^2 \hat{T}_1}{d\zeta^2} - \zeta \hat{T}_1 = -s^{-5/3}, \tag{9}$$

where $\zeta \equiv s^{1/3} y$ and $\hat{T}_1(s, y)$ is the Laplace transform of $T_1'(u, y)$. The general solution of Eq. (9), under the condition $\hat{T}_1(s, \infty) = 0$, reads

$$\hat{T}_1(s, \zeta) = -3^{1/3}\Gamma(1/3) s^{-1/3} \hat{\phi}(s) \mathrm{Ai}(\zeta) + \pi s^{-5/3}\left[\mathrm{Bi}(\zeta)\int_\zeta^\infty \mathrm{Ai}(t)dt + \mathrm{Ai}(\zeta)\int_0^\zeta \mathrm{Bi}(t)dt + 3^{-1/2}\mathrm{Ai}(\zeta)\right], \tag{10}$$

where $\mathrm{Ai}(\zeta)$ and $\mathrm{Bi}(\zeta)$ are Airy functions and $\hat{\phi}(s)$ is the Laplace transform of the (unknown) derivative of $T_1'(u, y)$ with respect to y at $y = 0$, that is, $\phi(u) \equiv \partial T_1'(u, y)/\partial y|_{y=0}$. If we now set $\zeta = 0$ in Eq. (10) and invert the Laplace transform, we obtain the following expression for $T_1'(u, 0)$:

$$T_1'(u, 0) = \frac{-1}{3^{1/3}\Gamma(2/3)} \int_0^u \frac{\phi(z)}{(u - z)^{2/3}} dz + \frac{\pi}{3^{1/6}\Gamma^2(2/3)} u^{2/3}. \tag{11}$$

This is a formal expression for $T_1'(u, 0)$, since the function $\phi(z)$ remains unknown. From the definition of $\phi(u)$ and the second matching condition (5) we see that $\phi(u) = -\phi(1-u)$. Therefore, the substitution of Eq. (11) into the first matching condition (5) shows that the unknown function $\phi(u)$ satisfies the integral equation

$$\int_0^1 \frac{\phi(z)}{|u-z|^{2/3}}\,dz = \frac{3^{2/3}\Gamma(1/3)}{2}[u^{2/3} - (1-u)^{2/3}]. \tag{12}$$

We will show elsewhere [13] that the solution to this equation is given by

$$\phi(z) = Mz^{-1/6}(1-z)^{-1/6}[F(1,-\tfrac{2}{3};\tfrac{5}{6};1-z) - F(1,-\tfrac{2}{3};\tfrac{5}{6};z)], \tag{13}$$

where $F(a,b;c;z)$ is the Gauss hypergeometric function and $M = 3^{1/6}\Gamma(3/2)/2\Gamma(5/6)\Gamma(4/3)$.

Having obtained the explicit expression of $\phi(u)$ we are now in the position of evaluating $T(x, v=0)$. In effect, the substitution of Eq. (13) into Eq. (11) yields the exact expression of $T(x,0)$. In the original units [cf. Eq. (6)] we have

$$T(x,0) = N\left(\frac{2L^2}{D}\right)^{1/3}\left(\frac{x}{L}\right)^{1/6}\left(1-\frac{x}{L}\right)^{1/6} \times\left[F\left(1,-\frac{1}{3};\frac{7}{6};\frac{x}{L}\right) + F\left(1,-\frac{1}{3};\frac{7}{6};1-\frac{x}{L}\right)\right], \tag{14}$$

where $N = (4/3)^{-5/6}/\Gamma(4/3)$. This constitutes one key result of this paper. Figure 2 shows the complete agreement between the expression of $T(x,0)$ given by Eq. (14) and simulation data. Monte Carlo values were obtained by simulating a free inertial system driven by Markovian dichotomous noise [9] of value $\pm a$ and average switching time λ^{-1}. This noise is known to converge in distribution to a Gaussian white noise of intensity D when $a \to \infty$ and $\lambda \to \infty$ provided that $D = a^2/\lambda$. Several simulations were run for growing values of a and λ and checked to converge.

Another interesting quantity, closely related to the exit time problem, is the averaged mean exit time $\overline{T}_L(v)$ over all initial positions x. If we assume that x is uniformly distributed on the interval $(0,L)$ then

$$\overline{T}_L(v) = \frac{1}{L}\int_0^L T(x,v)\,dx. \tag{15}$$

When $v=0$ this averaged time reads

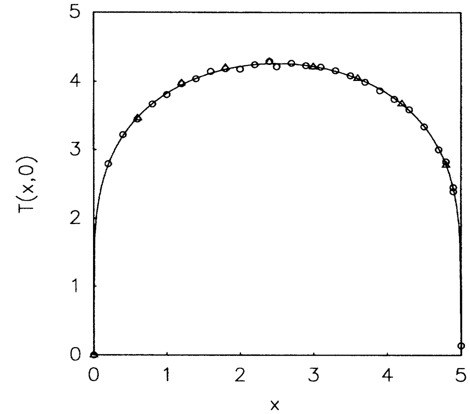

FIG. 2. $T(x,0)$ as a function of x, for noise intensity $D=1$ and $L=5$ (solid line). Simulation data correspond to $a=10$ and $\lambda=100$ (circle symbols) and $a=200$ and $\lambda=40\,000$ (triangles). Error bars are of the order of symbol sizes ($\sigma \sim 10^{-2}$).

$$\overline{T}_L(0) = \frac{3^{7/3}\Gamma(1/6)}{40\sqrt{\pi}}\left(\frac{2L^2}{D}\right)^{1/3}. \tag{16}$$

An interesting feature of this expression is that the dependence of $\overline{T}_L(0)$ on the size L of the interval is $L^{2/3}$. We recall that the dynamical exponent ν of a random process $X(t)$, which we assume to be zero centered, is given by $\langle X^2(t)\rangle \sim t^{2\nu}$. We have shown elsewhere [14] that for free inertial processes driven by white noise, the dynamical exponent of the displacement $X(t)$ is $\nu = 3/2$. Therefore, Eq. (16) clearly demonstrates the reciprocity between exponents. This reciprocity had been conjectured in a previous work by a scaling argument [9].

We will now present the second key result of the paper, that is, the exact expression of the MET, $T(x,v)$, for all values of the velocity v. To this end we must invert the Laplace transform expression of $T_1'(u,v)$ given by Eq. (10). At this point it is useful to recall that Airy functions $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$ are linear combinations of modified Bessel functions of order 1/3. Once we write the Airy functions in terms of Bessel functions, all terms appearing in the resulting equation can be Laplace inverted in closed form. We write the result as

$$T(x,v) = \left(\frac{2L^2}{D}\right)^{1/3}\left[A\left(\frac{x}{L},(2/LD)^{1/3}|v|\right)\Theta(-v) + A\left(1-\frac{x}{L},(2/LD)^{1/3}|v|\right)\Theta(v)\right], \tag{17}$$

where $\Theta(v)$ is the Heaviside step function and

$$A(u,y) \equiv \frac{3^{1/6}\Gamma(1/3)}{2\pi}\int_0^u \frac{e^{-y^3/9z}}{z^{2/3}}\left[\phi(u-z) + \frac{\pi 3^{1/6}}{\Gamma^2(1/3)}(u-z)^{1/3}\right]dz + (\pi^{1/2}/6)y^{1/2}\int_0^u \frac{e^{-y^3/18z}}{z^{1/2}}\left[I_{-1/6}\left(\frac{y^3}{18z}\right) + I_{1/6}\left(\frac{y^3}{18z}\right)\right]dz, \tag{18}$$

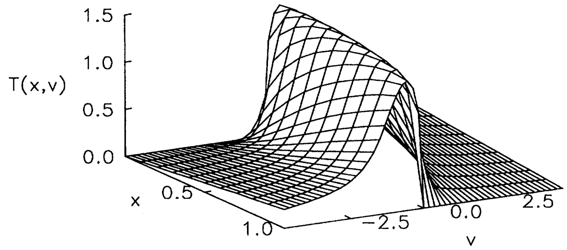

FIG. 3. Plot of the exact expression for $T(x,v)$ as a function of x and v, for $D = 1$ and $L = 1$.

where $\phi(z)$ is given by Eq. (13). We plot the complete solution (17) and (18) in Fig. 3.

Let us finally evaluate the averaged mean exit time over the initial displacement, $\overline{T}_L(v)$. The substitution of Eq. (17) into Eq. (15) reads

$$\overline{T}_L(v) = \left(\frac{2L^2}{D}\right)^{1/3} \int_0^1 A(u,(2/LD)^{1/3}|v|)du. \quad (19)$$

Note that when $L \to \infty$ and $|v| < \infty$ we have

$$\lim_{L\to\infty} A(u,(2/LD)^{1/3}|v|) = A(u,0).$$

Now the integral on the right hand side of Eq. (19) does not depend on L. Therefore $\overline{T}_L(v) \sim L^{2/3}$ as $L \to \infty$. This asymptotic relation is valid for all values of velocity v provided that $|v|$ is finite.

We now briefly summarize the main results achieved. The mean exit time out of an interval for a free inertial process driven by Gaussian white noise has been exactly obtained up to a quadrature [cf. Eqs. (17) and (18)]. We have obtained a complete explicit expression of the MET when $v = 0$ [cf. Eq. (14)]. Moreover, when the initial position is uniformly randomized over the interval $(0, L)$, we have shown that the resulting mean exit time satisfies the following asymptotic relation for large L, $\overline{T}_L(v) \sim L^{1/\nu}$, where ν is the dynamical exponent of the inertial process and this relation becomes exact for all values of L when $v = 0$. We finally note that the procedure we have developed for solving the boundary value problem (2) and (3) may open a new way of dealing with a variety of similar problems with profound physical implications.

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